Zad 43 Oblicz całki iterowane:

1. $$\int_1^2 dx \int_0^3 (x+y^2x)\,dy = \int_1^2 \left(\int_0^3 (x+y^2x)\,dy \right) dx = \int_1^2 \left[xy + \frac{y^3}{3}x \right]_{y=0}^{y=3} dx$$

$$= \int_1^2 3x + \frac{27}{3}x\,dx = \int_1^2 12x\,dx = \left[6x^2\right]_1^2 = 6\cdot 4 - 6 = 18$$

2. $$\int_0^3 dy \int_1^2 (x+y^2x)\,dx = \int_0^3 \left(\int_1^2 (x+y^2x)\,dx \right) dy = \int_0^3 \left[\frac{x^2}{2} + \frac{y^2x^2}{2} \right]_{x=1}^{x=2} dy$$

$$= \int_0^3 2 + 2y^2 - \frac{1}{2} - \frac{1}{2}y^2\,dy = \int_0^3 \frac{3}{2}y^2 + \frac{3}{2}\,dy = \left[\frac{y^3}{2} + \frac{3}{2}y \right]_0^3 = \frac{27}{2} + \frac{9}{2} =$$

$$= \frac{36}{2} = 18$$

Zad ~~59~~ 44, Oblicz całki podwójne

Korzystamy ze wzoru: $$\iint\limits_{[a,b]\times[c,d]} g(x)h(y)\,dx\,dy = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

1. $$\iint\limits_{\substack{R\\ \shortparallel \\ [0,1]\times[0,1]}} e^{x+y}\,dx\,dy = \iint\limits_{[0,1]\times[0,1]} e^x\cdot e^y\,dx\,dy = \left(\int_0^1 e^x\,dx\right)\left(\int_0^1 e^y\,dy\right) = \left[e^x\right]_0^1\left[e^y\right]_0^1 =$$
$$= (e-1)^2$$

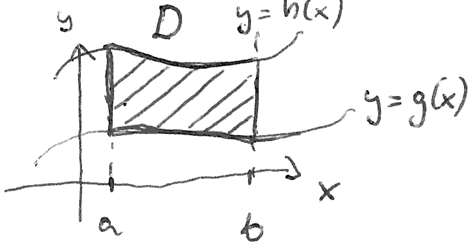
2. $$\iint\limits_{\substack{R\\ \shortparallel \\ [-\frac{\pi}{4},\frac{\pi}{4}]\times[0,\frac{\pi}{4}]}} \sin(x+y)\,dx\,dy = \iint\limits_R (\sin x\cos y + \sin y\cos x)\,dx\,dy = \iint\limits_R \sin x\cos y\,dx\,dy + \iint\limits_R \sin y\cos x\,dx\,dy$$
$$= \Bigg(\underbrace{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin x\,dx}_{=0}\Bigg)\left(\int_0^{\frac{\pi}{4}} \cos y\,dy\right) + \left(\int_0^{\frac{\pi}{4}} \sin y\,dy\right)\Bigg(\overbrace{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x\,dx}^{2\int_0^{\frac{\pi}{4}}\cos x\,dx}\Bigg) = 2\left[-\cos y\right]_0^{\frac{\pi}{4}}\left[\sin x\right]_0^{\frac{\pi}{4}} = 2\left(1-\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} = \sqrt{2}-1$$

Definicja

Jak ogólnie liczyć całki podwójne?

Obszar $D \subseteq \mathbb{R}^2$ taki, że $D = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\ g(x) \leq y \leq h(x)\}$

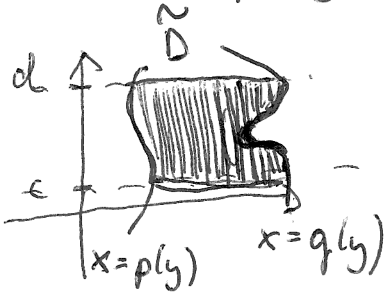
nazywamy obszarem normalnym względem osi OX



Wówczas $$\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

Podobnie, obszar $\tilde{D} \subseteq \mathbb{R}^2$ taki, że $\tilde{D} = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d,\ p(y) \leq x \leq q(y)\}$

nazywamy obszarem normalnym względem osi OY



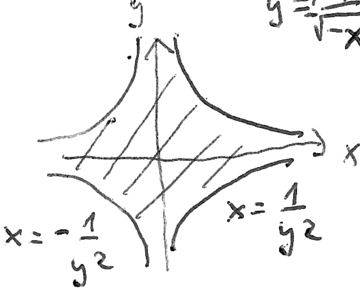
Wówczas $$\iint_{\tilde{D}} f(x,y)\,dx\,dy = \int_c^d \left(\int_{p(y)}^{q(y)} f(x,y)\,dx \right) dy$$

Obszar $G \subseteq \mathbb{R}^2$ nazywamy regularnym, jeśli $G = D_1 \cup \ldots \cup D_n$, gdzie D_i – ~~regularne~~ normalne i $\operatorname{Int} D_i \cap \operatorname{Int} D_j = \emptyset$ dla $i \neq j$. Wówczas $$\iint_G f(x,y)\,dx\,dy = \sum_{i=1}^{n} \iint_{D_i} f(x,y)\,dx\,dy$$

<u>Zad ~~60~~ 45</u> Zamienić $\iint_D f(x,y)\,dx\,dy$ na całki iterowane, gdy $D \subseteq \mathbb{R}^2$ ograniczony przez: 105

1. $x^2y^4 = 1$ $\quad x^2y^4 - 1 = (xy^2-1)(xy^2+1) = 0$ $\quad x = \frac{1}{y^2}$ lub $x = -\frac{1}{y^2}$

$y = \pm\frac{1}{\sqrt{x}}$ $\qquad y = \pm\frac{1}{\sqrt{-x}}$



$D = \{(x,y) \in \mathbb{R}^2 : -\infty < y < \infty,\ -\frac{1}{y^2} \le x \le \frac{1}{y^2}\}$

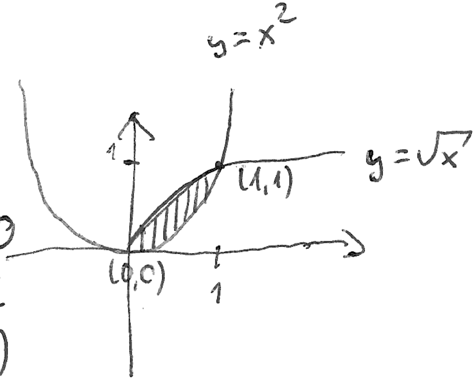
$$\iint_D f(x,y)\,dx\,dy = \int_{-\infty}^{+\infty}\left(\int_{-\frac{1}{y^2}}^{\frac{1}{y^2}} f(x,y)\,dx\right)dy$$

2. $y = x^2,\ y = \sqrt{x}$ $\qquad$ Punkty przecięcia spełniają

$$\begin{cases} y = x^2 \\ y = \sqrt{x} \end{cases} \Rightarrow \begin{matrix} y^2 = x^4 \\ y^2 = x \\ x > 0 \end{matrix} \Rightarrow x(x^3-1) = 0$$

$x = 0 \lor x = 1$

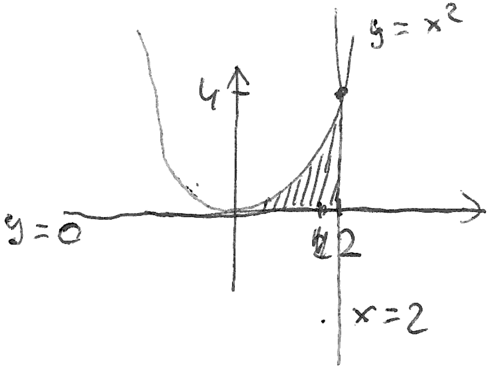
$(0,0) \qquad (1,1)$



$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$

$$\iint_D f(x,y)\,dx\,dy = \int_0^1\left(\int_{x^2}^{\sqrt{x}} f(x,y)\,dy\right)dx$$

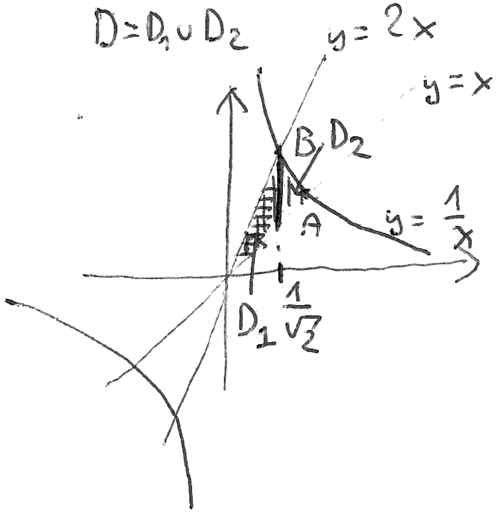
3. $y=0,\ x=2,\ y=x^2$



$D=\{(x,y):\ 0\le x\le 2,\ 0\le y\le x^2\}$

$$\iint_D f(x,y)\,dx\,dy = \int_0^2\left(\int_0^{x^2} f(x,y)\,dy\right)dx$$

4. $y=\frac{1}{x},\ y=x,\ y=2x\ (x>0)$



Znajdujemy punkty przecięcia krzywych

$A\ \begin{cases} y=\frac{1}{x} \\ y=x \end{cases} \Rightarrow x^2=1 \Rightarrow x=\pm 1 \Rightarrow A=(1,1)$ $\quad \begin{matrix} x=1 \\ y=x=1 \end{matrix}$

$B\ \begin{cases} y=\frac{1}{x} \\ y=2x \end{cases} \Rightarrow 2x^2=1 \Rightarrow x=\pm\frac{1}{\sqrt2} \Rightarrow B=\left(\frac{1}{\sqrt2},\sqrt2\right)$ $\quad \begin{matrix} x=\frac{1}{\sqrt2} \\ y=\frac{1}{x}=\sqrt2 \end{matrix}$

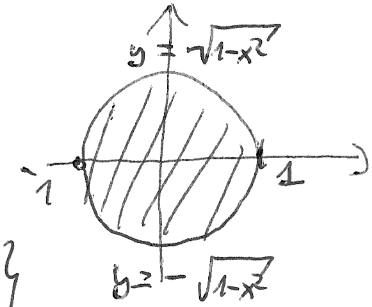
$D_1=\{(x,y)\in\mathbb{R}^2:\ 0\le x\le\frac{1}{\sqrt2},\ x\le y\le 2x\}$, $D_2=\{(x,y)\in\mathbb{R}^2:\ \frac{1}{\sqrt2}\le x\le 1:\ x\le y\le\frac{1}{x}\}$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f(x,y)\,dx\,dy + \iint_{D_2} f(x,y)\,dx\,dy =$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \left(\int_x^{2x} f(x,y)\,dy \right) dx + \int_{\frac{1}{\sqrt{2}}}^{1} \left(\int_x^{\frac{1}{x}} f(x,y)\,dy \right) dx$$

5.

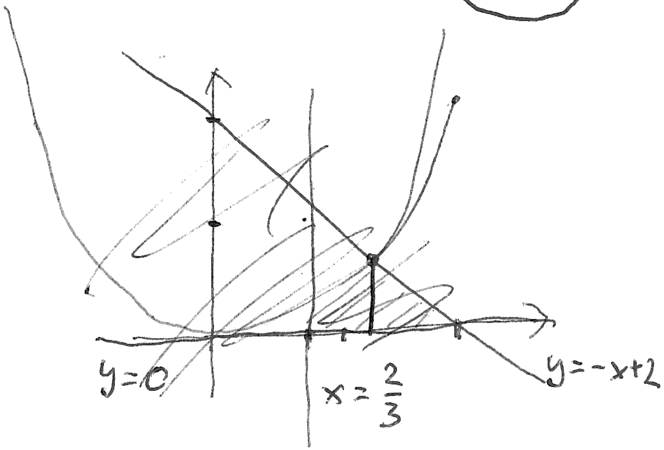
$x^2+y^2=1$



$D = \{(x,y) \in \mathbb{R}^2 : -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$

$$\iint_D f(x,y)\,dx\,dy = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy \right) dx$$

6. $y = x^2, \quad y = -x+2, \quad y = 0, \quad x = \frac{2}{3}$

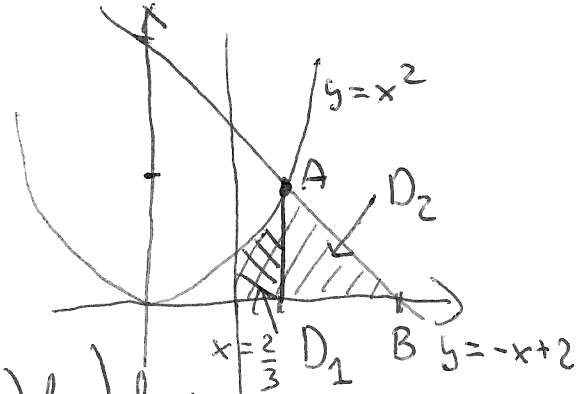


~~A=~~

$$A \begin{cases} y = x^2 \\ y = -x+2 \end{cases}$$

$$x^2 + x - 2 = (x+2)(x-1) = 0 \quad \underbrace{x = 1}_{y=1} \vee \cancel{x} = -2$$

$A = (1,1)$ $\qquad$ $B = (2,0)$



$$D_1 = \{(x,y) \in \mathbb{R}^2 : \tfrac{2}{3} \le x \le 1, \ 0 \le y \le x^2\}$$

$$D_2 = \{(x,y) \in \mathbb{R}^2 : 1 \le x \le 2, \ 0 \le y \le -x+2\}$$

$$\iint_D f(x,y)\,dx\,dy = \int_{\frac{2}{3}}^{1} \left(\int_0^{x^2} f(x,y)\,dy \right) dx + \int_1^2 \left(\int_0^{-x+2} f(x,y)\,dy \right) dx$$

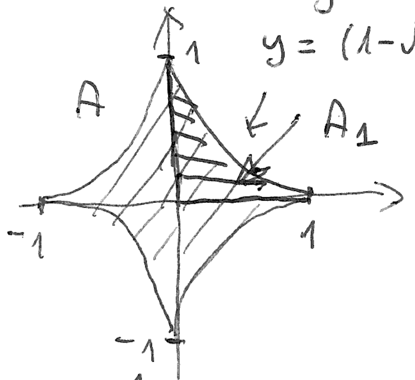
Zad ~~1~~ 46 Oblicz miarę zbioru

Ogólnie: $\mu(A) = \iint\limits_A 1\,dx\,dy$ dla $A \subseteq \mathbb{R}^2$

$x \geq 0, y \geq 1$

$\sqrt{x} + \sqrt{y} = 1$

$y = (1-\sqrt{x})^2$



1 $A = \{(x,y) : \sqrt{|x|} + \sqrt{|y|} < 1\}$

$\mu(A) = 4\mu(A_1)$

$A_1 = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq (1-\sqrt{x})^2\}$

$$\mu(A_1) = \iint\limits_A dx\,dy = \int_0^1 \left(\int_0^{(1-\sqrt{x})^2} dy \right) dx = \int_0^1 (1-\sqrt{x})^2 dx = \int_0^1 1 - 2\sqrt{x} + x \, dx$$

$$= \left[x - \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^2}{2} \right]_0^1 = \left[x - \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 \right]_0^1 = \frac{1}{6} \qquad \mu(A) = 4\mu(A_1) = \frac{2}{3}$$

2 $\mu(B) = ?$

Obszar normalny:

tylko stałe / tylko zależne od x / tylko zależne od (x,y)

$$B = \{(x,y,z) \in \mathbb{R}^3 : -1 < x < 1,\ -1 < y < 1,\ 0 < z < x^2+y^2\}$$

Ogólnie:

$$D = \{(x,y,z) \in \mathbb{R}^3 : a < x < b,\ g_1(x) < y < g_2(x),\ h_1(x,y) < z < h_2(x,y)\}$$

Wówczas

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} \left(\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz \right) dy \right) dz$$

W naszym przypadku

$$\mu(B) = \iiint_D dx\,dy\,dz = \int_{-1}^{1} \left(\int_{-1}^{1} \left(\int_0^{x^2+y^2} dz \right) dy \right) dx =$$

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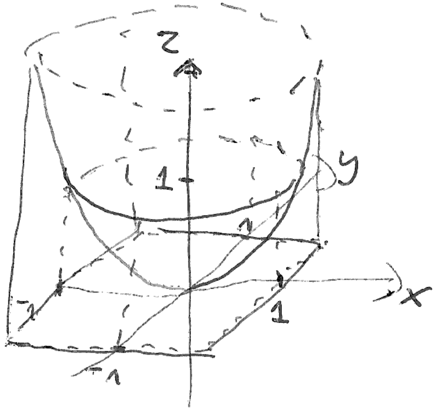
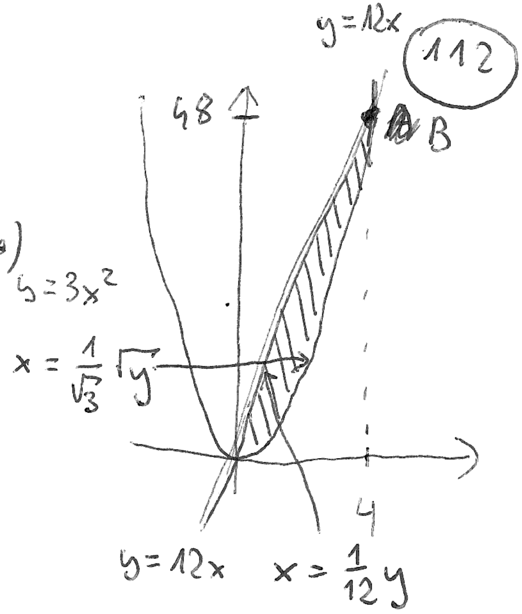
$$= \int_{-1}^{1} \left(\int_{-1}^{1} x^2 + y^2 \, dy \right) dx = \int_{-1}^{1} \left(\left[x^2 y + \frac{1}{3} y^3 \right]_{y=-1}^{y=1} \right) dx =$$

$$= \int_{-1}^{1} x^2 + \frac{1}{3} - \left(-x^2 - \frac{1}{3} \right) dx = \int_{-1}^{1} 2x^2 + \frac{2}{3} \, dx = \left[\frac{2}{3} x^3 + \frac{2}{3} x \right]_{-1}^{1} = \frac{2}{3} + \frac{2}{3} - \left(-\frac{2}{3} - \frac{2}{3} \right)$$

$$= \frac{8}{3}$$



Zad ~~62~~ 47 Zamienić kolejność całkowania



1. $\int_0^4 \left(\int_{3x^2}^{12x} f(x,y)\,dy \right) dx = \iint_A f(x,y)\,dx\,dy \overset{(*)}{=}$

$A = \{ (x,y) : 0 \le x \le 4,\ 3x^2 \le y \le 12x \}$

B $\begin{cases} y = 3x^2 \\ y = 12x \end{cases} \Rightarrow$ $12x - 3x^2 = 0$ $\quad (0,0),\ (4,48)$

$3x(4-x) = 0$

$x = 0 \ \vee \ x = 4$

Należy zapisać zbiór A jako obszar normalny względem osi OY:

$A = \{ (x,y) : 0 \le y \le 48,\ \frac{1}{12} y \le x \le \frac{1}{\sqrt{3}} \sqrt{y}$

$(*) = \int_0^{48} \left(\int_{\frac{1}{12}y}^{\frac{1}{\sqrt{3}}\sqrt{y}} f(x,y)\,dx \right) dy$

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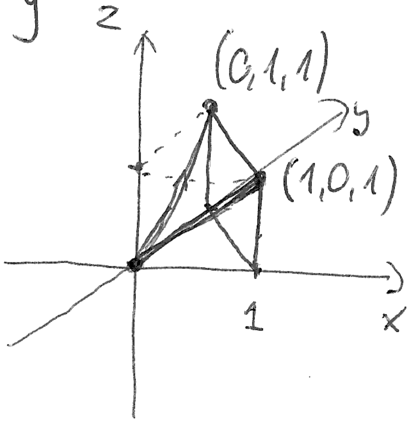
2. $\int_0^1 \left(\int_0^{1-x} \left(\int_0^{x+y} f(x,y,z)\,dz \right) dy \right) dx = \iiint_B f(x,y,z)\,dx\,dy\,dz = (*)$

$B = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le x+y\}$



~~$B = \{(x,y,z) : 0 \le z \le 1$~~,

Zatem z takie, że $0 \le z \le 1$

Wówczas: $0 \le y \le 1$

x takie, że $\begin{matrix} x \le 1-y \\ x \ge z-y \\ x \ge 0 \end{matrix} \Rightarrow \begin{matrix} \text{dla } y \le z : & z-y \le x \le 1-y \\ \text{dla } y \ge z : & 0 \le x \le 1-y \end{matrix}$

$B = \{(x,y,z) : 0 \le z \le 1,\ 0 \le y \le z,\ z-y \le x \le 1-y\} \cup$

$\cup \{(x,y,z) : 0 \le z \le 1,\ z \le y \le 1,\ 0 \le x \le 1-y\}$

Stąd

$$(*) = \int_0^1 \left(\int_0^z \left(\int_{z-y}^{1-y} f(x,y,z)\,dx \right) dy \right) dz + \int_0^1 \left(\int_z^1 \left(\int_0^{1-y} f(x,y,z)\,dx \right) dy \right) dz$$

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Zad ~~63~~ 48 Oblicz podane całki:

1 $\iint\limits_D (x^2 - xy)\,dx\,dy$

$D = \{(x,y): y \geq x,\ y \leq 3x - x^2\}$

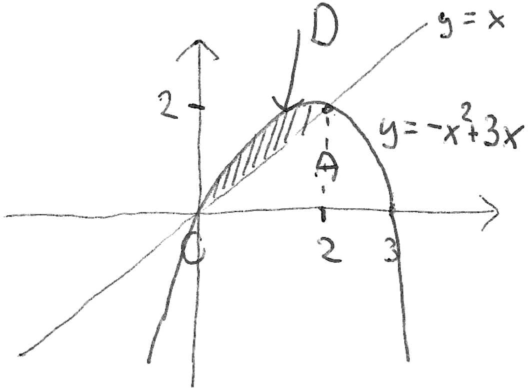
A $\begin{cases} y = x \\ y = 3x - x^2 \end{cases}$

$\Downarrow$

$x^2 - 2x = 0$

$x(x-2) = 0$

$\begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 2 \\ y = 2 \end{cases}$



$D = \{(x,y): 0 \leq x \leq 2,\ x \leq y \leq 3x - x^2\}$

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$$D = \{(x,y) : 0 \le x \le 2,\ x \le y \le 3x - x^2$$

$$\iint_D (x^2 - xy)\,dx\,dy = \int_0^2 \left(\int_x^{3x-x^2} x^2 - xy\,dy \right) dx = \int_0^2 \left[x^2 y - x\frac{y^2}{2} \right]_{y=x}^{y=3x-x^2} dx =$$

$$= \int_0^2 x^2(3x - x^2) - \frac{x}{2}(3x - x^2)^2 - x^3 + \frac{x^3}{2}\,dx = \int_0^2 3x^3 - x^4 - \frac{9x^3}{2} + 3x^4 - \frac{x^5}{2} - \frac{x^3}{2}\,dx$$

$$= \int_0^2 -\frac{x^5}{2} + 2x^4 - 2x^3\,dx = \left[-\frac{x^6}{12} + \frac{2x^5}{5} - 2\frac{x^4}{4} \right]_0^2 = -\frac{2^4}{3} + \frac{64}{5} - 8 =$$

$$= -\frac{16}{3} + \frac{64}{5} - 8 = -\frac{80}{15} + \frac{192}{15} - \frac{120}{15} = -\frac{8}{15}$$

2. $\iint_D y\,dx\,dy$, $\quad D=\{(x,y):\ x\le \arcsin y,\ y\le \frac{1}{\sqrt2},\ x\ge 0\}$

(116)

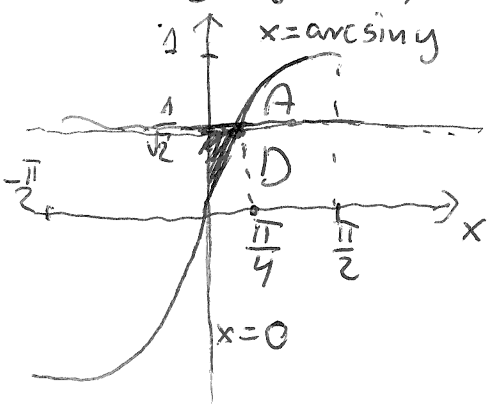
$$A\ \begin{cases} y=\frac{1}{\sqrt2} \\ x=\arcsin y \end{cases}$$

$\Downarrow$

$\sin x = \frac{1}{\sqrt2}=\frac{\sqrt2}{2} \Rightarrow x=\frac{\pi}{4}$

$x\in(-\frac{\pi}{2},\frac{\pi}{2}) \qquad y=\frac{1}{\sqrt2}$

$A=(\frac{\pi}{4},\frac{1}{\sqrt2})$



$D=\{(x,y):\ 0\le x\le\frac{\pi}{4},\ \sin x\le y\le\frac{1}{\sqrt2}\}$

$$\iint_D y\,dx\,dy = \int_0^{\frac{\pi}{4}}\left(\int_{\sin x}^{\frac{1}{\sqrt2}} y\,dy\right)dx = \int_0^{\frac{\pi}{4}}\left[\frac{y^2}{2}\right]_{y=\sin x}^{y=\frac{1}{\sqrt2}} dy = \int_0^{\frac{\pi}{4}} \frac14-\frac{\sin^2 x}{2}\,dx$$

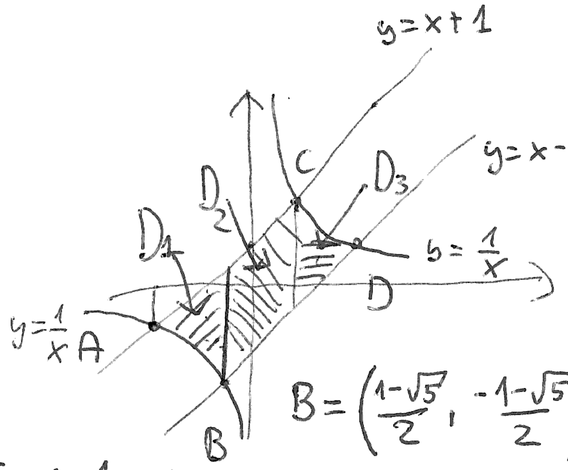
(117)

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} 1 - 2\sin^2 x \, dx = \frac{1}{4}\int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \, dx = \frac{1}{4}\int_0^{\frac{\pi}{4}} \cos 2x \, dx$$

$$= \frac{1}{4}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{8}\cdot \sin\frac{\pi}{2} = \frac{1}{8}$$

3

$$\iint_D xy \, dx\, dy \qquad \begin{cases} xy = 1 \\ |x-y| = 1 \end{cases} \Rightarrow \begin{cases} y = \frac{1}{x} \\ y = x - 1 \\ y = x + 1 \end{cases}$$



A, C

$$\begin{cases} y = x + 1 \\ y = \frac{1}{x} \end{cases}$$

$$\frac{1}{x} = x + 1$$

$$x^2 + x - 1 = 0$$

$\Delta = 1 + 4 = 5$

$x_1 = \frac{-1-\sqrt{5}}{2}$

$x_2 = \frac{-1+\sqrt{5}}{2}$

$$A = \left(\frac{-1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$$

$$C = \left(\frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$

B, D

$$\begin{cases} y = x - 1 \\ y = \frac{1}{x} \end{cases}$$

$$\frac{1}{x} = x - 1$$

$$x^2 - x - 1 = 0$$

$x_1 = \frac{1-\sqrt{5}}{2}$

$x_2 = \frac{1+\sqrt{5}}{2}$

$$B = \left(\frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right)$$

$$D = \left(\frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$$

$$\text{(118)}$$

$$D = \{(x,y)\in\mathbb{R}^2 : \underbrace{\tfrac{-1-\sqrt5}{2} \le x \le \tfrac{1-\sqrt5}{2},\ \tfrac{1}{x} \le y \le x+1}_{D_1}\} \cup$$

$$\cup \{(x,y)\in\mathbb{R}^2 : \underbrace{\tfrac{1-\sqrt5}{2} \le x \le \tfrac{-1+\sqrt5}{2},\ x-1 \le y \le x+1}_{D_2}\} \cup$$

$$\cup \{(x,y)\in\mathbb{R}^2 : \underbrace{\tfrac{-1+\sqrt5}{2} \le x \le \tfrac{1+\sqrt5}{2},\ x-1 \le y \le \tfrac{1}{x}}_{D_3}\}$$

$$\iint_D xy\,dx\,dy = \int_{\frac{-1-\sqrt5}{2}}^{\frac{1-\sqrt5}{2}} x\left(\int_{\frac1x}^{x+1} y\,dy\right)dx + \int_{\frac{1-\sqrt5}{2}}^{\frac{-1+\sqrt5}{2}} x\left(\int_{x-1}^{x+1} y\,dy\right) + \int_{\frac{-1+\sqrt5}{2}}^{\frac{1+\sqrt5}{2}} x\left(\int_{x-1}^{\frac1x} y\,dy\right)dx$$

$$= \int_{\frac{-1-\sqrt5}{2}}^{\frac{1-\sqrt5}{2}} x\left[\frac{y^2}{2}\right]_{y=\frac1x}^{y=x+1} dx + \int_{\frac{1-\sqrt5}{2}}^{\frac{-1+\sqrt5}{2}} x\left[\frac{y^2}{2}\right]_{y=x-1}^{y=x+1} dx + \int_{\frac{-1+\sqrt5}{2}}^{\frac{1+\sqrt5}{2}} x\left[\frac{y^2}{2}\right]_{y=x-1}^{y=\frac1x} dx =$$

$$= \int_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{5}}{2}} \frac{x}{2}\left(x^2+2x+1-\frac{1}{x^2}\right)dx + \int_{\frac{1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}} \frac{x}{2}\left(x^2+2x+1-x^2+2x-1\right)dx$$

$$+ \int_{\frac{-1+\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} \frac{x}{2}\left(\frac{1}{x^2}-x^2+2x-1\right)dx = \int_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{5}}{2}} \frac{x^3}{2}+x^2+\frac{x}{2}-\frac{1}{2x}\,dx$$

$$+ \int_{\frac{1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}} 2x^2\,dx + \int_{\frac{-1+\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} -\frac{x^3}{2}+x^2-\frac{x}{2}+\frac{1}{2x}\,dx = \left[\frac{x^4}{8}+\frac{x^3}{3}+\frac{x^2}{4}-\frac{1}{2}\ln|x|\right]_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{5}}{2}}$$

$$+ \left[\frac{2}{3}x^3\right]_{\frac{1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}} + \left[-\frac{x^4}{8}+\frac{x^3}{3}-\frac{x^2}{4}+\frac{1}{2}\ln x\right]_{\frac{-1+\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} =$$

$$= \frac{\left(\frac{-1+\sqrt{5}}{2}\right)^4}{8} - \frac{\left(\frac{\sqrt{5}-1}{2}\right)^3}{3} + \frac{\left(\frac{\sqrt{5}-1}{2}\right)^2}{4} - \frac{1}{2}\ln\frac{\sqrt{5}-1}{2} - \frac{\left(\frac{1+\sqrt{5}}{2}\right)^4}{8} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^3}{3} - \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2}{4} + \frac{1}{2}\ln\frac{1+\sqrt{5}}{2}$$

$$+ \frac{2}{3}\left(\frac{\sqrt{5}-1}{2}\right)^3 + \frac{2}{3}\left(\frac{\sqrt{5}-1}{2}\right)^3 - \frac{\left(\frac{1+\sqrt{5}}{2}\right)^4}{8} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^3}{3} - \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2}{4} + \frac{1}{2}\ln\frac{1+\sqrt{5}}{2} + \frac{\left(\frac{\sqrt{5}-1}{2}\right)^4}{8} - \frac{\left(\frac{\sqrt{5}-1}{2}\right)^3}{3}$$

$$+ \frac{\left(\frac{\sqrt{5}-1}{2}\right)^2}{4} - \frac{1}{2}\ln\frac{\sqrt{5}-1}{2} = \frac{1}{4}\left(\frac{\sqrt{5}-1}{2}\right)^4 - \frac{1}{4}\left(\frac{\sqrt{5}+1}{2}\right)^4 + \frac{2}{3}\left(\frac{1+\sqrt{5}}{2}\right)^3 + \frac{2}{3}\left(\frac{\sqrt{5}-1}{2}\right)^3$$

$$+ \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^2 - \frac{1}{2}\left(\frac{\sqrt{5}+1}{2}\right)^2 + \ln\frac{1+\sqrt{5}}{2} - \ln\frac{\sqrt{5}-1}{2}$$

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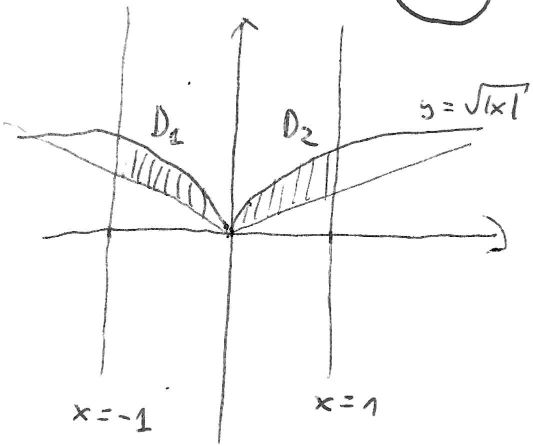
$\cancel{4}$ 3. $\iint\limits_D x+y \, dx\,dy$ $\qquad y=\sqrt{|x|} \quad 2y=|x| \; ; \; |x|=1$



$$D=\{(x,y): -1\le x\le 0,\ -\tfrac{x}{2}\le y\le \sqrt{-x}\}\ \cup$$

$$\cup\{(x,y): 0\le x\le 1,\ \tfrac{x}{2}\le y\le \sqrt{x}\}$$

$x = -x$

$$\iint\limits_D x+y\,dx\,dy = \int_{-1}^{0}\left(\int_{-\frac{x}{2}}^{\sqrt{-x}} x+y\,dy\right)dx + \int_{0}^{1}\left(\int_{\frac{x}{2}}^{\sqrt{x}} x+y\,dy\right)dx =$$

$$= \int_{0}^{1}\left(\int_{\frac{x}{2}}^{\sqrt{x}} -x+y\,dy\right)dx + \int_{0}^{1}\left(\int_{\frac{x}{2}}^{\sqrt{x}} x+y\,dy\right)dx = \int_{0}^{1}\left(\int_{\frac{x}{2}}^{\sqrt{x}} 2y\,dy\right)dx$$

$$= \int_{0}^{1}\left[y^2\right]_{y=\frac{x}{2}}^{y=\sqrt{x}} dx = \int_{0}^{1} x-\frac{x^2}{4}\,dx = \left[\frac{x^2}{2}-\frac{x^3}{12}\right]_0^1 = \frac{1}{2}-\frac{1}{12} = \frac{5}{12}$$

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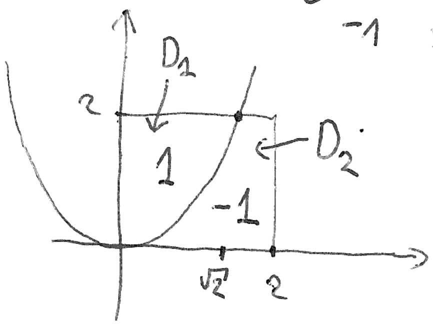
4. $\iint\limits_D \operatorname{sgn}(y-x^2)\,dx\,dy\,,\quad D=[0,2]\times[0,2]$

$$\operatorname{sgn} x = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$



$0\le x\le \sqrt{y}$

$D_1=\{(x,y)\in\mathbb{R}^2 : 0\le y\le 2,\ \cancel{\sqrt{y}\le x\le 2}\}$

$D_2=\{(x,y)\in\mathbb{R}^2 : 0\le y\le 2,\ \sqrt{y}\le x\le 2\}$

$$\iint\limits_D \operatorname{sgn}(y-x^2)\,dx\,dy = \iint\limits_{D_1} \operatorname{sgn}(y-x^2)\,dx\,dy + \iint\limits_{D_2} \operatorname{sgn}(y-x^2)\,dx\,dy = \iint\limits_{D_1} dx\,dy - \iint\limits_{D_2} dx\,dy =$$

$$= \int_0^2 \left(\int_0^{\sqrt{y}} dx\right) dy - \int_0^2 \left(\int_{\sqrt{y}}^2 dx\right) dy = \int_0^2 \sqrt{y}\,dy - \int_0^2 2-\sqrt{y}\,dy =$$

$$= 2\int_0^2 \sqrt{y}\,dy - 2\int_0^2 dy = \left[2\,\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 - \left[2y\right]_0^2 = \frac{4}{3}\cdot 2^{3/2} - 4 = 4\left(\frac{2\sqrt{2}}{3}-1\right)$$