(21)

Zad 14 $\quad g(x,y,z) = x + yz, \quad f(t) = g(\underset{x}{\sin t}, \underset{y}{\cos t}, \underset{z}{t^2}), \quad f'(t) = ?$

$$f'(t) = \frac{\partial g}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial g}{\partial z}\cdot\frac{dz}{dt} = 1\cdot\cos t + t^2(-\sin t) + \cos t\cdot 2t$$

$\frac{dx}{dt} = \cos t \qquad \frac{\partial g}{\partial x}\cancel{(\sin t, \cos t)} = 1$

$\frac{dy}{dt} = -\sin t \qquad \frac{\partial g}{\partial y} = z = t^2$

$\frac{dz}{dt} = 2t \qquad \frac{\partial g}{\partial z} = y = \cos t$

zad 15

$u(x,y) = x^n f\left(\frac{y}{x^2}\right) \qquad x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} \overset{?}{=} nu$

$$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x^2}\right) + x^n f'\left(\frac{y}{x^2}\right)\left(-2\frac{y}{x^3}\right), \quad \frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x^2}\right)\cdot\frac{1}{x^2}$$

$$L = x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x^2}\right) - 2yx^{n-2} f'\left(\frac{y}{x^2}\right) + 2yx^{n-2} f'\left(\frac{y}{x^2}\right) = nx^n f\left(\frac{y}{x^2}\right) = nu = P$$

Zad 16, $u(x,y,z) = \frac{1}{12}x^4 - \frac{1}{6}x^3(y+z) + \frac{1}{2}x^2yz + f(y-x, z-x)$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = ?$$

$$\frac{\partial u}{\partial x} = \frac{1}{3}x^3 - \frac{1}{2}x^2(y+z) + xyz - D_1 f(y-x, z-x) - D_2 f(y-x, z-x)$$

$$\frac{\partial u}{\partial y} = -\frac{1}{6}x^3 + \frac{1}{2}x^2 z + D_1 f(y-x, z-x)$$

$$\frac{\partial u}{\partial z} = -\frac{1}{6}x^3 + \frac{1}{2}x^2 y + D_2 f(y-x, z-x)$$

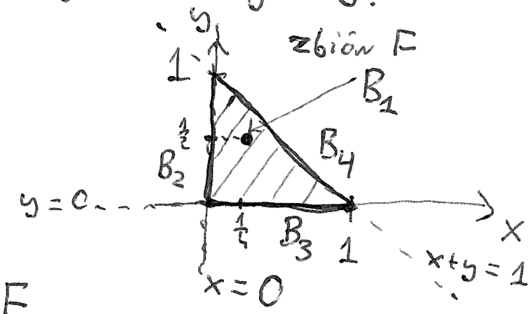
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{1}{3}x^3 - \frac{1}{6}x^3 - \frac{1}{6}x^3 - \frac{1}{2}x^2(y+z) + \frac{1}{2}x^2 z + \frac{1}{2}x^2 y + xyz = xyz$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$$

Zad 17

Niech $f(x,y) = 2x^2 - x + y^2 - y$, $F = \{(x,y) : x \geq 0,\ y \geq 0,\ x+y \leq 1\}$.

Wyznaczyć $\max\limits_F f$, $\min\limits_F f$



Niech

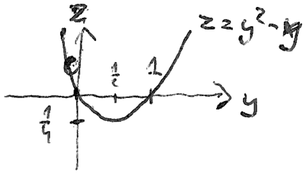
$$B = \{(x,y) \in \operatorname{Int} F : \frac{\partial f}{\partial x}(x,y) = 0,\ \frac{\partial f}{\partial y}(x,y) = 0\} \cup \partial F$$

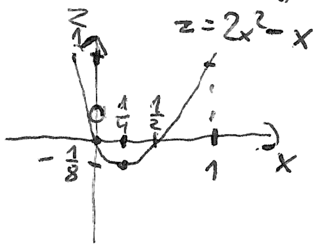
Wówczas $\max\limits_F f = \max\limits_B f$, $\min\limits_F f = \min\limits_B f$

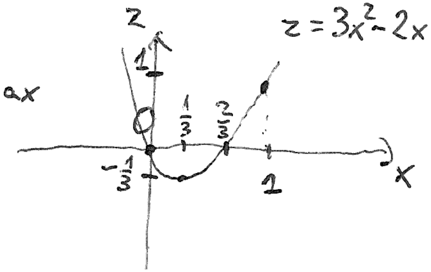
Znajdujemy zbiór B:
$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 4x - 1 = 0 \\ \frac{\partial f}{\partial y}(x,y) = 2y - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{4} \\ y = \frac{1}{2} \end{cases} \quad \left(\tfrac{1}{4}, \tfrac{1}{2}\right) \in \operatorname{Int} F$$

Zatem
$$B = \underbrace{\{(\tfrac{1}{4}, \tfrac{1}{2})\}}_{B_1} \cup \underbrace{\{(0,y) : 0 \leq y \leq 1\}}_{B_2} \cup \underbrace{\{(x,0) : 0 \leq x \leq 1\}}_{B_3} \cup \underbrace{\{(x, 1-x) : 0 \leq x \leq 1\}}_{B_4}$$

Szukamy ekstremów na B_1, B_2, B_3, B_4:







B_1: $f\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{8} - \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = -\frac{3}{8} \leftarrow \min$

B_2: $\max\limits_{B_2} f = \max\limits_{0\le y\le 1} f(0,y) = \max\limits_{0\le y\le 1} y^2 - y = f(0,0) = f(0,1) = 0$

$\min\limits_{B_2} f = \min\limits_{0\le y\le 1} f(0,y) = \min\limits_{0\le y\le 1} y^2 - y = f\left(0,\frac{1}{2}\right) = -\frac{1}{4}$

B_3: $\max\limits_{B_3} f = \max\limits_{0\le x\le 1} f(x,0) = \max\limits_{0\le x\le 1} 2x^2 - x = f(1,0) = 1 \leftarrow \max$

$\min\limits_{B_3} f = \min\limits_{0\le x\le 1} f(x,0) = \min\limits_{0\le x\le 1} 2x^2 - x = f\left(\frac{1}{4},0\right) = -\frac{1}{8}$

B_4: $\max\limits_{B_4} f = \max\limits_{0\le x\le 1} f(x,1-x) = \max\limits_{0\le x\le 1} 2x^2 - x + (1-x)^2 - 1 + x = \max\limits_{0\le x\le 1} 3x^2 - 2x = f(1,0) = 1 \leftarrow \max$

$\min\limits_{B_4} f = \min\limits_{0\le x\le 1} f(x,1-x) = \min\limits_{0\le x\le 1} 3x^2 - 2x = f\left(\frac{1}{3},\frac{2}{3}\right) = -\frac{1}{3}$

Odp: $\max\limits_{F} f(x,y) = f(1,0) = 1$, $\min\limits_{F} f(x,y) = f\left(\frac{1}{4},\frac{1}{2}\right) = -\frac{3}{8}$

Zad 18

Niech $f(x,y) = (x+y)e^{-x^2-y^2}$, $F = \{(x,y): x \geq 0, y \geq 0\}$. Wyznaczyć $\inf_F f$, $\sup_F f$

Jeśli zbiór F nie jest ograniczony, to funkcja nie musi przyjmować na nim kresów.
Natomiast zachodzi

$$\sup_F f = \max\{c, f(B)\},$$

gdzie $c = \overline{\lim}_{\|x\|\to\infty} f(x)$, $B = \{x \in \operatorname{Int} F: D_1 f(x) = \ldots = D_n f(x) = 0\} \cup \partial F$

$$\inf_F f = \min_F f = f(0,0) = 0 \quad \left(\text{bo } \forall_{(x,y)\in F} \; f(x,y) \geq 0\right)$$

$\sup_F f = ?$

$r = \|(x,y)\| = \sqrt{x^2+y^2}$

Zauważmy, że $\lim_{\|(x,y)\|\to\infty} \left|(x+y)e^{-x^2-y^2}\right| \leq \lim_{\|(x,y)\|\to\infty} \left|\sqrt{2}\sqrt{x^2+y^2}\,e^{-x^2-y^2}\right| = \lim_{r\to\infty} \sqrt{2}\, r e^{-r^2} = 0$

Zatem $\sup_F f = \max_B f$, $B = \underbrace{\{(x,y): x>0, y>0, \tfrac{\partial f}{\partial x}(x,y) = 0, \tfrac{\partial f}{\partial y}(x,y) = 0\}}_{B_1} \cup \underbrace{\{(x,0): x \geq 0\}}_{B_2} \cup \underbrace{\{(0,y): y \geq 0\}}_{B_3}$

B_1:

$f(x,y) = (x+y)e^{-x^2-y^2}$ (26)

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = e^{-x^2-y^2} + (x+y)(-2x)e^{-x^2-y^2} = 0 & / \cdot e^{x^2+y^2} \\ \frac{\partial f}{\partial y}(x,y) = e^{-x^2-y^2} + (x+y)(-2y)e^{-x^2-y^2} = 0 & / \cdot e^{x^2+y^2} \end{cases}$$

$$\begin{cases} 1-2x(x+y)=0 \\ 1-2y(x+y)=0 \end{cases} \Rightarrow \begin{cases} x+y = \frac{1}{2x} \\ 1-2y\cdot\frac{1}{2x}=0 \end{cases} \Rightarrow \begin{cases} x+y=\frac{1}{2x} \\ 1-\frac{y}{x}=0 \end{cases} \Rightarrow \begin{cases} 4x^2=1 \\ y=x \end{cases} \Rightarrow \underset{\in F}{\begin{cases} x=\frac{1}{2} \\ y=\frac{1}{2} \end{cases}} \text{ lub } \underset{\notin F}{\begin{cases} x=-\frac{1}{2} \\ y=-\frac{1}{2} \end{cases}}$$

$B_1 = \{(\frac{1}{2}, \frac{1}{2})\}$ $f(\frac{1}{2},\frac{1}{2}) = e^{-(\frac{1}{4}+\frac{1}{4})} = e^{-\frac{1}{2}}$ ← max

B_2: $\max_{x\geq 0} f(x,0) = \max_{x\geq 0} xe^{-x^2} = f(\frac{1}{\sqrt{2}},0) = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$

Aby znaleźć ekstremum liczymy pochodną:

$(xe^{-x^2})' = e^{-x^2} + x(-2x)e^{-x^2} = e^{-x^2}(1-2x^2)=0$

$1-2x^2=0$

$x=\frac{1}{\sqrt{2}}$ lub $x=-\frac{1}{\sqrt{2}}$ $\Rightarrow$ supremum dla $x=\frac{1}{\sqrt{2}}$ (bo $x\geq 0$)

B_3: podobnie:

$\max_{y\geq 0} f(0,y) = \max_{y\geq 0} ye^{-y^2} = f(0,\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$

Ostatecznie

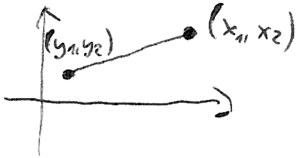
Odp: $\inf_F f = f(0,0) = 0$

$\sup_F f = f(\frac{1}{2},\frac{1}{2}) = e^{-1/2}$

Zad 19

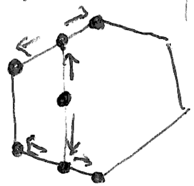
Stwierdzenie

Jeżeli brzeg zbioru zwartego $F \subset \mathbb{R}^2$ jest sumą skończonej liczby odcinków, to funkcja postaci $f(x_1, x_2) = a x_1 + b x_2 + d$ dla $(x_1, x_2) \in F$ przyjmuje wartości największą i najmniejszą w końcach tych odcinków



Dowód: Wystarczy wykazać, że dla dowolnego odcinka $I \subset F$,

$I = \{(t x_1 + (1-t) y_1,\ t x_2 + (1-t) y_2) : t \in [0,1]\}$



Faktycznie $\sup_I f =$

$$\cancel{\sup_{(x,y)\in I} f(x,y)} = \sup_{t\in[0,1]} a(t x_1 + (1-t) y_1) + b(t x_2 + (1-t) y_2) + d$$ – funkcja liniowa od t, ekstremum na końcu odcinka

□

Rezultat ten można uogólnić na $\mathbb{R}^k$:

Stwierdzenie

Niech $f(x_1,\dots,x_k)=a_1x_1+\dots+a_kx_k+d$. Wówczas dla dowolnego odcinka I:

$$\max_I f = \max_{\partial I} f \qquad \min_I f = \min_{\partial I} f$$

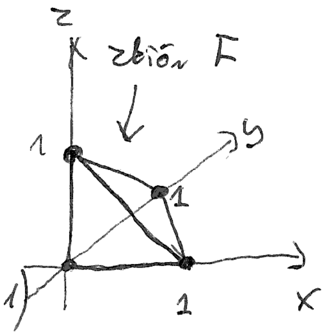
Wniosek

Jeśli F - wielościan, to ekstrema są przyjmowane w jego wierzchołkach

Wracamy do Zad 19:

$f(x,y,z)=2x-3y+z$, $F=\{(x,y,z): x\geq 0, y\geq 0, z\geq 0, x+y+z\leq 1\}$

Znaleźć $\max_F f$, $\min_F f$



F jest czworościanem o wierzchołkach $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

$f(0,0,0)=0$, $f(\overset{\max}{1},0,0)=2$, $f(\overset{\min}{0},1,0)=-3$, $f(0,0,1)=1$. Zatem

Odp: $\max_F f = f(1,0,0)=2$ i $\min_F f = f(0,1,0)=-3$