Zad 51.

$f: G \overset{\text{otw}}{\subseteq} \mathbb{R}^n \to \mathbb{R}$ klasy $C^2$, $x_0 \in G$

Lokalny wzór Taylora: $f(x_0+h) = f(x_0) + Df(x_0)h + \frac{1}{2} D^2 f(x_0)hh + \|h\|^2 \psi(h)$

$\psi(h)$ - ciągła w $(0,0)$

$\lim\limits_{h \to 0} \psi(h) = 0$

Niech $f: \mathbb{R}^3 \to \mathbb{R}$ klasy $C^2$, $0 = f(0) = D_i f(0) = D_i D_j f(0) = 0$ $\quad i,j = 1,2,3$, $i \neq j$

$D_1^2 f(0) = D_2^2 f(0) = D_3^2 f(0) = c$

Oblicz $\lim\limits_{x \to 0} \frac{f(x)}{\|x\|^2}$

Korzystając z lokalnego wzoru Taylora dostajemy

$$f(x) = f(0) + Df(0)x + \frac{1}{2} D^2 f(0)xx + \|x\|^2 \psi(x)$$

$f(0) = 0$

$Df(0) = (0, 0, 0)$

$$D^2 f(0) = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} = c \cdot I$$

~~Zaten~~

$$\cancel{f(x)} \; \frac{1}{2} D^2 f(0) x x = \frac{1}{2} \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2} c x_1^2 + \frac{1}{2} c x_2^2 + \frac{1}{2} c x_3^2 = \frac{1}{2} c \|x\|^2$$

Zatem

$$f(x) = \frac{1}{2} c \|x\|^2 + \|x\|^2 \psi(x), \quad \lim_{x \to 0} \psi(x) = 0$$

A stąd

$$\lim_{x \to 0} \frac{f(x)}{\|x\|^2} = \lim_{x \to 0} \frac{\frac{1}{2} c \|x\|^2 + \|x\|^2 \psi(x)}{\|x\|^2} = \lim_{x \to 0} \frac{1}{2} c + \psi(x) = \frac{1}{2} c + \lim_{x \to 0} \psi(x) = \frac{1}{2} c$$

Jak znaleźć ekstrema lokalne funkcji?

Ogólnie: $f: G \overset{otw}{\subseteq} \mathbb{R}^n \to \mathbb{R}$, $f$ klasy $C^2$

Warunek konieczny istnienia ekstremum w $x_0$: $Df(x_0) = 0$

Dodatkowy warunek wystarczający istnienia ekstremum w $x_0$:

Jeśli forma kwadratowa $h \mapsto D^2 f(x_0)hh$ jest dodatnio określona to $x_0$ - minimum lokalne

ujemnie określona to $x_0$ - maksimum lokalne

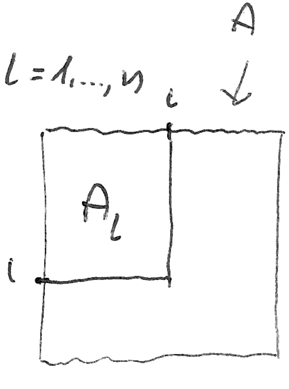
Kiedy forma kwadratowa jest dodatnio/ujemnie określona?

$D^2 f(x_0)hh = hAh$   $A$ - dodatnio określona $\iff \det A_l > 0$, $l = 1, \dots, n$

$A$ - ujemnie określona $\iff (-1)^l \det A_l > 0$

~~Dla~~ $\mathbb{R}^2$:

$A$ - nieokreślona gdy przyjmuje zarówno wartości dodatnie jak i ujemne



Dla $\mathbb{R}^2$



$$D^2 f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$ – macierz Hessego

$$W(x,y) = \det D^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$D^2 f(x,y)$ dodatnio określona, gdy:

1) $\frac{\partial^2 f}{\partial x^2} > 0$

2) $W(x,y) > 0$

$D^2 f(x,y)$ ujemnie określona, gdy

1) $\frac{\partial^2 f}{\partial x^2} < 0$

2) $W(x,y) > 0$

Schemat postępowania przy szukaniu ekstremów lokalnych funkcji $f = f(x,y)$ 89

1) Znajdujemy wszystkie pary $(x,y)$ spełniające układ równań

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \quad (*)$$

2) Dla każdej takiej pary ~~sp~~ $(x,y)$ spełniającej $(*)$ sprawdzamy, czy:

a) $\frac{\partial^2 f}{\partial x^2}(x,y) > 0$, $W(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$

Jeśli tak, to $f$ ma w $(x,y)$ minimum lokalne. Wyliczamy to minimum

b) $\frac{\partial^2 f}{\partial x^2}(x,y) < 0$, $W(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$

Jeśli tak, to $f$ ma w $(x,y)$ maksimum lokalne. Wyliczamy to maksimum.

c) $W(x,y) < 0 \Rightarrow$ w $(x,y)$ nie ma ekstremum

Zad 52. Znaleźć ekstrema lokalne

1. $f(x,y) = x^2+y^2-2x+6y$

Warunek konieczny

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 2x-2=0 \\ \frac{\partial f}{\partial y}(x,y) = 2y+6=0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=-3 \end{cases}$$

Warunek wystarczający

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 2 \qquad D^2f(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$W(x,y) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \qquad \frac{\partial^2 f}{\partial x^2}(x,y) = 2 > 0$$

Zatem $D^2f(1,-3)$ jest dodatnio określona, czyli $f$ ma w $(1,-3)$ minimum lokalne równe $f(1,-3) = -10$

2. $f(x,y) = x^3 + 3xy^2 - 6xy$ (91)

Warunek konieczny

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6y = 0 \\ \frac{\partial f}{\partial y} = 6xy - 6x = 0 \end{cases}$$

$6x(y-1) = 0 \Rightarrow x = 0 \vee y = 1$

$$\begin{cases} x = 0 \\ 3y^2 - 6y = 3y(y-2) = 0 \Rightarrow y = 0 \vee y = 2 \end{cases} \quad \begin{cases} x_1 = 0 \\ y_1 = 0 \end{cases} \quad \begin{cases} x_2 = 0 \\ y_2 = 2 \end{cases}$$

$$\begin{cases} y = 1 \\ 3x^2 - 3 = 3(x-1)(x+1) = 0 \Rightarrow x = -1 \vee x = 1 \end{cases} \quad \begin{cases} x_3 = -1 \\ y_3 = 1 \end{cases} \quad \begin{cases} x_4 = 1 \\ y_4 = 1 \end{cases}$$

Warunek wystarczający

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial x \partial y} = 6y - 6, \quad \frac{\partial^2 f}{\partial y^2} = 6x \quad D^2 f(x,y) = \begin{pmatrix} 6x & 6y-6 \\ 6y-6 & 6x \end{pmatrix}$$

$$W(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 36x^2 - 36(y-1)^2$$

$(x_1, y_1) = (0,0)$: $\quad \frac{\partial^2 f}{\partial x^2} = 6x$ , $W(x,y) = 36x^2 - 36(y-1)^2$ 92

$W(x_1, y_1) = -36 < 0 \Rightarrow$ nie ma ekstremum $\quad f(x,y) = x^3 + 3xy^2 - 6yx$

$(x_2, y_2) = (0,2)$

$W(x_2, y_2) = -36 < 0 \Rightarrow$ nie ma ekstremum

$(x_3, y_3) = (-1, 1)$

$W(x_3, y_3) = 36 > 0$ , $\frac{\partial^2 f}{\partial x^2}(x_3, y_3) = -6 < 0 \Rightarrow D^2 f(x_3, y_3)$ ujemnie określona

Zatem $f$ ma w $(x_3, y_3) = (-1, 1)$ maksimum lokalne równe $f(-1,1) = 2$

$(x_4, y_4) = (1,1)$

$W(x_4, y_4) = 36 > 0$, $\frac{\partial^2 f}{\partial x^2}(x_4, y_4) = 6 > 0 \Rightarrow D^2 f(x_4, y_4)$ dodatnio określona

Zatem $f$ ma w $(x_4, y_4) = (1,1)$ minimum lokalne równe $f(1,1) = -2$

3. $f(x,y) = xy + e^{-x^2-y^2}$ (93)

Warunek konieczny

$$\begin{cases} \frac{\partial f}{\partial x} = y - 2x e^{-x^2-y^2} = 0 \quad / \cdot y \\ \frac{\partial f}{\partial y} = x - 2y e^{-x^2-y^2} = 0 \quad / \cdot x \end{cases} \xRightarrow{\text{odejmując równania}} y^2 - x^2 = 0 \Rightarrow (y-x)(y+x) = 0 \quad y = x \lor y = -x$$

1) $y = x$

$$x - 2x e^{-2x^2} = x(1 - 2e^{-2x^2}) = 0 \Rightarrow x = 0 \lor 2e^{-2x^2} = 1$$

$$-2x^2 = -\ln 2$$

$$x = \pm\sqrt{\frac{\ln 2}{2}}$$

$$\begin{cases} x_1 = 0 \\ y_1 = 0 \end{cases}, \quad \begin{cases} x_2 = \sqrt{\frac{\ln 2}{2}} \\ y_2 = \sqrt{\frac{\ln 2}{2}} \end{cases}, \quad \begin{cases} x_3 = -\sqrt{\frac{\ln 2}{2}} \\ y_3 = -\sqrt{\frac{\ln 2}{2}} \end{cases}$$

2) $y = -x$

$$x + 2x e^{-2x^2} = x(1 + 2e^{-2x^2}) = 0 \Rightarrow x = 0 \lor 2e^{-2x^2} = -1$$

to jest niemożliwe!

to rozwiązanie już znaleźliśmy

Warunek wystarczający $\frac{\partial f}{\partial x} = y - 2x e^{-x^2-y^2}$, $\frac{\partial f}{\partial y} = x - 2y e^{-x^2-y^2}$ 34

$f(x,y) = xy + e^{-x^2-y^2}$

$$\frac{\partial^2 f}{\partial x^2} = -2e^{-x^2-y^2} + 4x^2 e^{-x^2-y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1 + 4xy e^{-x^2-y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = -2e^{-x^2-y^2} + 4y^2 e^{-x^2-y^2}$$

• $(x_1,y_1) = (0,0)$ $\quad D^2 f(0,0) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ $\quad \frac{\partial^2 f}{\partial x^2}(0,0) = -2 < 0$, $\quad W(0,0) = 3 > 0$ $\Rightarrow D^2 f(0,0)$ ujemnie określona

$f$ ma maksimum lokalne w $(0,0)$ równe $f(0,0) = 1$

• $(x_2,y_2) = \left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right)$ $\quad D^2 f\left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right) = \begin{pmatrix} -1+\ln 2, & 1+\ln 2 \\ 1+\ln 2 & -1+\ln 2 \end{pmatrix}$ $\quad \frac{\partial^2 f}{\partial x^2} = -1+\ln 2 < 0$

$W\left(\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right) = (-1+\ln 2)^2 - (1+\ln 2)^2 = -4\ln 2 < 0 \Rightarrow$ nie ma ekstremum

• $(x_3,y_3) = \left(-\sqrt{\frac{\ln 2}{2}}, -\sqrt{\frac{\ln 2}{2}}\right)$, $\quad D^2 f\left(-\sqrt{\frac{\ln 2}{2}}, -\sqrt{\frac{\ln 2}{2}}\right) = \begin{pmatrix} -1+\ln 2 & 1+\ln 2 \\ 1+\ln 2 & -1+\ln 2 \end{pmatrix}$ $\quad \frac{\partial^2 f}{\partial x^2} = -1+\ln 2 < 0$

$W\left(-\sqrt{\frac{\ln 2}{2}}, \sqrt{\frac{\ln 2}{2}}\right) = -4\ln 2 < 0$ $\Rightarrow$ nie ma ekstremum

Zad ~~6~~5, 53 (95) $\boxed{\cancel{27}}$

1. $f(x,y) = e^{2x+3y}$

$\dfrac{\partial^{k+l} f}{\partial x^k \partial y^l} = ?$ , $\dfrac{\partial f}{\partial x} = 2e^{2x+3y}$ , $\dfrac{\partial f}{\partial y} = 3e^{2x+3y}$ – każda pochodna po $x$ to mnożenie przez 2, a każda pochodna po $y$ to mnożenie przez 3, zatem

$$\frac{\partial^{k+l} f}{\partial x^k \partial y^l} = 2^k 3^l e^{2x+3y}$$

$$D^3 f(x,y) h h' h'' = \frac{\partial^3 f}{\partial x^3} h_1 h_1' h_1'' + \frac{\partial^3 f}{\partial x^2 \partial y} \left( h_1 h_1' h_2'' + h_1 h_2' h_1'' + h_2 h_1' h_1'' \right) + \frac{\partial^3 f}{\partial x \partial y^2} \left( h_1 h_2' h_2'' + h_2 h_1' h_2'' + h_2 h_2' h_1'' \right) + \frac{\partial^3 f}{\partial y^3} h_2 h_2' h_2''$$

$$D^3 f(x,y) h h' h'' = 2^3 e^{2x+3y} h_1 h_1' h_1'' + 2^2 \cdot 3 e^{2x+3y} \left( h_1 h_1' h_2'' + h_1 h_2' h_1'' + h_2 h_1' h_1'' \right) + 2 \cdot 3^2 e^{2x+3y} \left( h_1 h_2' h_2'' + h_2 h_1' h_2'' + h_2 h_2' h_1'' \right) + 3^3 e^{2x+3y} h_2 h_2' h_2''$$

96 ~~28~~

2. $f(x,y) = xye^{x+y}$

$$\frac{\partial f}{\partial x} = ye^{x+y} + xye^{x+y} = (x+1)ye^{x+y} \qquad \frac{\partial}{\partial x}: \; xye^{x+y} \mapsto (x+1)ye^{x+y}$$

$$\frac{\partial f}{\partial y} = xe^{x+y} + xye^{x+y} = x(y+1)e^{x+y} \qquad \frac{\partial}{\partial y}: \; xye^{x+y} \mapsto x(y+1)e^{x+y}$$

Zgadujemy, że $\displaystyle\frac{\partial^{k+l} f}{\partial x^k \partial y^l} = (x+k)(y+l)e^{x+y}$

Korzystając z indukcji po $k$ oraz $l$ pokażemy, że powyższy wzór jest prawdziwy:

dla $k=0$, $l=0$ ok

Załóżmy, że jest prawdziwy dla $k, l$

$$\frac{\partial^{k+1+l} f}{\partial x^{k+1} \partial y^l} = \frac{\partial}{\partial x}\left[(x+k)(y+l)e^{x+y}\right] = (y+l)e^{x+y} + (x+k)(y+l)e^{x+y} = (x+k+1)(y+l)e^{x+y} \text{ ok}$$

$$\frac{\partial^{k+l+1} f}{\partial x^k \partial y^l} = \frac{\partial}{\partial y}\left[(x+k)(y+l)e^{x+y}\right] = (x+k)e^{x+y} + (x+k)(y+l)e^{x+y} = (x+k)(y+l+1)e^{x+y} \text{ ok}$$

$$D^3f(x,y)hh'h'' = (x+3)ye^{x+y}h_1h_1'h_1'' + (x+2)(y+1)e^{x+y}(h_1h_1'h_2'' + h_1h_2'h_1'' + h_2h_1'h_1'') + (x+1)(y+2)e^{x+y}(h_1h_2'h_2'' + h_2h_1'h_2'' + h_2h_2'h_1'') + x(y+3)e^{x+y}h_2h_2'h_2''$$

Zad ~~28~~ 54 $f: \mathbb{R}^n \to \mathbb{R}$ $\quad D^p f(x) = 0 \quad \forall_{x\in\mathbb{R}^n} \quad f(x) = ?$

Ze wzoru Taylora mamy:

$$f(x) = f(0) + \frac{Df(0)}{1!}x + \frac{D^2f(0)}{2!}x^2 + \ldots + \frac{D^{p-1}f(0)}{(p-1)!}x^{p-1} + \underbrace{\frac{D^pf(\theta x)}{p!}}_{\overset{0}{=}\ \text{(z założenia)}}x^p$$

Zatem $f(x) = f(x_1, \ldots, x_n)$ – dowolny wielomian zmiennych $x_1, \ldots, x_n$ stopnia co najwyżej $p-1$

Zad ~~28~~ 55

$c \in [a, b]$

wzór Taylora $\quad f(b) = f(a) + \frac{Df(a)}{1!}(b-a) + \frac{D^2f(a)}{2!}(b-a)^2 + \ldots + \frac{D^nf(a)}{n!}(b-a)^n + \frac{D^{n+1}f(c)}{n!}(b-a)^{n+1}$

$f(x,y) = x^y$ względem $(e,1)$ do pochodnych rzędu 2

$$f(x,y) = f(e,1) + \frac{Df(e,1)}{1!}\begin{pmatrix} x-e \\ y-1 \end{pmatrix} + \frac{D^2f(e,1)}{2!}\begin{pmatrix} x-e \\ y-1 \end{pmatrix}\begin{pmatrix} x-e \\ y-1 \end{pmatrix} + R$$

$f(e,1) = e$

$\frac{\partial f}{\partial x} = yx^{y-1}$, $\frac{\partial f}{\partial y} = \ln x \, x^y \quad Df(x,y) = (yx^{y-1}, \ln x \, x^y)$, $\quad Df(e,1) = (1, e)$

$$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = x^{y-1} + y x^{y-1} \ln x$$

$$\frac{\partial^2 f}{\partial y^2} = \ln^2 x \; x^y$$

$$D^2 f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \qquad D^2 f(e,1) = \begin{pmatrix} 0 & 2 \\ 2 & e \end{pmatrix}$$

98 $\boxed{\cancel{30}}$

$$x^y = e + (1,e)\begin{pmatrix} x-e \\ y-1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 2 & e \end{pmatrix}\begin{pmatrix} x-e \\ y-1 \end{pmatrix}\begin{pmatrix} x-e \\ y-1 \end{pmatrix} + R = e + (x-e) + e(y-1)$$

$$+ 2(x-e)(y-1) + \frac{e}{2}(y-1)^2 + R$$

59 $\boxed{\cancel{31}}$

Zad ~~31~~ 56 $f(x,y) = 2x^2 - xy - y^2 - 6x - 3y + 5$ - rozwinąć w szereg Taylora względem $(1,-2)$

To jest wielomian stopnia 2, zatem wystarczy rozwinąć do pochodnych rzędu 2

$$f(x,y) = f(1,-2) + \frac{Df(1,-2)}{1!}\begin{pmatrix} x-1 \\ y+2 \end{pmatrix} + \frac{D^2f(1,-2)}{2!}\begin{pmatrix} x-1 \\ y+2 \end{pmatrix}\begin{pmatrix} x-1 \\ y+2 \end{pmatrix}$$

$$f(1,-2) = 2+2-4-6+6+5 = 5$$

$$\frac{\partial f}{\partial x} = 4x - y - 6, \quad \frac{\partial f}{\partial x}(1,-2) = 0 \qquad \frac{\partial^2 f}{\partial x^2} = 4, \quad \frac{\partial^2 f}{\partial x \partial y} = -1, \quad \frac{\partial^2 f}{\partial y^2} = -2$$

$$\frac{\partial f}{\partial y} = -x - 2y - 3, \quad \frac{\partial f}{\partial y}(1,-2) = 0$$

Zatem

$$f(x,y) = 5 + (0,0)\begin{pmatrix} x-1 \\ y+2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4 & -1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x-1 \\ y+2 \end{pmatrix}\begin{pmatrix} x-1 \\ y+2 \end{pmatrix} =$$

$$= 5 + 2(x-1)^2 - (x-1)(y+2) - (y+2)^2$$

(100) $\boxed{\cancel{32}}$

Zad ~~26~~ 57 Rozwinąć w szereg Maclaurina

$$f(x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\partial^{k+l} f(0,0)}{\partial x^k \partial y^l} \cdot \frac{1}{k!\,l!} x^k y^l$$

1. $f(x,y) = e^{4x-5y}$

$$\frac{\partial^{k+l} f(x,y)}{\partial x^k \partial y^l} = 4^k(-5)^l e^{4x-5y}$$

$$\frac{\partial^{k+l} f(0,0)}{\partial x^k \partial y^l} = 4^k(-5)^l$$

Zatem $f(x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 4^k(-5)^l \frac{x^k y^l}{k!\,l!}$

Uwaga: Zauważmy, że można to też policzyć inaczej:

$$f(x,y) = e^{4x}\cdot e^{-5y} = \left(\sum_{k=0}^{\infty} \frac{4^k x^k}{k!}\right)\left(\sum_{l=0}^{\infty} \frac{(-5)^l y^l}{l!}\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 4^k(-5)^l \frac{x^k y^l}{k!\,l!}$$

2. $f(x,y) = e^x \sin y$

$$\frac{\partial^k f}{\partial x^k}(x,y) = e^x \sin y$$

$$\frac{\partial f}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 f}{\partial y^2} = -e^x \sin y$$

$$\frac{\partial^3 f}{\partial y^3} = -e^x \cos y$$

$$\frac{\partial^4 f}{\partial y^4} = e^x \sin y = f$$

$$\frac{\partial^l f}{\partial y^l}(x,y) = \begin{cases} e^x \sin y & l = 4m \\ e^x \cos y & l = 4m+1 \\ -e^x \sin y & l = 4m+2 \\ -e^x \cos y & l = 4m+3 \end{cases}$$

101 ~~33~~

$$\frac{\partial^{k+l} f(0,0)}{\partial x^k \partial y^l} = \begin{cases} 0 & l = 2m \\ (-1)^m & l = 2m+1 \end{cases} \qquad f(x,y) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{x^k}{k!} \frac{y^{2m+1}}{(2m+1)!}$$

3. $f(x,y) = \ln(1+x)\ln(1+y)$

Jeśli $k=0$ lub $l=0$ $\frac{\partial^{k+l} f(0,0)}{\partial x^k \partial y^l} = 0$

$$\frac{\partial f}{\partial x} = \frac{\ln(1+y)}{(1+x)}$$

$$\frac{\partial^2 f}{\partial x^2} = (-1) \frac{\ln(1+y)}{(1+x)^2}$$

$$\frac{\partial^k f}{\partial x^k} = \frac{(-1)^{k-1}(k-1)!\ln(1+y)}{(1+x)^k}$$

Dla $k,l > 0$:

$$\frac{\partial^{k+l} f(x,y)}{\partial x^k \partial y^l} = \frac{(-1)^{k-1}(k-1)!(-1)^{l-1}(l-1)!}{(1+x)^k (1+y)^l} \qquad \frac{\partial^{k+l} f(0,0)}{\partial x^k \partial y^l} = (-1)^{k+l}(k-1)!(l-1)!$$

$$f(x,y) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (-1)^{k+l}(k-1)!(l-1)! \frac{x^k}{k!} \frac{y^l}{l!}$$

$$f(x,y) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(-1)^{k+l}}{k \cdot l} x^k y^l$$