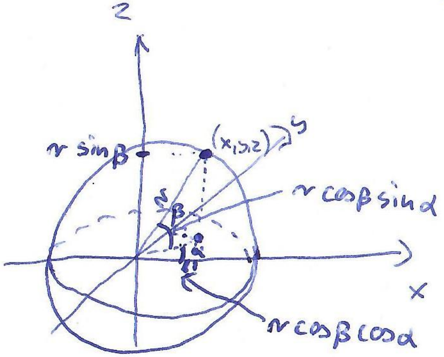
Zad 27

$$\Phi(r,\alpha,\beta)=\Big(\overset{x}{\overset{\shortparallel}{r\cos\beta\cos\alpha}},\ \overset{y}{\overset{\shortparallel}{r\cos\beta\sin\alpha}},\ \overset{z}{\overset{\shortparallel}{r\sin\beta}}\Big)$$



$$P=\{(r,\alpha,\beta):\ r>0,\ -\pi<\alpha<\pi,\ -\tfrac{\pi}{2}<\beta<\tfrac{\pi}{2}\}$$

$$\Phi:\ P \xrightarrow[na]{1-1} \mathbb{R}^3-\{(x,0,z):\ x\le 0,\ z\in\mathbb{R}\}$$

z konstrukcji;

Pokażemy, że Φ jest dyfeomorfizmem

klasy C^1

$$D\Phi(r,\alpha,\beta)=\begin{pmatrix}\cos\beta\cos\alpha & -r\cos\beta\sin\alpha & -r\sin\beta\cos\alpha\\ \cos\beta\sin\alpha & r\cos\beta\cos\alpha & -r\sin\beta\sin\alpha\\ \sin\beta & 0 & r\cos\beta\end{pmatrix}$$

Φ jest 1-1, na, C^1; nieosobliwe, zatem

$$\Phi:\ P \xrightarrow[dyfeo]{1-1,\ na} \mathbb{R}^3\setminus\{(x,0,z):\ x\le 0\}$$

$$J\Phi(r,\alpha,\beta)=r^2\cos^3\beta\cos^2\alpha+r^2\cos\beta\sin^2\beta\sin^2\alpha+r^2\cos\beta\sin^2\beta\cos^2\alpha+r^2\cos^3\beta\sin^2\alpha=$$

$$r^2\cos\beta\left(\underline{\cos^2\beta\cos^2\alpha}+\underline{\sin^2\beta\sin^2\alpha}+\underline{\sin^2\beta\cos^2\alpha}+\underline{\cos^2\beta\sin^2\alpha}\right)=r^2\cos\beta\left(\underline{\cos^2\alpha}+\underline{\sin^2\alpha}\right)=r^2\cos\beta\underset{na\ P}{>}0$$

Zad 28 35

$$D\Phi(r,\alpha,\beta)=\begin{pmatrix}\cos\beta\cos\alpha & -r\cos\beta\sin\alpha & -r\sin\beta\cos\alpha\\ \cos\beta\sin\alpha & r\cos\beta\cos\alpha & -r\sin\beta\sin\alpha\\ \sin\beta & 0 & r\cos\beta\end{pmatrix}$$

$$D\Phi(r,\alpha,\beta)^{-1}=\frac{1}{r^2\cos\beta}\begin{pmatrix}r^2\cos\alpha\cos^2\beta & r^2\cos^2\beta\sin\alpha & r^2\cos\beta\sin\beta\\ -r\sin\alpha & r\cos\alpha & 0\\ -r\cos\beta\cos\alpha\sin\beta & -r\cos\beta\sin\beta\sin\alpha & r\cos^2\beta\end{pmatrix}=\begin{pmatrix}\cos\alpha\cos\beta & \sin\alpha\cos\beta & \sin\beta\\ -\frac{\sin\alpha}{r\cos\beta} & \frac{\cos\alpha}{r\cos\beta} & 0\\ -\frac{\cos\alpha\sin\beta}{r} & -\frac{\sin\beta\sin\alpha}{r} & \frac{\cos\beta}{r}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{r\cos\alpha\cos\beta}{r} & \frac{r\sin\alpha\cos\beta}{r} & \frac{r\sin\beta}{r}\\ -\frac{r\cos\beta\sin\alpha}{r^2\cos^2\beta} & \frac{r\cos\alpha\cos\beta}{r^2\cos^2\beta} & 0\\ -\frac{r^2\cos\alpha\cos\beta\sin\beta}{r^3\cos\beta} & -\frac{r^2\cos\beta\sin\alpha\sin\beta}{r^3\cos\beta} & \frac{\cos\beta\, r}{r^2}\end{pmatrix}=\begin{pmatrix}\frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}}\\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0\\ \frac{-xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} & \frac{-yz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} & \frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}\end{pmatrix}$$

$x=r\cos\beta\cos\alpha$ $\quad z=r\sin\beta$ $\quad \sqrt{x^2+y^2}=r\cos\beta$

$y=r\cos\beta\sin\alpha$ $\quad x^2+y^2+z^2=r^2$

Zad 29 $\quad \varphi(x,y) = \left(\frac{ax}{\sqrt{1+x^2+y^2}}, \frac{by}{\sqrt{1+x^2+y^2}} \right) \quad a, b > 0$

1-1 ?

Niech $\varphi(x_1,y_1) = \varphi(x_2,y_2)$. Wówczas

$$\begin{cases} \frac{ax_1}{\sqrt{1+x_1^2+y_1^2}} = \frac{ax_2}{\sqrt{1+x_2^2+y_2^2}} \; /:a \\ \frac{by_1}{\sqrt{1+x_1^2+y_1^2}} = \frac{by_2}{\sqrt{1+x_2^2+y_2^2}} \; /:b \end{cases} \quad \frac{x_1}{y_1} = \frac{x_2}{y_2} \Rightarrow \begin{matrix} x_2 = \alpha x_1 \\ y_2 = \alpha y_1 \end{matrix}$$

dla pewnego $\alpha > 0$

$$\frac{x_1^2}{\cancel{\sqrt{}}1+x_1^2+y_1^2} = \frac{\alpha^2 x_1^2}{\cancel{\sqrt{}}1+\alpha^2x_1^2+\alpha^2y_1^2} \; /: x_1^2$$

$$\Downarrow$$

$$1+\alpha^2x_1^2+\alpha^2y_1^2 = \alpha^2+\alpha^2x_1^2+\alpha^2y_1^2$$

$$\alpha^2 = 1 \Rightarrow \alpha = 1 \Rightarrow \begin{cases} x_2 = x_1 \\ y_2 = y_1 \end{cases}$$

φ jest 1-1

klasy C^1 ?

$$\frac{\partial\varphi_1}{\partial x} = \frac{a\sqrt{1+x^2+y^2} - ax\frac{x}{\sqrt{1+x^2+y^2}}}{1+x^2+y^2} = \frac{a(1+x^2+y^2)-ax^2}{(1+x^2+y^2)^{3/2}} = \frac{a(1+y^2)}{(1+x^2+y^2)^{3/2}}$$

$$\frac{\partial\varphi_1}{\partial y} = -\frac{axy}{(1+x^2+y^2)^{3/2}} \qquad \frac{\partial\varphi_2}{\partial x} = \frac{-bxy}{(1+x^2+y^2)^{3/2}} \qquad \frac{\partial\varphi_2}{\partial y} = \frac{b(1+x^2)}{(1+x^2+y^2)^{3/2}}$$

$$D\varphi(x,y) = \begin{pmatrix} \frac{a(1+y^2)}{(1+x^2+y^2)^{3/2}} & \frac{-axy}{(1+x^2+y^2)^{3/2}} \\ \frac{-bxy}{(1+x^2+y^2)^{3/2}} & \frac{b(1+x^2)}{(1+x^2+y^2)^{3/2}} \end{pmatrix}$$ klasy C^1

$$J\varphi(x,y) = \frac{ab(1+x^2)(1+y^2) - abx^2y^2}{(1+x^2+y^2)^3} = \frac{ab(1+x^2+y^2)}{(1+x^2+y^2)^3} = \frac{ab}{(1+x^2+y^2)^2} > 0$$

φ jest nieosobliwe $\Rightarrow$ $\varphi : \mathbb{R}^2 \xrightarrow[\text{na}]{1-1} \varphi(\mathbb{R}^2)$

Co to jest $\varphi(\mathbb{R}^2)$?

Niech $u = \frac{ax}{\sqrt{1+x^2+y^2}}$ Wówczas: $\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = \frac{x^2+y^2}{1+x^2+y^2} < 1$, ponadto

$v = \frac{by}{\sqrt{1+x^2+y^2}}$ $\forall_{(u,v)\in\mathbb{R}^2}$ t. że $\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 < 1$ $\exists_{(x,y)\in\mathbb{R}^2}$ t. że $\varphi(x,y) = (u,v)$

Obszarem ograniczonym

zatem

$\varphi(\mathbb{R}^2)$ jest elipsą o półosiach $a, b > 0$