Symbolic Methods for Solving Systems of Linear Ordinary Differential Equations

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Introduction

What are these lectures about?

◊ **Main objective:** introduce symbolic methods for studying matrix linear ordinary differential equations in the complex domain with rational function coefficients.

◊ **Main topics:**

- Local Problems: Classification of Singularities, Computing Formal Invariants, Computing Formal Solutions.
- Global Problems: Finding Closed Form Solutions (Polynomial, Rational, Exponential Solutions . . .), Factorization.

◊ **General strategy:**

- Develop and use appropriate tools of local analysis to compute efficiently local data (around singularities).
- Global problems are solved by piecing together the local information around the different singularities.

Linear ODE's

• Linear differential equations of arbitrary order n:

$$(E) \qquad y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) \ldots + a_0(x)y(x) = b(x)$$

where b and the coefficients a_i's are in some *differential field* K, e.g. $K = \mathbb{C}(x)$.

• Systems of first order linear differential equations:

$$\begin{cases} \frac{dy_1}{dx} & = & a_{11}(x)y_1(x) + \ldots + a_{1n}(x)y_n(x) + b_1(x) \\ & \vdots & \\ \frac{dy_n}{dx} & = & a_{n1}(x)y_1(x) + \ldots + a_{nn}(x)y_n(x) + b_n(x) \end{cases}$$

Or in matrix notation:

$$Y' = AY + B$$

with $Y = {}^t(y_1, \ldots, y_n)$, $A = (a_{ij}) \in M_n(K)$, $B = {}^t(b_1, \ldots, b_n) \in K^n$.

Differential Fields

A *differential field* (K, ∂) is a field K with a map (*derivation*) $\partial : K \to K$, satisfying $\partial(a+b) = \partial(a) + \partial(b)$, $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in K$.

Notation: $\partial(a) = a'$

Examples:

- $(\mathbb{C}(x), \partial = \frac{d}{dx})$ = field of rational functions
- $(\mathbb{C}(x, e^x), \partial = \frac{d}{dx})$
- $\mathbb{C}[[x]]$ = ring of formal power series

  $$(\mathbb{C}((x)), \tfrac{d}{dx}) = \text{ quotient field of } \mathbb{C}[[x]] = \mathbb{C}[[x]][\tfrac{1}{x}]$$
- $(\mathcal{M}(\Omega), \frac{d}{dx})$ = field of fncs merom on Ω $\text{open, connected} \subset \mathbb{C}$

Solutions

Consider a differential system of dimension n with coefficients in K:

$$[A]: \quad Y' = AY, \quad A \in M_n(K).$$

• We should describe the class of functions in which the solutions are to be found.

• A (rational) solution: a vector $Y \in K^n$ such that $Y' = AY$.

• The set $\mathcal{S}_K = \{Y \in K^n \mid Y' = AY\}$ is a vector-space of dim $\leq n$ over the field of constants.

• In general, $\dim \mathcal{S}_K < n$. However, there always exists a differential field extension $K \subset L$ such that over L the solution space has dimension n.

• Fundamental solution matrix of $[A]$: an n by n invertible matrix W (with entries in some extension L of K) satisfying $W' = AW$.

Closed Form Solutions of Linear Differential Equations

- Rational solutions: Functions lying in K.
  - Algorithm by Liouville (1833) for $K = \mathbb{Q}(x)$ in the scalar case
  - Singer (1991) for more general fields.
  - Barkatou (1997) for the matrix case, $K = \mathbb{C}(x)$, Barkatou-Raab (2012) for more general fields.
- Algebraic solutions: Functions lying in an algebraic extension of K.
  - Investigated by Pépin (1881), Fuchs(1875), Klein, Jordan(1878), ...
  - Algorithms by Singer (1979, 1993).

Liouvillian Solutions

Functions that can be generated from the rational functions by successively substituting nested algebraic functions, integrals and exponential of integrals.

An example of such a construction is

$$\frac{1}{x^2-1} \xrightarrow{\sqrt{\ }} \frac{1}{\sqrt{x^2-1}} \xrightarrow{e^{\int}} \exp\left(\int \frac{dx}{\sqrt{x^2-1}}\right)$$

- Algorithm for second-order equations by Kovacik (1986)
- Singer (1991) using differential Galois theory: the problem of finding Liouvillian solutions is decidable, in theory, for equations of arbitrary order. However no practical algorithm.

Exponential Solutions

- Any function f whose logarithmic derivative $\dfrac{f'}{f}$ lies in K.
- They form a subclass of Liouvillian solutions.
- Algorithms for exponential solutions are used in all known methods for finding Liouvillian solutions.
- Algorithm by Beke (1894).
- van Hoeij (1997) for $K = \bar{\mathbb{Q}}(x)$.
- Pflügel, Barkatou (1997) for matrix equations.

Examples of Closed Form Solutions

Closed form solutions: can be written in terms of functions in some differential field K using: field operations $(+, -, \times, /)$, algebraic extensions $\sqrt{}$, composition $\circ$, differentiation $'$, integration $\int dx$

- What are the solutions of

$$y'' + \frac{3(x^2 + x + 1)}{16(x-1)^2x^2}y = 0?$$

Answer:

$$y = c_1 \sqrt[4]{x - x^2}\sqrt{1 + \sqrt{x}} + c_2 \sqrt[4]{x - x^2}\sqrt{1 - \sqrt{x}}$$

Algebraic function: root of a polynomial with coefficients in $\mathbb{C}(x)$.

Examples

- What are the solutions of

$$y'' + \frac{x}{x^2-1}y' - \frac{1}{2\nu^2(x^2-1)}y = 0?$$

Answer:

$$y = c_1 \exp\left(\int \frac{dx}{2\nu\sqrt{x^2-1}}\right) + c_2 \exp\left(\int \frac{-dx}{2\nu\sqrt{x^2-1}}\right)$$

Liouvillian function.

$$y = c_1 \mathrm{e}^{\frac{\ln\left(x+\sqrt{x^2-1}\right)}{2\nu}} + c_2 \mathrm{e}^{-\frac{\ln\left(x+\sqrt{x^2-1}\right)}{2\nu}}$$

$$y = c_1\left(x+\sqrt{x^2-1}\right)^{\frac{1}{2\nu}} + c_2\left(x+\sqrt{x^2-1}\right)^{\frac{-1}{2\nu}}$$

Solutions are algebraic when ν is rational.

- What are the solutions of

$$y'' + \frac{x}{x^2-1}y' - \frac{1}{2\nu^2(x^2-1)}y = 0?$$

- The answer given by the Maple *dsolve* command is:

$$y(x) = c_1 \sin\left(1/2\,\frac{\sqrt{2}\arcsin(x)}{\nu}\right) + c_2 \cos\left(1/2\,\frac{\sqrt{2}\arcsin(x)}{\nu}\right)$$

Airy Equation

- What are the solutions of

$$y'' - xy = 0?$$

- Answer :

$$y(x) = c_1 \operatorname{Ai}(x) + c_2 \operatorname{Bi}(x)$$

Special functions.

- If we ask for *Power series solutions*, we get

$$y = c_1 \sum_{n=0}^{\infty} \frac{(1/9)^n x^{3n}}{\Gamma(n+1)\Gamma(n+2/3)} + c_2 \sum_{n=0}^{\infty} \frac{(1/9)^n x^{3n+1}}{\Gamma(n+4/3)\Gamma(n+1)}$$

More General Closed Form Solutions

- A function is in closed form if it can be expressed in terms of well know special functions: Airy, Bessel, Hermite, Legendre, Laguerre, Kummer, Whittaker, ${}_2F_1-$hypergeometric functions, etc.

- These special functions satisfy a second order differential equation

- A lot of information is known about these functions [Abramowitz and Stegun : Handbook of Mathematical Functions (1972)].

- Open Problem: Given any linear differential equation with rational function coefficients, decide if it is solvable in closed form, and if so, find its closed form solutions.

- It is possible to decide with the help of differential Galois theory wheteher or not *Liouvillian* solutions exist (Singer 1991).

Example

- What are the solutions of

$$y'' + \frac{2}{3x} y' - \frac{1}{9x} y = 0?$$

Answer:

$$y(x) = c_1\, x^{1/6} \mathrm{I}_{1/3}\left(2/3\sqrt{x}\right) + c_2\, x^{1/6} \mathrm{K}_{1/3}\left(2/3\sqrt{x}\right)$$

where I_ν, K_ν are Bessel functions.

- If we ask for *Generalized series solutions*, we get

$$y = c_1 \sum_{n=0}^{\infty} \frac{(1/9)^n\, x^n}{\Gamma(n+1)\,\Gamma(n+2/3)} + c_2\, x^{1/3} \sum_{n=0}^{\infty} \frac{(1/9)^n\, x^n}{\Gamma(n+4/3)\,\Gamma(n+1)}$$

(*Frobenius* series)

Example

- What are the solutions of

$$y''' - 4xy' - 2y = 0?$$

Answer:

$$y = c_1(A_1(x))^2 + c_2A_1(x)A_2(x) + c_3(A_2(x))^2$$

where A_1, A_2 are two linearly independent solutions of

$$y'' - xy = 0 \quad (\textit{Airy Equation})$$

- Problem: Given any linear differential equation with rational function coefficients, decide if it is solvable in terms of solutions of linear equations of lower order.

Formal Solutions

- Formal (or asymptotic) solution describe the behavior of the actual solutions in in the neighborhood of a given point.
- At an ordinary point x_0, Taylor series are sufficient : $\sum_{n=0}^{\infty} a_n(x-x_0)^n$
- At singular points more general expansions must be used: Frobenius series, log-exp expansions)

$$e^{q(\frac{1}{t})}t^{\lambda}(\phi_0(t)+\phi_1(t)\log t+\cdots+\phi_s(t)(\log t)^s)$$

$$t=(x-x_0)^{1/r}, \quad r\in\mathbb{N}^* \quad \text{(ramification)},$$
$$q\in\mathbb{C}[X], \quad \lambda\in\mathbb{C}, \quad s\in\mathbb{N}$$
$$\text{and } \phi_j\in\mathbb{C}[[t]].$$

- local-to-global approach: global problems are solved by piecing together the local information around the different singularities.

Examples

- Euler Equation: $x^2y' + y = x$

Formal solutions at $x = 0$: $\hat{f} + ce^{-1/x}$, $c \in \mathbb{C}$,

$$\hat{f} = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1}$$

- Airy Equation: $y'' = xy$

Formal solutions at $x = \infty$:

$$\hat{A}_1(x) = \left(\sum_{n=0}^{\infty} (-1)^n a_n t^{-3n} \right) t^{-1/2} e^{-2t^3/3}$$

$$\hat{A}_2(x) = \left(\sum_{n=0}^{\infty} a_n t^{-3n} \right) t^{-1/2} e^{2t^3/3}$$

where $t = x^{1/2}$, $a_0 = 1$, $a_n = \frac{1}{2\pi} \frac{1}{n!} \Gamma(n + \frac{5}{6}) \Gamma(n + \frac{1}{6})$

Example : Euler Equation

- What are the solutions of

$$x^2\frac{dy}{dx}+y=x. \tag{1}$$

at the singular point $x=0$?

- Answer $y=\hat{f}+ce^{-1/x}$, $c\in\mathbb{C}$ where

$$\hat{f}=\sum_{n=0}^{\infty}(-1)^n n! x^{n+1}$$

- This is a divergent series!
- It cannot be "summed" in the usual sense to a function which is a solution of (1) in the neighborhood of $x=0$.
- What does this formal solution represent exactly? What can it tell us about the *actual* solutions?

Example : Euler Equation

- Using the method of variation of parameter, we get the particular actual solution:

$$f(x) = e^{1/x} \int_0^x \frac{e^{-1/t}}{t} dt.$$

- Which can be rewritten (for $Re(x) > 0$) as:

$$f(x) = \int_0^\infty \frac{e^{-\frac{z}{x}}}{1+z} dz, \tag{2}$$

- One can prove the following inequalities (for $Re(x) > 0$):

$$|f(x) - \sum_{n=0}^{N-1} (-1)^n \, n! x^{n+1}| \quad \leq \quad N! \, |x|^{N+1}.$$

- This proves that $\hat{f}$ is an asymptotic expansion of $f(x)$ as $x \to 0$ in the half-plane $Re(x) > 0$ (Poincaré 1886).

Organization of the lectures

◊ **Main objective:** to present symbolic methods for manipulating, in the broad sense, systems $Y' = AY$ with rational function coefficients.

◊ **Main Focus:** Formal aspects of Local Analysis and Related Computer Algebra Algorithms.

◊ **Not Discussed**: Analytic aspects: Asymptotic and Summability Theory (see R. Schäfke's lectures) .

▷ These lectures are divided into three parts:

- Part 1: Basic Tools for Local Analysis - Systems of First Kind.
- Part 2: Systems of Second Kind - Fundamental Algorithms.
- Part 3: Applications to Solving Systems with Rational Function Coefficients.

Part 1: Basic Tools of Local Analysis- Systems of First Kind

Outline

- Correspondance Matrix/Scalar Equations
- Classification of Singularities
- Systems of First Kind

Correspondance Matrix/Scalar Equations

Linear Differential Equations over $(K, ')$

Three equivalent forms:

- a scalar linear differential equation:

$$D(y) = a_n y^{(n)} + \ldots + a_0 y = 0, \; a_i \in K$$

- a matrix linear differential equation:

$$Y' = AY, \; A \in M_n(K)$$

- a differential module of dimension n:

an n dim. K-vector space V with an additive map $\partial : V \to V$ satisfying

$$\partial(\alpha v) = \alpha' v + \alpha \partial v$$

for all $\alpha \in K, v \in V$.

Linear Differential Systems

We consider a system of first order linear differential equations of the form

$$[A] \qquad Y' = AY,$$

where Y is column-vector of length n,

A is an $n \times n$ matrix with entries in K,

K is a differential field of characteristic zero with constant field $\mathcal{C} \supset \mathbb{Q}$.
In this talk

$$K = \mathcal{C}((x)) = \mathcal{C}[[x]][x^{-1}], \quad \text{or} \quad K = \mathcal{C}(x)$$

with $' = \frac{d}{dx}$ the standard derivation.

Equivalent Systems

Consider a system $[A]$ $\quad Y' = AY, \; A \in M_n(K)$.

Gauge transformation: $Y = TZ$, $T \in GL(n, K)$, leads to

$$[B] \quad Z' = BZ,$$

$$B = T[A] := T^{-1}AT - T^{-1}T'.$$

Systems $[A]$ and $[B]$ are called equivalent (over K).

If $T \in GL(n, L)$ for some differential field extension L of K then $[A]$ and $[B]$ are called equivalent over L.

Correspondance Systems/Equations

Scalar $\longrightarrow$ Matrix

Consider a scalar linear differential equation:

$$D(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0, \quad a_i \in K$$

Let

$$Y = (y, y', \ldots, y^{(n-1)})^T$$

Then

$$Y' = CY$$

where

$$C = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix}$$

Notation: $C = companion(a_i)_{0 \le i \le n-1}$

Matrix $\longrightarrow$ Scalar

Thm: (Cyclic Vector Lemma) Assume $\exists a \in K, a' \neq 0$. Then every system $Y' = AY$ is equivalent to a scalar equation $D(y) = 0$.

In other words, given a system $Y' = AY$ over K, one can always construct a gauge transformation $T \in GL(n, K)$ such that

$$T[A] := T^{-1}AT - T^{-1}T' = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}$$

- Many proofs : Loewy 1918, Cope 1936, Deligne 1970, Ramis 1978, Katz 1989, Barkatou 1993, Churchill-Kovacic 2002 ...

Cyclic Vectors

Consider a system $[A]: \quad \partial Y = AY$ over a differential field (K, ∂).

Let $\Lambda = (\lambda_1, \cdots, \lambda_n) \in K^n$.

Put

$$y = \Lambda Y = \lambda_1 y_1 + \cdots + \lambda_n y_n$$

Computing successively $\partial y, \cdots, \partial^n y$ and using the equation $\partial Y = AY$ we obtain

$$\partial^i y = \Lambda_i Y \qquad \text{for} \quad i = 0, \cdots, n \tag{3}$$

where the sequence of row vectors $\{\Lambda_i\}$ is defined inductively as :

$$\Lambda_0 = \Lambda, \qquad \Lambda_i = \partial \Lambda_{i-1} + \Lambda_{i-1} A \quad \text{for} \quad i = 1, \cdots, n.$$

Let

$$P = \begin{pmatrix} \Lambda_0 \\ \Lambda_1 \\ \vdots \\ \Lambda_{n-1} \end{pmatrix}, \quad Z = \begin{pmatrix} y \\ \partial y \\ \vdots \\ \partial^{n-1} y \end{pmatrix}, \quad B = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_n \end{pmatrix}.$$

Then (3) can be written as

$$Z = PY \qquad \text{and} \qquad \partial Z = BY \tag{4}$$

Note that $B = \partial(P) + PA$.

Def: The vector Λ is said to be a cyclic vector for system $[A]$ if the matrix P is nonsingular (i.e. $\det P \neq 0$).

If Λ is a cyclic vector then equation (4) can be written

$$Y = P^{-1}Z \qquad \text{and} \qquad \partial Z = CZ$$

$C := BP^{-1} = \partial(P)P^{-1} + PAP^{-1}$ is a companion matrix
$C = companion(a_i)_{0 \leq i \leq n-1}$.

- Hence it follows that the system $\partial Y = AY$ is equivalent to the scalar differential equation :

$$D_\Lambda(y) = \partial^n y + a_{n-1}\partial^{n-1}y + \cdots + a_1\partial y + a_0 y = 0$$

- Note that this equation is by no means uniquely determined by the system $\partial Y = AY$. It depends on the choice of the cyclic vector Λ.
- It is always possible (Cope, Ramis) to choose a cyclic vector Λ whose components are polynomials in x of degree $\leq n-1$.

Example

Let

$$A := \begin{bmatrix} -x + 2\,x^{-1} & x^3 + x^2 & 4\,x^{-1} \\ x^{-3} + 2\,x^{-2} & 1 - x & 4\,x^{-1} \\ x & 3\,x^2 & 2\,x^{-1} + x^2 \end{bmatrix}$$

Take $\Lambda = [1, 0, 0]$. Then

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{x^2-2}{x} & x^3 + x^2 & 4\,x^{-1} \\ \frac{2\,x^2+2+x^4+2\,x^3+x}{x^2} & -x\left(-6\,x - 16 + 2\,x^3 + x^2\right) & 4\,\frac{3-x^2+x^4+2\,x^3}{x^2} \end{bmatrix}$$

$\det P = -12\,x + 12\,x^3 + 4\,x^5 + 12\,x^4 - 52 \neq 0$. Hence $\Lambda = [1, 0, 0]$ is a cyclic vector.

Example

Compute $T := P^{-1}$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{-2x^4-2x^3+7x^2-4+2x^5+x+x^6}{x^3(3x^4+3x^3-3x-13+x^5)} & \frac{3-x^2+x^4+2x^3}{x^2(3x^4+3x^3-3x-13+x^5)} & -\frac{1}{x(3x^4+3x^3-3x-13+x^5)} \\ 1/4\,\frac{-14x^3-21x^2+x^5-2x^4+9x+30}{3x^4+3x^3-3x-13+x^5} & 1/4\,\frac{x\left(-6x-16+2x^3+x^2\right)}{3x^4+3x^3-3x-13+x^5} & 1/4\,\frac{x^2(x+1)}{3x^4+3x^3-3x-13+x^5} \end{bmatrix}$$

Compute

$$C := T[A] = T^{-1}AT - T^{-1}T' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_0 & c_1 & c_2 \end{bmatrix}$$

$$c_0 = \frac{-156+269x^4-146x^5-215x^6+38x^9+15x^{10}+16x^8+4x^{11}-109x^7-243x+432x^3}{x^3\left(3x^4+3x^3-3x-13+x^5\right)}$$

$$c_1 = \frac{130-156x^4-24x^5+67x^6+4x^9+2x^{10}+10x^8+42x^7+65x-311x^3-124x^2}{x^2\left(3x^4+3x^3-3x-13+x^5\right)}$$

$$c_2 = \frac{-52+24x^4+6x^5-2x^6+x^8+23x^2+x^7-28x+14x^3}{x\left(3x^4+3x^3-3x-13+x^5\right)}$$

Algebraic complexity

◇ Arithmetic (operations in C) complexity estimate:

- Computation of a cyclic vector and the corresponding scalar equation: $\mathcal{O}(n^5)$

◇ Bounds on degrees and arithmetic sizes:

- Degree of the coefficients c_i's: $n \deg \Lambda + \frac{n(n+1)}{2} \deg A$
- Size of the coefficients c_i's: $n^2 \deg \Lambda + n^3 \deg A$

Comments on use of cyclic vectors

◊ **Interest**: Many algorithmic problems are easily solvable for scalar equations.

◊ **Drawbacks**:

- for systems with "large" dimension n (in practice $n \geq 10$), the construction of an equivalent scalar equation may take a "long time";
- the scalar equation (when it can be computed) has often "too complicated" coefficients compared with the entries of the input system (even for small dimensions) and in consequence solving this equation can be costly.
- The reduction to a scalar equation may destroy the symmetries or structure of the input matrix equation (e. g. Linear Hamiltonian systems)

◊ Direct methods are to be preferred.

Singularities

Singularities

Consider a linear differential equation in the complex plane $\mathbb{C}$ with analytic (e.g. meromorphic or rational function) coefficients:

$$[A] \quad \frac{dY}{dx} = A(x)Y.$$

Def: $x_0 \in \mathbb{C}$ is an ordinary point if all the entries of $A(x)$ are holomorphic in some nbhd of x_0, otherwise x_0 is a singular point.

• x_0 is an ordinary point $\Rightarrow$ there exists a fund soln matrix W whose entries are holomorphic in in some nbhd of x_0.

Classification of Singularities

◊ Suppose that $A(x)$ is holomorphic in a punctured nbhd of x_0, $\Omega = \{x \in \mathbb{C} \,|0 < |x - x_0| < \rho\}$, with at most a pole at the point x_0.

◊ Since Ω is not simply connected, the solutions of Eqn $[A]$ need not be single-valued, but we have the following result (cf. Wasow):

Every fund soln matrix W of $[A]$ has the form:

$$W(x) = \Phi(x)(x - x_0)^{\Lambda}$$

where $\Phi(x)$ is holomorphic on Ω, and Λ is a constant matrix.

Def: The point x_0 is called a regular singular point for $[A]$ if $\Phi(x)$ has at most a pole at the point x_0, otherwise x_0 is called an irregular singular point.

Examples

- $\frac{dy}{dx} = \frac{\alpha}{x} y, \alpha \in \mathbb{C}$

  fund soln: x^{α} so $x = 0$ is a regular singular point

- $\frac{dy}{dx} = \frac{-a}{px^{p+1}} y$, $a \in \mathbb{C}^*, p \in \mathbb{N}^*$

  fund soln: $e^{\frac{a}{x^p}}$ so $x = 0$ is an irregular singular point.

- $\frac{dY}{dx} = \frac{\Lambda}{x} Y$, $\Lambda \in M_n(\mathbb{C})$

  fund soln matrix x^{Λ} so $x = 0$ is a regular singular point.

Classification of the point at ∞

The change of variable $x \mapsto \frac{1}{x}$ permits to classify the point $x = \infty$ as an ordinary, regular singular or irregular singular point for $[A]$:

Let $z = \frac{1}{x}$ $\frac{dY}{dx} = A(x)Y \Rightarrow \frac{dY}{dz} = -\frac{1}{z^2}A(\frac{1}{z})Y$

Type of $x = \infty$ for $\frac{dY}{dx} = A(x)Y$

$= $ Type of $z = 0$ for $\frac{dY}{dz} = -\frac{1}{z^2}A(\frac{1}{z})Y$

Examples

$\frac{dy}{dx} = \frac{1}{3x}y$, $z = \frac{1}{x} \Rightarrow \frac{dy}{dz} = -\frac{1}{3z}y$

$z = 0$ is a regular singular point $\Rightarrow x = \infty$ is a regular singular point

$y' = \frac{2}{x^3}y$ $z = \frac{1}{x} \Rightarrow \frac{dy}{dz} = -2zy$

$z = 0$ is an ordinary point $\Rightarrow x = \infty$ is an ordinary point.

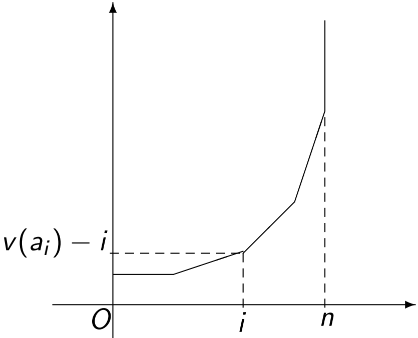
The scalar case

Newton Polygon of a Scalar Equation

Let

$$D = \sum_{i=0}^{n} a_i \left(\frac{d}{dx}\right)^i, \quad a_i \in \mathbb{C}[[x]].$$

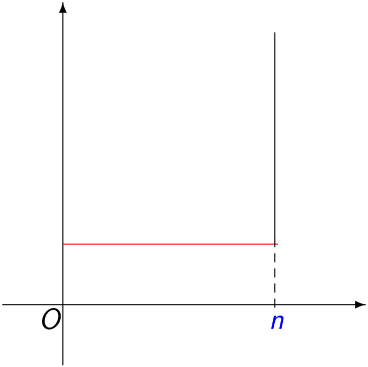
Def: $N(D) :=$ Newton polygon of D = Convex hull with nonnegative slopes of the points $(i, v(a_i) - i)$



$0 \neq f = \sum_j f_j x^j \in \mathbb{C}((x)),\ v(f) = min\{j \mid f_j \neq 0\},\ v(0) = +\infty.$

Fuchs Criterion for Scalar Equations

The point $x = 0$ is regular singular $\iff$ $v(a_i) - i \geq v(a_n) - n$ for all i (Fuchs criterion).



Formal solutions: Scalar case

Let

$$D(y) = a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$$

where the $a_j(x) \in \mathbb{C}[[x]]$ (formal power series).

There exists n linearly independent formal solutions of the form:

$$y_i = e^{q_i(\frac{1}{t})} t^{\lambda_i} (\phi_{i0} + \phi_{i1} \log t + \cdots + \phi_{is_i} (\log t)^{s_i})$$

$$x = t^{r_i}, \quad r_i \in \mathbb{N}^* \quad (\text{ramification})$$
$$q_i \in \mathbb{C}[X], \quad \lambda_i \in \mathbb{C}, \quad s_i \in \mathbb{N}$$
$$\text{and } \phi_{ij} \in \mathbb{C}[[t]].$$

Existence: Fabry, Poincaré, Malgrange, etc.

The regular case

$x = 0$ is regular singular $\iff$ all the q_i's are zero.

$$y_i = x^{\lambda_i} \left(\phi_{i0} + \phi_{i1} \log x + \cdots + \phi_{is_i} (\log x)^{s_i}\right)$$

In that case:

- no need of ramification: $t = x$ ($r_i = 1$)
- $\phi_{ij} \in \mathbb{C}\{x\}$ whenever the a_i's are convergent,
- λ_i called local exponent (at $x = 0$) = root of indicial equation,

Construction: Frobenius method.

The irregular case

$x = 0$ is Irregular singular $\iff$ at least one of the q_i's is not zero.

$$y_i = e^{q_i(\frac{1}{t})} t^{\lambda_i} (\phi_{i0} + \phi_{i1} \log t + \cdots + \phi_{is_i} (\log t)^{s_i})$$

In that case:

- the "degree" of q_i in x^{-1} = slope of the Newton polygon of D and its leading coefficient = a root of the corresponding Newton polynomial.
- may need ramification: $t = x^{1/r_i}$, $r_i > 1$
- the involved power series are in general divergent.

Construction : Newton Algorithm based on Newton polygon calculations.

Matrix Case: Another Classification of Singularities

• In the scalar case, the nature of a singular point x_0 can be read off from the leading terms of the coefficients of the equation (Fuchs' Criterion).

• In the matrix case, there is no analogue of the Fuchs' Criterion.

• The nature of a singular point x_0 is based upon the knowledge of fundamental matrix solution and hence is not immediately checkable for a given system.

• **Important Problem:** Give an algorithm to decide for any system whether it has regular singularity.

- First step: Give a characterization of regular singularities which is not based on a prior knowledge of solutions.

From now we will assume that $x_0 = 0$ (unless otherwise specified).

A Classification easier to check

Consider a linear differential system with a singularity at $x = 0$:

$$[A]: \quad Y' = A(x)Y,$$

$$A(x) = \frac{1}{x^{p+1}} \sum_{i=0}^{\infty} A_i x^i, \quad A_i \in M_n(\mathbb{C}), \;\; A_0 \neq 0, \;\; p \in \mathbb{N}.$$

The integer $p(A) := p$ is called the Poincaré rank of the system $[A]$.

Def:

- If $p(A) = 0$, the point $x_0 = 0$ is called a singularity of first kind for system $[A]$.
- If $p(A) > 0$, the point $x_0 = 0$ is called a singularity of second kind for system $[A]$.

Comparison of the two classifications

◊ The two classifications are not directly comparable. However we have the following

- $x = 0$ is a singularity of the first kind $\Rightarrow$ $x = 0$ is a regular singularity.
- The converse is false, in general: a singularity of second kind ($p > 0$) may be a regular singularity.
- System $[A]$ has a regular singularity at $x = 0$ iff $\exists T \in GL(n, K)$ s.t. $T[A]$ has a singularity of first kind at $x = 0$.

◊ **Important Problem:** Give an algorithm to decide whether a system of second kind has a regular singularity (Part 2 of this Tutorial).

Singular Differential Systems

$K = \mathbb{C}((x)) = \mathbb{C}[[x]][x^{-1}]$ field formal Laurent series.

$$[A] \qquad \frac{dY}{dx} = A(x)Y,$$

$A(x)$ an $n \times n$ matrix with entries in K:

$$A(x) = x^{-p-1}\sum_{i=0}^{\infty} A_i x^i, \quad A_0 \neq 0, p = p(A) \in \mathbb{N} \textit{ Poincaré rank of [A]}$$

Gauge transformation: $Y = TZ$, $T \in GL(n, K)$, leads to

$$[B] \quad \frac{Z}{dx} = BZ, \qquad B = T[A] := T^{-1}AT - T^{-1}T'.$$

Systems $[A]$ and $[B]$ are called equivalent over K.

Exp. Parts and Formal Solutions

$[A]$ has a formal fundamental matrix solution of the form

$$\Phi(x^{1/s})x^{\Lambda}\exp\left(Q(x^{-1/s})\right)$$

$s \in \mathbb{N}^*$, $\Phi \in GL(n, \mathbb{C}((x^{1/s})))$,
$Q(x^{-1/s}) = \mathrm{diag}\left(q_1(x^{-1/s}), \ldots, q_n(x^{-1/s})\right)$ where the q_i's are polynomials in $x^{-1/s}$ over $\mathbb{C}$ without constant term,
Λ is a constant matrix commuting with Q.

- Q is invariant under all gauge transformations $T \in GL(n, \overline{\mathrm{K}})$.

$$\overline{\mathrm{K}} = \bigcup_{m \in \mathbb{N}^*} \mathbb{C}((x^{1/m}))$$ field of formal Puiseux series in x.

Classification of singularities

- When $Q(x^{-1/s}) \not\equiv 0$, the origin is an *irregular singular point* of the system $[A]$. In this case the elements of $Q(x^{-1/s})$ determine the main asymptotic behavior of actual solutions as $x \to 0$ in sectors of sufficiently small angular opening (Asymptotic existence theorem (cf. Wasow)).

- If $Q(x^{-1/s}) \equiv 0$, the point $x = 0$ is *regular singular* point of $[A]$. In this case $s = 1$ and the formal series $\Phi(x)$ converges whenever the series for $A(x)$ does.

- Existence of Formal Solutions goes back to Turritin, Hukuhara, Levelt, Balser-Jurkat-Lutz

- Efficient computation of Q [Barkatou 1997], [Pflügel 2000], [Barkatou-Pflügel 2007]

Example: $y''(z) = zy(z)$ (Airy Equation)

$z = \infty$ is a singular point at ∞

$x = \frac{1}{z} \Rightarrow x^5 y'' + 2x^4 y' - y = 0$

$\Rightarrow \mathbf{y}' = \begin{pmatrix} 0 & 1 \\ \frac{1}{x^5} & -\frac{2}{x} \end{pmatrix} \mathbf{y}$

Formal Solution Matrix: $Y = \Phi(x) U x^J U^{-1} e^{Q(x)}$

$$\Phi(x) = \ldots, \quad U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{3}{4} \end{pmatrix}$$

$$Q = \begin{pmatrix} -\frac{2}{3x^{3/2}} & 0 \\ 0 & \frac{2}{3x^{3/2}} \end{pmatrix}$$

Hukuhara-Turritin's Normal Form

Any differential system $[A]$ is equivalent over $\overline{K}$ to a system $[B]$ of the form:

$$B = \Gamma_1 x^{-k_1-1} + \Gamma_2 x^{-k_2-1} + \cdots + \Gamma_m x^{-k_m-1} + (\Gamma_0 + N)x^{-1}$$

where

(a) $k_1 > k_2 > \cdots > k_m > 0$ are rational numbers (k_1 is Katz-invariant of $[A]$),

(b) the Γ_j's and N are constant matrices,

(c) the Γ_j's are diagonal and commute with N.

Existence : Hukuhara (1930's), Turritin (1950's) , Wasow (1960's), Levelt, Jurkat, Lutz, Balser (1970's) , Babbit& Varadajan (1980's), etc.

Algorithms: Moser (1960's), Dietrich (1970's), Levelt, Wagenfuhrer, Hilali&Wazner (1980's), Chen , Barkatou, (1990's), Pflügel (2000)

Efficient computation: [Barkatou 1997], [BP98], [Pflügel 2000]

What's the point of knowing the Exp. Parts?

- Local analysis near a singularity:
  - Asymptotic behavior of actual solutions
  - Summability problems,
  - Stokes multipliers, etc.
- Local data are useful for solving global problems:
  - Computing exponential solutions of systems with coefficients in $\mathbb{C}(x)$ or in an extension of $\mathbb{C}(x)$
  - Factorisation problems, Computing eigenrings
  - Computing Hom(A,B)
  - Testing the equivalence of two systems with rational function coefficients

Systems of First Kind

Structure of Solutions

Consider a linear differential system of first kind: $[A]: \quad Y' = A(x)Y,$

$$A(x) = \frac{1}{x}\sum_{i=0}^{\infty} A_i x^i, \quad A_i \in M_n(\mathbb{C}), \;\; A_0 \neq 0.$$

Thm Any system of first kind has a formal fundamental matrix solution of the form

$$S(x)x^{\Lambda}$$

where Λ is a constant matrix and $S \in GL(n, \mathbb{C}((x)))$.
Moreover, the formal series $S(x)$ converges whenever the series for $A(x)$ does.

• Many Proofs: Sauvage (1886) developed a method, analogous to Frobenius method for the scalar case.

Sauvage's method

It constructs n linearly independent vector solutions $\mathbf{y}_1, \cdots, \mathbf{y}_n$ of the form

$$\mathbf{y}_j = x^{\lambda_j} \sum_{k=0}^{n_j-1} \mathbf{f}_{k,j}(x) \frac{\log^k x}{k!} \tag{5}$$

where $\lambda_j \in \mathbb{C}$, n_j is a positive integer and the $\mathbf{f}_{kj}$'s are vectors of power series in x.

- the characteristic polynomial of the matrix A_0 plays the role of the *indicial equation*,
- each exponent λ_j is an eigenvalue of A_0,
- the integer n_j is the dimension of a Jordan block corresponding to λ, $\mathbf{f}_{0,j}(0), \mathbf{f}_{1,j}(0), \ldots, \mathbf{f}_{n_j-1,j}(0)$ is a Jordan chain of generalized eigenvectors of A_0 associated to λ_j;
- and the other coefficients in the power series $\mathbf{f}_{k,j}$ are given by linear recurrence formulas.

Eulerian systems : $Y' = \frac{\Lambda}{x} Y$, $\Lambda \in M_n(\mathbb{C})$

- A fundamental solution is given by $x^{\Lambda} = \exp(\Lambda \log x)$
- The matrix Λ can be brought to its Jordan normal form $J = P^{-1}\Lambda P$ by a suitable $P \in GL_n(\mathbb{C})$.
- Hence Px^J is as well a fundamental solution the system.
- The matrix x^J is a block-diagonal matrix, with blocks of the form

$$x^{\lambda} \begin{pmatrix} 1 & \log x & & \frac{\log^{\nu-1} x}{(\nu-1)!} \\ 0 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \log x \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

- λ is an eigenvalue of Λ and ν is the size of Jordan block associated with λ.
- The entries of the fundamental matrix Px^J are of the form $x^{\lambda} p(x)$ where λ is an eigenvalue of Λ and $p(x)$ is a polynomial in $\log x$ with degree less than the size of a Jordan block associated to λ.

Another proof

- It is possible to construct a formal gauge transformation $S \in GL(n, \mathbb{C}((x)))$ such that the transformed system $S[A]$ has the simple form of a Eulerian system $Y' = x^{-1}\Lambda Y$.
- Two cases are distinguished:
  1. the *non-resonant case*: the residue matrix A_0 has no eigenvalues that differ from each other by positive integers,
  2. the *resonant case*: A_0 has eigenvalues differing by positive integers.
- In the first case, the exponent matrix Λ can be taken equal to the residue matrix A_0 itself and $S_0 = \mathrm{I}_n$,
- In the second case S_0 is a singular matrix and Λ depends not only on A_0 but in all the coefficients $A_0, A_1, \cdots, A_\ell$ where ℓ is the largest positive integer which is the difference of two eigenvalues of A_0.

The non-resonant case

$$[A]: \ Y' = A(x)Y, \quad A(x) = \frac{A_0}{x} + \sum_{i=1}^{\infty} A_i x^{i-1}, \ \ A_i \in M_n(\mathbb{C}), \ \ A_0 \neq 0.$$

Thm 1 If the eigenvalues of A_0 do not differ by nonzero integers, then there exists $T \in GL(n, \mathbb{C}[[x]])$ with $T(0) = I_n$ such that $T[A] := T^{-1}AT - T^{-1}T' = \frac{A_0}{x}$

Proof: Look for $T = \sum_{i=0}^{\infty} T_i x^i$ satisfying: $xT' = xAT - TA_0$.

Inserting the series of A and T in the above equation yields:

$$A_0 T_0 - T_0 A_0 = 0, \quad (A_0 - jI_n)T_j - T_j A_0 = -\sum_{i=0}^{j-1} A_{j-i} T_i, \ \ j \geq 1. \qquad (6)$$

By choosing $T_0 = I_n$, the T_j's are determined recursively by (6) which has a unique solution since $A_0 - jI_n$ and A_0 have no common eigenvalues for $j \geq 1$ (see next slide).

Sylvester Equation

$\Diamond$ Uniqueness of solution of (6) follows from the following well-known Linear Algebra result :

- Let M and N be two square matrices of order m and n with entries in a field $\mathcal{C}$ and having no common eigenvalues (in the algebraic closure of $\mathcal{C}$). Then for every matrix $L \in Mat_{m,n}(\mathcal{C})$ there exists a unique matrix $X \in Mat_{m,n}(\mathcal{C})$ such that $MX - XN = L$.

$\Diamond$ The matrix X can be determined by solving a sparse system of mn linear equations in mn unknowns (More efficient algorithms exist).

Systems of First Kind: The resonant case

$$[A]: \; Y' = A(x)Y, \quad A(x) = \frac{A_0}{x} + \sum_{i=1}^{\infty} A_i x^{i-1}, \; A_i \in M_n(\mathbb{C}), \; A_0 \neq 0.$$

Thm 2 There exists $T \in GL(n, \mathbb{C}[x])$ with $\det T(x) \neq 0$ for $x \neq 0$ such that $B = T[A] = x^{-1}(B_0 + xB_1 + \dots)$ where the eigenvalues of B_0 do not differ by nonzero integers.

Proof of Thm 2.

• Arrange eigenvalues of A_0 in disjoint sets so that the elements in each set differ only by integers.

• Let $\mu_1, \ldots, \mu_s$ the elements of such a set:

$$\Re\mu_1 > \Re\mu_2 > \ldots > \Re\mu_s, \quad \mu_i - \mu_{i+1} = \ell_i \in \mathbb{N}^*, \quad i = 1, \ldots, s-1.$$

• Let $\mu_{s+1}, \ldots, \mu_r$ denote the other eigenvalues of A_0.

• For $i = 1, \ldots, r$, denote by m_i the multiplicity of μ_i.

• By applying a constant gauge transformation we can assume that:

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ 0 & A_0^{22} \end{pmatrix},$$

where A_0^{11} is an m_1 by m_1 matrix having one single eigenvalue μ_1 :

$$A_0^{11} = \mu_1 I_{m_1} + N_1$$

N_1 being nilpotent matrix.

• Apply the gauge transformation $U = diag(xI_{m_1}, I_{n-m_1})$ yields the new system:

$$Z' = x^{-1}B(x)Z, \quad B(x) = xU^{-1}A(x)U - xU^{-1}U'$$

with the leading matrix:

$$B(0) = \left(A_0 + xU^{-1}A_1U - xU^{-1}U'\right)_{|x=0}.$$

• Let A_1 be partitioned as A_0 :

$$A_1 = \begin{pmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{pmatrix}, \quad A_1^{11} \in \mathbb{C}^{m_1 \times m_1}$$

Then

$$B(0) = \begin{pmatrix} A_0^{11} - I_{m_1} & A_1^{12} \\ 0 & A_0^{22} \end{pmatrix}.$$

Hence the eigenvalues of $B(0)$ are: $\mu_1 - 1, \mu_2, \ldots, \mu_s, \ldots, \mu_r$, each with the same initial multiplicity m_i.

• By repeating this process ℓ_1 times, the eigenvalues become:

$$\mu_1 - \ell_1 = \mu_2, \mu_2, \ldots, \mu_s, \ldots, \mu_r.$$

• Thus, after $\ell_1 + \ldots + \ell_{s-1}$ steps, the eigenvalues $\mu_1, \ldots, \mu_s$ are reduced to one single eigenvalue μ_s of multiplicity $m_1 + \ldots + m_s$.

• By applying the same process to the other groups of eigenvalues, one obtains a matrix B_0 whose eigenvalues do not differ by positive integer. •

The matrix T in Thm2 is obtained as a product of invertible constant matrices or matrices of type U.
Hence T is a polynomial matrix whose determinant is nonzero.

• The degree of $T(x)$ is bounded by ℓ the largest integer difference between the eigenvalues of the residue matrix A_0.

Example

$$A(x) = \begin{pmatrix} 2x^{-1}-2 & -2+x^{-1} & 0 \\ 2 & 2x^{-1}+2 & x \\ 3 & 3 & x^{-1} \end{pmatrix} \quad A_0 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Put $U = diag(x, x, 1)$ and let $B := xU[A]$

$$B(x) = xU^{-1}AU - xU^{-1}U' = \begin{pmatrix} -2x+1 & -2x+1 & 0 \\ 2x & 1+2x & x \\ 3x^2 & 3x^2 & 1 \end{pmatrix}.$$

$$B_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has 1 as eigenvalue of multiplicity 3.

Part 2 : Systems of Second Kind - Fundamental Algorithms

Outline

Formal Solutions of a Singular Differential System

Formal Solutions

A differential equation

$$Y' = AY, \quad A \in M_n(\mathbb{C}[[x]][x^{-1}])$$

has a formal fundamental matrix solution of the form

$$\Phi(x^{1/s})x^{\Lambda} \exp\left(Q(x^{-1/s})\right)$$

$s \in \mathbb{N}^*$, $\Phi \in GL(n, \mathbb{C}[[x^{1/s}]])$,
$Q(x^{-1/s}) = \mathrm{diag}\left(q_1(x^{-1/s}), \ldots, q_n(x^{-1/s})\right)$

the q_i's are polynomials in $x^{-1/s}$

Λ is a constant matrix commuting with Q.

Exponential Part

- Q is invariant under gauge transformations $T \in GL(n, \overline{\mathbb{C}((x))})$.
- $x = 0$ is regular singular $\iff$ $Q(x^{-1/s}) \equiv 0$. In this case $s = 1$ and the formal series $\Phi(x)$ converges whenever the series for $A(x)$ does.
- When $Q(x^{-1/s}) \not\equiv 0$, the origin is an irregular singular point of the system. In this case $\Phi(x)$ need not be convergent even if the series for $A(x)$ does.
- The elements of $Q(x^{-1/s})$ determine the main asymptotic behavior of actual solutions as $x \to 0$ in sectors of sufficiently small angular opening (Asymptotic existence theorem (cf. Wasow)).

- Existence :
  - Matrix Case: Hukuhara (1930's), Turritin (1950's) , Wasow (1960's), Levelt , Jurkat, Lutz, Balser (1970's) , Babbit& Varadajan (1980's), ...
  - Scalar Case: Fabry (1885), Poincaré (1886), Malgrange (1978), ...
- Algorithms (for related problems):
  - Matrix Case: Moser (1960's), Dietrich (1970's), Levelt, Wagenfuhrer, Hilali&Wazner (1980's), Chen , Barkatou, (1990's), Pflügel (2000), Barkatou (2004), Barkatou-Pflügel (2007)
  - Scalar Case: Frobenius method- Newton Algorithm : Della-Dora et al. (1986), Dietrich (1986), Barkatou (1988), ...
- Efficient computation (Matrix case): [Barkatou 1997], [Barkatou-Pflügel 1999], [Pflügel 2000], [Barkatou-Pflügel 2007]

Examples

- Euler Equation: $x^2y' + y = x$

It has a formal power series solution $\hat{f} = \sum_{n=0}^{\infty}(-1)^n n! x^{n+1}$

Homogeneous Equation: $x^3y'' + (x^2 + x)y' - y = 0$

$$\Rightarrow \frac{dY}{dx} = \begin{pmatrix} 0 & 1 \\ \frac{1}{x^3} & -(\frac{1}{x} + \frac{1}{x^2}) \end{pmatrix} Y, \quad Y = \begin{pmatrix} y \\ y' \end{pmatrix}$$

Formal Solution Matrix: $Y = \Phi(x)e^Q$

$$Q = \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & 0 \end{pmatrix} \quad \Phi(x) = \begin{pmatrix} 1 & \hat{f} \\ -\frac{1}{x^2} & \hat{f}' \end{pmatrix}$$

Systems of Second Kind

A matrix linear differential equation with Poincaré rank $p > 0$:

$$[A] \qquad\qquad Y' = AY,$$

$$A(x) = \frac{1}{x^{p+1}} \sum_{i=0}^{\infty} A_i x^i, \quad A_i \in M_n(\mathbb{C}), \;\; A_0 \neq 0.$$

- System $[A]$ has a regular singularity at $x = 0$ if it is equivalent to a system of the first kind (for which p=0).
- Problem: Give an algorithm to decide for any system of second kind whether it has regular singularity.

Systems of Second Kind with Regular Singularity

How to recognize a regular singular system?

Problem 1: Given a system $[A]$ of second kind, i.e. with Poincaré rank $p(A) > 0$, to decide whether it is regular singular or not.

In other words, to decide if the Poincaré rank of the given system can be reduced to 0 or not?

Problem 2: Given a system $[A]$ with Poincaré rank $p(A) > 0$, to decide whether there exists $T \in GL(n, K)$ such that $p(T[A]) < p(A)$.

There is an algorithm due to Moser (1960) which transforms a given system $[A]$ to an equivalent one with minimal Poincaré rank.

Other methods for reducing Poincaré rank (to its minimal value): Levelt (1992), Wagenfurer (1989), . . . , Barkatou &El Bacha (2012).

Moser Reduced Systems

$$A(x) = \frac{1}{x^{p+1}} \sum_{i=0}^{\infty} A_i x^i, \quad A_i \in M_n(\mathbb{C}), \quad A_0 \neq 0, \; p \in \mathbb{Z}.$$

Moser rank: $m(A) = p + \frac{rank(A_0)}{n}$ if $p > 0$, otherwise $m(A) = 1$.

Moser invariant: $\mu(A) = \min \{ m(T[A]) \mid T \in GL(n, \mathbb{C}((x))) \}$

Definition. $[A]$ is said to be Moser-reducible if $m(A) > \mu(A)$.

- $[A]$ is Moser-reducible $\iff \exists T \in GL(n, \mathbb{C}((x)))$ such that $m(T[A]) < m(A)$.

- $x = 0$ is regular singular for $[A] \iff \mu(A) = 1$.

A Criterion for Moser-reducibility

Theorem. [Moser 1960]

1. If $p > 0$ then A is Moser-reducible iff the polynomial

$$\Theta_A(\lambda) := x^{rank(A_0)} \det\left(\lambda I - A_0/x - A_1\right)_{|x=0} \equiv 0.$$

2. If A is Moser reducible then the reduction can be carried out with a transformation of the form
$T = (P_0 + xP_1) diag(1, \ldots, 1, x, \ldots, x), \quad P_i \in \mathbb{C}^{n \times n}, det P_0 \neq 0.$

- Applying Moser's Theorem several times, if necessary, $\mu(A)$ can be determined.

- Further, a matrix polynomial $T \in GL(n, K)$ such that $m(T[A]) = \mu(A)$ can be computed in this way

Remarks

- Moser's initial intention: *classification of singularity*
- Barkatou (1997): also useful for computing formal solutions in the irregular singular case.
- Moser's Theorem can be applied to a system $[A]$ for diminishing the number $p(A)$, when it is possible.
- A necessary condition that there exist a gauge transformation $T \in GL(n, \mathbb{C}((x)))$ such that $T[A] = \frac{1}{x^{p'+1}}(B_0 + B_1 x + \cdots)$ with $p' < p$ $(B_0 \neq 0)$, is that A_0 is nilpotent.

Review: Moser Reduction Algorithms

- There are various algorithms to compute T such that $T[A]$ is Moser-reduced.
- Moser's paper: no constructive algorithm given
- Dietrich (1978), Hilali/Wazner (1987): first efficient algorithms,
- Barkatou (1995): version for rational function coefficients, implemented in ISOLDE
- Barkatou-Pflügel (2007): New reduction algorithm + complexity analysis.

Description of Moser Algorithm

- By a constant gauge transformation we can put A_0 in the form:

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ A_0^{21} & 0 \end{pmatrix}, \quad A_0^{11} \in \mathbb{C}^{r \times r} \quad r = rank(A_0).$$

- Let A_1 be partitioned so that A_1^{11} is a square matrix of order r:

$$A_1 = \begin{pmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{pmatrix},$$

- Consider

$$G_\lambda(A) = \begin{pmatrix} A_0^{11} & A_1^{12} \\ A_0^{21} & A_1^{22} + \lambda I_{n-r} \end{pmatrix}.$$

- Then $\det G_\lambda(A)) = \Theta_A(\lambda)$.
- A is Moser-reducible $\iff$ $\det G_\lambda(A) \equiv 0$.

Case 1: $rank(A_0^{11} \quad A_1^{12}) < r$

A is Moser-reducible $\iff \begin{vmatrix} A_0^{11} & A_1^{12} \\ A_0^{21} & A_1^{22} + \lambda I_{n-r} \end{vmatrix} = 0.$

Proposition 1 If $m(A) > 1$ and $rank(A_0^{11} \quad A_1^{12}) < r$, then A is reducible and the reduction can be carried out with the gauge transformation

$$T = diag(xI_r, I_{n-r}).$$

Proof: Let $B = T[A] = T^{-1}AT - T^{-1}\frac{dT}{dx}$.

$$B = x^{-p-1}[B_0 + xB_1 + \cdots] + x^{-1}diag(I_r, 0)$$

where

$$B_0 = \begin{pmatrix} A_0^{11} & A_1^{12} \\ 0 & 0 \end{pmatrix},$$

Since $p > 0$, then $m(B) = p + rank(B_0)/n < m(A) = p + r/n$.

Example

$$A = x^{-2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} + x^{-1}\begin{pmatrix} 4 & 0 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}.$$

Here $p = 1,\ r = 1 \Rightarrow m(A) = 1 + 1/2 = 3/2 > 1.$

$\det G_\lambda(A) = \begin{vmatrix} 0 & 0 \\ 2 & -3+\lambda \end{vmatrix} = 0 \Rightarrow A$ is Moser-reducible.

Let

$$T = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

$$B := T[A] = T^{-1}AT - T^{-1}T' = \frac{1}{x}\begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}.$$

The system $Z' = BZ$ has a singularity of first kind at $x = 0$.

Hence $Y' = AY$ has a regular singularity at $x = 0$.

To solve $Y' = AY$, it suffices to solve $Z' = BZ$ whose solution can be obtained immediately since $B = x^{-1}B_0$ where B_0 is the constant matrix:

$$B_0 = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}.$$

The matrix B_0 is diagonalizable:

$$B_0 = P^{-1}JP, \quad \text{où} \quad P = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \quad \text{et} \quad J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\Rightarrow Px^J$ is a fundamental matrix solution for $Z' = BZ$.

It follows that

$$W = TPx^J = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} -1 & 2\,x^2 \\ -x^{-1} & x \end{pmatrix}$$

is a fundamental matrix solution of $Y' = AY$.

Case 2: $rank(A_0^{11} \;\; A_1^{12}) = r$

Proposition 2 If A is reducible and $rank(A_0^{11} \;\; A_1^{12}) = r$, then there exists a constant matrix Q such that the matrix $G_\lambda(Q[A])$ has the form has the following particular form:

$$G_\lambda(A) = \begin{pmatrix} A_0^{11} & U_1 & U_2 \\ V_1 & W_1 + \lambda I_{n-r-h} & W2 \\ 0 & 0 & W_3 + \lambda I_h \end{pmatrix}, \tag{7}$$

where $1 \leq h \leq n - r$, $W1, \; W3$ are square matrices of order $(n - r - h)$ and h respectively, W_3 is upper triangular with zero diagonal with the condition

$$rank(A_0^{11} \;\; U_1) < r \tag{8}$$

Proposition 3 If $m(A) > 1$ and $G_\lambda(A)$ has the form (7) with the condition (8), then A is reducible and the reduction can be carried out with the transformation

$$T = diag(xI_r, I_{n-r-h}, xI_h)$$

Proof: Put $B = T[A] = T^{-1}AT - T^{-1}\frac{dT}{dx}$. One has

$$B = x^{-p-1}[B_0 + xB_1 + \cdots] + x^{-1}\mathrm{diag}(I_r, 0, I_h)$$

where

$$B_0 = \left(\begin{array}{ccc} A_0^{11} & U_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right),$$

and then $rank(B_0) = rank(A_0^{11} \ \ U_1) < r = rank(A_0)$. On the other hand since $p > 0$, then $m(B) = p + rank(B_0)/n$. Hence $m(B) < m(A)$.

Summary

If A is Moser-reducible and $m(A) > 1$ then one can construct a matrix polynomial S of the form :

$$S = U\mathrm{diag}(x, x, \cdots, x, 1, 1, \cdots, 1)$$

where $U \in GL(n, \mathbb{C})$, such that $m(S[A]) < m(A)$.

- Moser's Theorem allows us to check whether A is Moser-reducible.
- If A is Moser-reducible then by the above theorem we can find a matrix S such that $m(S[A]) < m(A)$.
- After this reduction has been carried out we can apply Moser's Theorem to check whether further reduction is possible and so on.
- After a finite number of steps we obtain at most *np* an equivalent matrix B such that $m(B) = \mu(A)$.
- The nature of the singularity depends on the first *np* coefficients in the series expansion of A

Example

Consider the system $[A]$ $\frac{dY}{dx} = A(x)Y$

$$A(x) = \begin{bmatrix} -2\,x^{-1} & 0 & x^{-2} & 0 \\ x^2 & -\frac{-1+x^2}{x} & x^2 & -x^3 \\ 0 & x^{-2} & x & 0 \\ x^2 & x^{-1} & 0 & -\frac{x^2+1}{x} \end{bmatrix}$$

Here

$$p = 1, \quad r = rank(A_0) = 2.$$

Hence

$$m(A) = 1 + 2/4 = 3/2 > 1.$$

One can check that

$$\Theta_A(\lambda) := x^{rank(A_0)} \det\left(\lambda I - A_0/x - A_1\right)_{|x=0} \equiv 0$$

Hence A is Moser reducible.

The equivalent matrix B computed by our implementation is

$$B(x) = \begin{bmatrix} -\frac{x^2+1}{x} & x & 1 & -x \\ x^{-1} & \frac{-1+x^2}{x} & 0 & 0 \\ 0 & x^{-1} & -2\,x^{-1} & 0 \\ x & 0 & x^2 & -\frac{x^2+1}{x} \end{bmatrix}$$

The transformation T is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ x^2 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence $[A]$ is singular regular.

Consider the system $\frac{dY}{dx} = x^{-1}A(x)Y$ where

$$A(x) = \begin{pmatrix} 4 & x^3 & -2\,x^6 & -x^6 \\ 0 & -1 - x^{-1} & x^{-1} & 0 \\ x^{-7} & 0 & x^{-1} - 2 & x^{-1} \\ x^{-5} + x^{-6} & -x^{-2} & x^2 + x + x^{-2} & -3 \end{pmatrix}$$

Here $m(A) = 7 + 1/4 = 29/4$.

$$x^{-1}B(x) = \begin{pmatrix} -2 - x^{-1} & 0 & x^{-1} & 0 \\ x^{-2} - x^{-1} & x - 1 & x^3 + x^2 - 2\,x + x^{-1} & -3 - x \\ 0 & x^{-2} & x^{-1} - 3 & 0 \\ -x^{-1} & x + 1 & x^3 + x^2 + x^{-1} & -x - 4 \end{pmatrix}$$

The transformation T is

$$\begin{pmatrix} 0 & x^6 & 0 & -x^6 \\ x & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

One has $\mu(A) = m(B) = 2 + 2/4 = 5/2$.

Systems of Second Kind with Irregular Singularity

Formal Solutions

Consider a system $[A]$ $Y' = AY$ with minimal Poincaré rank $p > 0$:

$$A(x) = \frac{1}{x^{p+1}} \sum_{i=0}^{\infty} A_i x^i, \quad A_i \in M_n(\mathbb{C}), \ \ A_0 \neq 0.$$

System $[A]$ has a formal fundamental matrix solution of the form

$$\Phi(x^{1/s}) x^{\Lambda} \exp\left(Q(x^{-1/s})\right)$$

$s \in \mathbb{N}^*$, $\Phi \in GL(n, \mathbb{C}((x^{1/s})))$,

$Q(x^{-1/s}) = \mathrm{diag}\left(q_1(x^{-1/s}), \ldots, q_n(x^{-1/s})\right)$

the q_i's are polynomials in $x^{-1/s}$ over $\mathbb{C}$ without constant term

Λ is a constant matrix commuting with Q.

How to compute the formal solutions of $[A]$?

Formal Reduction : an algorithmic procedure that allows construction of formal solutions.

Main idea: Transformation of system into new system with smaller p or n

Important tools: Moser Algorithm, Splitting Lemma, Katz Invariant computation.

- Discussion depending on structure of A_0. We distinguish two cases:
  1. Case 1: A_0 has at least two eigenvalues.
  2. Case 2: A_0 has only one eigenvalue.

Case 1- The Splitting Lemma

Splitting Lemma

Theorem: Consider a system $[A] : Y' = A(x)Y$

$A(x) = x^{-p-1} \sum_{i=0}^{\infty} A_i x^i, \quad A_0 \neq 0, \;\; p > 0$ and assume that A_0 is block-diagonal

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ 0 & A_0^{22} \end{pmatrix} \text{ with } \operatorname{spec}(A_0^{11}) \cap \operatorname{spec}(A_0^{22}) = \emptyset.$$

Then there exists a gauge transformation of the form

$$T(x) = \sum_{j=0}^{\infty} T_j x^j \quad (T_0 = \mathrm{I}_n)$$

such that the matrix $B := T[A]$ is block-diagonal matrix with the same block partition as in A_0

$$B = x^{-p-1} \begin{pmatrix} B^{11}(x) & 0 \\ 0 & B^{22}(x) \end{pmatrix}.$$

Sketch of Proof

- Put $T_0 = \mathrm{I}_n$ and $B_0 = A_0$

- Look for matrices T_i of the special form

$$T_i = \begin{pmatrix} 0 & T_i^{12} \\ T_i^{21} & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} B_i^{11} & 0 \\ 0 & B_i^{22} \end{pmatrix}.$$

- Then for $i \geq 1$ the coefficients T_i and B_i can be obtained by successively solving Sylvester linear equations of the form

$$A_0^{11} X - X A_0^{22} = U_i \;\text{ or }\; A_0^{22} Y - Y A_0^{11} = V_i$$

where U_i and V_i depend only on A_j, B_j, T_j for $j = 0, \ldots, i-1$.

A very simple situation

$$A(x) = x^{p-1}\sum_{i=0}^{\infty} A_i x^i, \qquad A_0 \neq 0, \quad p > 0.$$

Corollary. If A_0 has all distinct eigenvalues, then there exists $T \in GL(n, \mathbb{C}[[x]])$ such that $T[A]$ is a diagonal matrix.

If $\quad B_0 := P^{-1}A_0P = \mathrm{diag}(\beta_1, \ldots, \beta_n) \quad$ with $\quad \beta_i \neq \beta_j \;$ for $\; i \neq j$

for some $P \in GL(n, \mathbb{C})$, then there exists a formal transformation

$$T(x) = \sum_{j \geq 0} T_j x^j \quad (T_0 = P)$$

such that

$$T[A] = \begin{pmatrix} \frac{\beta_1}{x^{p+1}} + O(\frac{1}{x^p}) & & 0 \\ & \ddots & \\ 0 & & \frac{\beta_n}{x^{p+1}} + O(\frac{1}{x^p}) \end{pmatrix}$$

Case 2- The Nilpotent Case

Reduction to the Case where A_0 is Nilpotent

Let

$$A(x) = \frac{1}{x^{p+1}} \sum_{i=0}^{\infty} A_i x^i, \quad A_0 \neq 0,\ p > 0.$$

- Apply the Splitting Lemma to decouple $[A]$ along the spectral subspaces of A_0:

  $$A = A^{(1)} \oplus \cdots \oplus A^{(k)}$$

  The leading matrix of each subsystem has only one eigenvalue.

- If $A_0 = \alpha \mathrm{I} \oplus N$, with N nilpotent then apply the substitution $Y = \exp\left(\frac{-\alpha}{px^p}\right) Z$ which replace A by $A - \dfrac{\alpha}{x^{p+1}} I$.
  This makes A_0 nilpotent.

- If necessary, apply the Moser algorithm to replace the system by an equivalent one with minimal Poincaré rank p.

The case A_0 nilpotent and $p > 0$ minimal

- In this case we need algebraic extension of K:

  Gauge transformations in $\mathbb{C}((x^{1/m}))$, for suitable integer $m \geq 2$, are applied to get an equivalent system $[\widetilde{A}]$ with leading coefficient $\widetilde{A}_0$ having distinct eigenvalues.

- How to choose m?

  Compute κ, the *Katz invariant* of $[A]$ (see below) and let m be the smallest positive integer such that $m\kappa$ is an integer.

- Using Moser Algorithm yields a system with Poincaré rank equal to $m\kappa$ and leading matrix A_0 with at least two eigenvalues.

- So we can again split the problem into problems of lower size, and so on.

Katz Invariant

Katz Invariant

Definition: The Katz Invariant of $[A]$ is the rational number

$$\kappa(A) = \max_{1\leq j\leq n} \deg_{1/x}(q_j)$$

where the q_j are the entries of the exponential part Q of $[A]$.

Fact: $\kappa(A) \leq p(A)$ with equality iff A_0 is non-nilpotent.

If $[A]$ is Moser-reduced and its leading coefficient A_0 is nilpotent then $\kappa(A)$ is not an integer.

Theorem[Bark05] Suppose A Moser reduced and A_0 nilpotent. Then

$$p(A) - 1 + \frac{r}{n-d} \leq \kappa(A) \leq p(A) - \frac{1}{r+1}$$

where $r = rank(A_0)$ and $d = deg\Theta_A(\lambda)$.

Example 1

$$A(x) = \frac{1}{x^4} \begin{bmatrix} 0 & 0 & x & 0 \\ 1 & -x^2 & x^2 & -x^2 \\ 0 & 1 & x^2 & 0 \\ x^2 & x^2 & 0 & -x^2 \end{bmatrix}$$ has Poincaré rank $p(A) = 3$.

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is nilpotent and has rank $r = 2$.

$\Theta_A(\lambda) = \lambda$ is not zero and has degree $d = 1$.

The above theorem tells us:

$$2 + 2/3 = p(A) - 1 + \frac{r}{n-d} \leq \kappa(A) \leq p(A) - \frac{1}{r+1} = 3 - 1/3.$$

Hence $\kappa(A) = 8/3$.

How to obtain Katz Invariant?

• If $D(y) = \partial^n y + c_{n-1}(x)\partial^{n-1}y + \cdots + c_1(x)\partial y + c_0(x)y = 0$ is an equation obtained from the system $[A]$ via a cyclic vector, then

$$\kappa(A) = \max\left(0, \max_{0 \le j < n}\left(\frac{-n + j - \mathrm{val}(c_j)}{n - j}\right)\right)$$

Theorem[Bark05] Let A be Moser-reduced, put $r = rank(A_0)$, $d = deg(\Theta_A(\lambda))$ and write

$$\det(\lambda I - A(x)) = \lambda^n + a_{n-1}(x)\lambda^{n-1} + \cdots + a_0(x).$$

Suppose

$$\text{(C)} \qquad p(A) \ge \left(1 - \frac{r}{n-d}\right)(r+1).$$

Then

$$\kappa(A) = \max\left(0, \max_{0 \le j < n}\left(\frac{-n + j - \mathrm{val}(a_j)}{n - j}\right)\right)$$

Back to Example 1

$[A]$ is Moser-reduced, $p = 3,\ r = 2,\ d = 1.$

Condition (C) in the above theorem is satisfied.

One can compute $\kappa(A)$ using the above formula:

$$\det\left(\lambda I - A(x)\right) = \lambda^4 + \frac{\lambda^3}{x^2} - \frac{\lambda^2}{x^6} + \frac{\left(-2\,x^7 - x^2 - x^5\right)\lambda}{x^{13}} + \frac{x^2 - 1}{x^{13}}.$$

One has

$$\mathrm{val}(a_3) = -2,\ \mathrm{val}(a_2) = -6,\ \mathrm{val}(a_1) = -11,\ \mathrm{val}(a_0) = -13.$$

Hence

$$\kappa(A) = \max\left(0, \max_{0 \le j < n}\left(\frac{-n + j - \mathrm{val}(a_j)}{n - j}\right)\right) = \max\left\{0, 1, 2, \frac{8}{3}, \frac{9}{4}\right\} = \frac{8}{3}.$$

Remarks

• It is always possible to come down to the case where Condition (C) is fulfilled.

Idea: If $p(A) < (1 - \frac{r}{n-d})(r+1)$, use a ramification $x = t^s$ where

$$s \geq \frac{n-r-d}{p-2+r/(n-d)}$$

What do we gain by computing Katz invariant?

Suppose $[A]$ be Moser-reduced and A_0 nilpotent and let $\kappa(A) = \frac{\ell}{m}$ with $(\ell, m) \in \mathbb{N} \times \mathbb{N}$ with $gcd(\ell, m) = 1$.

Put $t = x^{1/m}$ and let $[\widetilde{A}]$ denote the resulting system:

$$\frac{dY}{dt} = \widetilde{A}Y, \quad \widetilde{A}(t) = mt^{m-1}A(t^m).$$

Then there is a $T \in GL(n, \mathbb{C}((t)))$ such that

- $\widetilde{B} := T[\widetilde{A}]$ has Poincaré rank equal to ℓ
- its leading matrix $\widetilde{B}_0$ has at least m distinct eigenvalues.

Remark The transformation T is in fact polynomial in t and can be computed using Moser Algorithm.

Back to our example

We have

$$\kappa(A) = \frac{8}{3}.$$

The change of variable

$$x = t^3$$

yields

$$\frac{dY}{dt} = \widetilde{A}(t)Y$$

where

$$\widetilde{A}(t) = \frac{3}{t^{10}} \begin{bmatrix} 0 & 0 & t^3 & 0 \\ 1 & -t^6 & t^6 & -t^6 \\ 0 & 1 & t^6 & 0 \\ t^6 & t^6 & 0 & -t^6 \end{bmatrix}.$$

One can check that this system is not Moser-reduced.

Moser Algorithm produces the gauge transformation

$$Y = SZ$$

where

$$S = \mathrm{diag}(t^2, t, 1, 1),$$

and the equivalent system

$$\frac{dZ}{dt} = \widetilde{B}(t)Z, \quad \widetilde{B}(t) = \frac{1}{t^9}\begin{bmatrix} -2\,t^8 & 0 & 3 & 0 \\ 3 & -3\,t^5 - t^8 & 3\,t^4 & -3\,t^4 \\ 0 & 3 & 3\,t^5 & 0 \\ 3\,t^7 & 3\,t^6 & 0 & -3\,t^5 \end{bmatrix}$$

Its Poincaré rank is equal to 8 as expected.

The leading matrix is

$$\widetilde{B}_0 = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is non nilpotent and has 4 distinct eigenvalues

$$0, 3, 3j, 3j^2$$

with $j^3 = 1$.

The system can be then decoupled into 4 scalar equations.

One fundamental formal solution can be written as

$$\widehat{Y}(x) = \widehat{F}(x) \begin{bmatrix} e^{1/x} & 0 \\ 0 & x^J U e^{Q(1/x)} \end{bmatrix}$$

where $\widehat{F}(x)$is a meromorphic formal series in x,

$$J = -\frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j^4 \end{bmatrix}$$

and

$$Q(1/x) = \begin{bmatrix} q(1/t) & 0 & \\ 0 & q(1/(jt)) & 0 \\ 0 & 0 & q(1/(j^2 t)) \end{bmatrix}$$

with

$$t = x^{1/3} \quad q\left(\frac{1}{t}\right) = \frac{-3}{8t^8} - \frac{1}{4t^4}.$$

An Important Question

Given a matrix

$$A(x) = x^{-p-1}(A_0 + A_1 x + \cdots), \qquad p > 0$$

• **Question:** How many terms in $\sum_{i=0}^{\infty} A_i x^{i-p-1}$ are necessary for computing the exponential part $Q(x^{-1/s})$ of the system$[A]$?

• The answer can be found in Lutz-Schäfke (1986) or Babbit-Varadarajan (1983):
The exponential part $Q(x^{-1/s})$ is determined by the coefficients

$$A_0, A_1, \cdots, A_{np-1}$$

Example

$$A = \begin{bmatrix} -\frac{5}{x^2} & \frac{5}{x^2} & -x^{-3} & \frac{4}{x^2} \\ 0 & \frac{1-4\,x}{x^3} & -x^{-2} & -\frac{2}{x^2} \\ \frac{2\,x+1}{x^3} & \frac{1-5\,x}{x^3} & \frac{2-3\,x}{x^3} & \frac{1-4\,x}{x^3} \\ 0 & \frac{4}{x^2} & x^{-2} & \frac{1+x}{x^3} \end{bmatrix}$$

$> Rational\_Exponential\_Part(A, x);$

$$[x = \alpha t^2, \frac{-\frac{9}{8} + \frac{7\,\alpha}{4}}{t} + \frac{-1/4 - \frac{11\,\alpha}{4}}{t^2} + \frac{2\,\alpha}{3\,t^3} + \frac{1}{2\,t^4}]$$

where

$$\alpha = RootOf(\_Z^2 + 1)$$

Part 3: Applications to Solving Systems with Rational Function Coefficients

1. Polynomial Solutions

2. Rational Solutions

3. Exponential Solutions

4. Factorization Using Eigenrings

5. Implementations in Maple

Example (see Phd Thesis of C. Raab (2012))

$$\int Ai'(x)^2\,dx = y_0(x)Ai(x)^2 + y_1(x)Ai(x)Ai'(x) + y_2(x)Ai'(x)^2$$

with $y_0, y_1, y_2 \in \mathbb{Q}(x)$
Differentiate

$$\begin{aligned} Ai'(x)^2 &= (y_0'(x) + xy_1(x))Ai(x)^2 + \\ &\quad (y_1'(x) + 2y_0(x) + 2xy_2(x))Ai(x)Ai'(x) + \\ &\quad (y_2'(x) + y_1(x))Ai'(x)^2 \end{aligned}$$

Coefficient comparison

$$\begin{pmatrix} y_0(x) \\ y_1(x) \\ y_2(x) \end{pmatrix}' + \begin{pmatrix} 0 & x & 0 \\ 2 & 0 & 2x \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_0(x) \\ y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\int Ai'(x)^2\,dx = \frac{1}{3}\left(xAi'(x)^2 + 2Ai(x)Ai'(x) - x^2Ai(x)^2\right)$$

Example

$$\int x(e^{2x} - n^2) J_n(e^x)\, dx = y_0(x) J_n(e^x) + y_1(x) J_{n+1}(e^x)$$

with $\mathbf{y} = (y_0, y_1) \in C(x, e^x)^2$ s.t.

$$\begin{pmatrix} y_0(x) \\ y_1(x) \end{pmatrix}' + \begin{pmatrix} n & e^x \\ -e^x & -(n+1) \end{pmatrix} \begin{pmatrix} y_0(x) \\ y_1(x) \end{pmatrix} = \begin{pmatrix} x(e^{2x} - n^2) \\ 0 \end{pmatrix}$$

Translate to $C(x, t)$ with $C = \mathbb{Q}(n)$ and $t' = t$:

$$\mathbf{y}' + \begin{pmatrix} n & t \\ -t & -(n+1) \end{pmatrix} \mathbf{y} = \begin{pmatrix} x(t^2 - n^2) \\ 0 \end{pmatrix}$$

$$\mathbf{y} = \begin{pmatrix} -nx + 1 \\ xt \end{pmatrix}$$

$$\int x(e^{2x} - n^2) J_n(e^x)\, dx = (-nx + 1) J_n(e^x) + xe^x J_{n+1}(e^x)$$

Integration

- $$\int Ai'(x)^2\,dx = \tfrac{1}{3}\left(xAi'(x)^2 + 2Ai(x)Ai'(x) - x^2Ai(x)^2\right)$$
- $$\int x(e^{2x} - n^2)J_n(e^x)\,dx = (-nx+1)J_n(e^x) + xe^xJ_{n+1}(e^x)$$
- $$\int P_n(x) - x^{n-1}P_{n+1}(x)\,dx = \tfrac{x^n-1}{n}(xP_n(x) - P_{n+1}(x))$$

Polynomial Solutions

Polynomial Solutions [B99]

Let $K = \mathbb{C}(x)$ and $\vartheta = x\frac{d}{dx}$.

• Linear Differential System :

$$[A] \qquad\qquad \vartheta y = A(x)y,$$

$A(x)$ is an $n \times n$ matrix with entries in K.

• Polynomial Solutions : functions $y \in \mathbb{C}[x]^n$ such that $\vartheta y = Ay$.

• Problem : Given a system $[A]$ to construct the space of polynomial solutions of $[A]$.

- A first important step consists in computing a bound N on the degree of polynomial solutions.

Bound of The Degree of Polynomial Solutions

- A first important step consists in computing a bound N of the degree of polynomial solutions.
- Such a bound can be obtained from the so-called indicial equation (at $x = \infty$) of the system [A].
- But the indicial equation is not immediately apparent for a given system.
- Need to transform the given system to a suitable form called 'simple form' from which the indicial equation can be immediately obtained.
- Every system can be reduced to an equivalent simple one

Simple Systems

Simple Systems [B99, BP99]

Consider the system

$$[A] \quad \vartheta y = Ay, \;\; A = (a_{i,j}) \in M_n(\mathbb{C}(x)).$$

We are interested in *Frobenius series solutions* in x^{-1} of the form:

$$\hat{y} = \sum_{i=0}^{+\infty} x^{-i-\lambda_0} \hat{y}_i \;\; \lambda_0 \in \mathbb{C}, \;\; \hat{y}_i \in \mathbb{C}^n, \hat{y}_0 \neq 0.$$

• A polynomial solution of degree N can be viewed as a Frobenius series solution (at $x = \infty$) with exponent $\lambda_0 = -N$.

• Look for a condition on λ_0 in order that $\hat{y}$ be a solution of system $[A]$.

How to do this?

Consider the system

$$[A] \quad \vartheta y = Ay, \quad A = (a_{i,j}) \in M_n(\mathbb{C}(x)).$$

Let

$$D = diag(x^{-\alpha_1}, \ldots, x^{-\alpha_n}),$$

where

$$\alpha_i = \max_{1 \le j \le n} (\deg(a_{i,j}), 0), \quad \text{for} \;\; 1 \le i \le n$$

For $0 \neq f \in \mathbb{C}(x)$, $\deg(f) = \deg(num(f)) - \deg(denom(f))$.

Multiplying on the left by D, both sides of Equation $[A]$, we get

$$D(x)\vartheta y - C(x)y = 0, \quad C = DA.$$

where $D(x), C(x) \in \mathbb{C}[[x^{-1}]]$.

Let

$$\mathcal{L}(y) := D(x)\vartheta y - C(x)y,$$

where $D(x), C(x) \in \mathbb{C}[[x^{-1}]]$.
We are interested in formal solutions of $\mathcal{L}(y) = 0$ of the form:

$$\hat{y} = \sum_{i=0}^{+\infty} x^{-i-\lambda_0}\hat{y}_i \;\; \lambda_0 \in \mathbb{C}, \;\hat{y}_i \in \mathbb{C}^n, \hat{y}_0 \neq 0.$$

Put

$$C = C_0 + \mathrm{O}(x^{-1}), \;\; D = D_0 + \mathrm{O}(x^{-1}).$$

Then

$$\mathcal{L}(\hat{y}) = -x^{-\lambda_0}\left((\lambda_0 D_0 + C_0)\hat{y}_0 + \mathrm{O}(x^{-1})\right).$$

If $\mathcal{L}(\hat{y}) = 0$ then $(\lambda_0 D_0 + C_0)\hat{y}_0 = 0$
wich implies

$$\det\left(C_0 + \lambda D_0\right) = 0.$$

Indicial Equation of a Simple System

$\Diamond$ To system $[A]$ we associate the polynomial

$$E_\infty(\lambda) := \det\left(C_0 + \lambda D_0\right).$$

- If y is a nonzero polynomial solution of $[A]$ of degree N then $E_\infty(-N) = 0$.
- The degree of polynomial solution can be bounded by the biggest nonnegative integer root of $E_\infty(-\lambda)$.

$\Diamond$ It may happen that $E_\infty(\lambda)$ vanishes identically, in which case it is quite useless for our initial purpose. This motivates the following definition

Definition

The system $[A]$ is called simple at $x = \infty$ if $\det\left(C_0 + \lambda D_0\right) \neq 0$ (as a polynomial in λ).
In this case $E_\infty(\lambda)$ is called the indicial polynomial of $[A]$ at $x = \infty$.

Example: Systems of First Kind

$$[A] \qquad x\frac{d}{dx}y = A(x)y,$$

$$A(x) = \sum_{i=0}^{\infty} A_i x^{-i}.$$

In this case $x = \infty$ is at worst a singularity of the first kind.

- In this case $D_0 = I_n$ and $C_0 = A_0$. Hence

$$E_\infty(\lambda) = \det\left(A_0 + \lambda I_n\right) \neq 0.$$

- The system $[A]$ is simple and its indicial polynomial is the characteristic polynomial of the matrix $-A_0$.

Systems Associated with a Scalar Differential Equation

$$A = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{pmatrix}, \quad a_i \in \mathbb{C}(x)$$

Here $D_0 = diag(1, \dots, 1, \epsilon)$ where $\epsilon = 1$ or 0.
The matrix C_0 is given by

$$C_0 = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & & \\ & & & 1 \\ \bar{a}_0 & \bar{a}_1 & & \bar{a}_{n-1} \end{pmatrix}$$

with $\bar{a}_j = \ell c_\infty(a_j)$ if $ord_\infty a_j = \min_{0 \leq i \leq n} ord_\infty a_i$, and 0 otherwise.

• It then follows that

$$\det\left(C_0 + \lambda D_0\right) = \sum_{ord_\infty a_j = \min_{0 \le i \le n} ord_\infty a_i} \bar{a}_j (-\lambda)^j,$$

which is a nonzero polynomial in λ.

• Any companion differential system is simple.

• Consequence: Any differential system can be reduced to an equivalent simple one (Use cyclic vector lemma)

An Example of a Non Simple System

$$x\frac{dy}{dx} = Ay, \quad A = \begin{bmatrix} 1 & x^3 \\ 2x^{-1} & 1 \end{bmatrix}.$$

One has

$$D = \begin{bmatrix} x^{-3} & 0 \\ 0 & 1 \end{bmatrix} \text{ and } C = DA = \begin{bmatrix} x^{-3} & 1 \\ 2x^{-1} & 1 \end{bmatrix}.$$

Thus

$$D_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C_0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

One has

$$\det\left(C_0 + \lambda D_0\right) = 0.$$

Hence $[A]$ is not simple at ∞.

Is it possible to make it simple, and if yes how?

◊ The answer is 'YES': any differential system can be reduced to an equivalent simple one.
◊ It can be done by using polynomial gauge transformations
◊ In this example, we put

$$w = Ty$$

where

$$T = \begin{bmatrix} 0 & x^2 \\ 1 & 0 \end{bmatrix}$$

Then w satisfies the equivalent differential system

$$x\frac{dw}{dx} = \widetilde{A}w$$

where

$$\widetilde{A} = \left(TA + x\frac{dT}{dx}\right) T^{-1} = \begin{bmatrix} 3 & 2\,x \\ x & 1 \end{bmatrix}$$

$$x\frac{dy}{dx} = \widetilde{A}y, \quad \widetilde{A} = \begin{bmatrix} 3 & 2\,x \\ x & 1 \end{bmatrix}.$$

One has

$$\widetilde{D} = \begin{bmatrix} x^{-1} & 0 \\ 0 & x^{-1} \end{bmatrix} \text{ and } \widetilde{C} = \widetilde{D}\widetilde{A} = \begin{bmatrix} 3x^{-1} & 2 \\ 1 & x^{-1} \end{bmatrix}.$$

Thus

$$\widetilde{D}_0 = 0 \quad \text{and} \quad \widetilde{C}_0 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

One has

$$\det(\widetilde{C}_0 + \lambda\widetilde{D}_0) = \det(\widetilde{C}_0) = -2.$$

Hence $[\widetilde{A}]$ is simple at ∞.

Remark: The indicial polynomial at ∞ is constant, hence system $[A]$ has no nonzero polynomial solution.

Simple forms can be computed effeciently

Theorem (Bark1997)

Given a differential system $[A]\ \vartheta y = Ay$*, one can construct a nonsingular matrix* T *polynomial in* x *such that the gauge transformation* $w = Ty$ *takes* $[A]$ *into an equivalent system* $[\widetilde{A}]\ \vartheta w = \widetilde{A} w$ *which is simple at* ∞*.*

• Such a transformation T can be constructed using the *super-reduction* algorithm (Hilali and Wazner (1987), Barkatou-Pflügel 2007) or a more recent algorithm by Barkatou and C. El Bacha (2012).

Remark. The fact that the transformation T can be chosen polynomial is important: if y is a polynomial solution of $[A]$ $w = Ty$ is a polynomial solution of the equivalent system $[\widetilde{A}]$.

Remarks

- This notion of simple systems extends to the case of finite singularities $\rightarrow$ useful for computing denominators of rational solutions.
- Another application: computation of regular formal solution (Barkatou-Pflügel 1997), (El Bacha 2011).

Rational Solutions

Rational Solutions [B99]

Let $K = \mathbb{C}(x)$ and $\vartheta = x\frac{d}{dx}$.

• Linear Differential System :

$$[A] \qquad\qquad \vartheta y = A(x)y,$$

$A(x)$ is an $n \times n$ matrix with entries in K.

• Rational Solutions : functions $y \in K^n$ such that $\vartheta y = Ay$.

• Problem : Given a system $[A]$, to construct the space $\mathcal{S}_A$ of rational solutions of $[A]$.

Let $\mathcal{S}_A$ be the space of rational solutions of $[A]$.

We proceed in two steps :

STEP 1. Construct a universal denominator for $[A]$, i.e. a polynomial (or rational function) $u(x)$ such that

for all $y \in \mathbb{C}(x)$, if $y \in \mathcal{S}_A$ then uy is a polynomial.

STEP 2. If u is a universal denominator for $[A]$ then set

$$w = uy$$

and search for polynomial solutions of the resulting system in w :

$$\vartheta w = (A(x) + u^{-1}\vartheta u \; I_n)w.$$

Computing Denominators of Rational Solutions

Universal Denominator

The problem: Given a differential system

$$[A] \qquad\qquad \vartheta y = A(x)y,$$

to find a rational function u such that
for all $y \in \mathbb{C}(x)$, if $y \in \mathcal{S}_A$ then uy is a polynomial.

Some Facts:

- If $y \in \mathcal{S}_A$ then the finite poles of y are poles of A.
- Given a pole x_0 of A, one can reduce the system $[A]$ to an equivalent system which is simple at $x = x_0$.
- The reduction can be achieved by a polynomial gauge transformation.

- To each point x_0 corresponds an indicial polynomial $E_{x_0}(\lambda) \in \mathbb{C}[\lambda]$
- If y is a nonzero rational solution with a pole of order m at x_0 then $E_{x_0}(-m) = 0$.
- If for some pole x_0 of A the corresponding indicial polynomial has no integer root then $\mathcal{S}_A = \{0\}$.
- For each pole x_0 of A put:

$$m_{x_0} = \min\{\mu \in \mathbb{Z} : \; E_{x_0}(\mu) = 0\}$$

Then

$$u(x) = \prod (x - x_0)^{-m_{x_0}}$$

is a universal denominator for $[A]$.

Complexity Estimate

Rational solutions of Differential systems

◇ C computable field of char. zero, $K = C(x)$

$$Y' = A\,Y, \quad A \in M_n(K), \quad \mathrm{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1}$$

◇ Algorithm for computing rational solutions (*Barkatou'99*):

- Compute a universal denominator $U = \prod_{i=1}^{s} q_i(x)^{m_i}$:
  if Y is a rational solution then $Z = UY$ is a polynomial vector.
- Compute polynomial solutions of $Z' = (A + (U'/U)\,I_n)\,Z$

Complexity estimate

$$Y' = A\,Y, \quad A = (a_{i,j})_{i,j} \in M_n(K), \quad \mathrm{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1}$$

$$d := \sum_{i=1}^{s} (r_i + 1)\,\deg(q_i)$$

$$r_\infty := \max\left(\max_{i,j}\left(1 + \deg(\mathrm{num}(a_{i,j})) - \deg(\mathrm{den}(a_{i,j}))\right), 0\right)$$

◇ Arithmetic (operations in C) complexity estimate:

- Universal denominator: simple form at q_i, integer roots of the *indicial polynomial*: $\mathcal{O}(n^5 \max_i(r_i)\,d)$
- Polynomial solutions: degree bound (simple form at ∞), coefficients: $\mathcal{O}(n^5\, r_\infty^2 + n^3\, N^2)$

$\rightsquigarrow$ rational solutions of $Y' = A\,Y$: $\mathcal{O}(n^5\,(\max_i(r_i)\,d + r_\infty^2) + n^3\,N^2)$

◇ Main tool: simple forms (*El Bacha's PhD'11*, *Barkatou-El Bacha'12* direct method for computing simple forms, arithmetic complexity.)

Exponential Solutions

Exponential solutions

$$y = \exp\left(\int u\right) z$$

with $u \in \mathbb{C}(x)$, $z \in \mathbb{C}[x]^n$.

For $x_0 \in \mathbb{C} \cup \{\infty\}$ define the *singular part* $S_{x_0}(u)$ of u as the principal part of the Laurent series expansion of u at $x = x_0$.

Idea: there exist local exponential part w such that $w = S_{x_0}(u)$

1. Compute all exponential parts of ramification 1 at all singularities (Use algorithms from Part 2)
2. Reconstruct u from
$$u = \sum_{x_0} S_{x_0}(u).$$
Find candidates $\tilde{u}$, do a change of exponential and search for polynomial solutions.

Drawbacks

- Exponential number of combinations to be checked,
- Large algebraic extensions possible (splitting field).
- Can be improved using the approach of Cluzeau and van Hoeij, 2004: reduce mod p to find the “good" combinations!

Example

$$
\begin{matrix}
\frac{-12+3x+3x^2}{(x-1)x^2} & \frac{12}{(x-1)x^2} & \frac{3+6x}{x(x-1)} & 0 & 0 & 0 & 0 & 0 \\
\frac{2x-4}{(x-1)x^2} & \frac{5x+4}{x^2} & \frac{2x^2+1}{x(x-1)} & \frac{8}{(x-1)x^2} & \frac{2+4x}{x(x-1)} & 0 & 0 & 0 \\
\frac{3-x}{(x-1)x^2} & \frac{-4}{(x-1)x^2} & \frac{-9+x+x^2}{(x-1)x^2} & 0 & \frac{8}{(x-1)x^2} & \frac{2+4x}{x(x-1)} & 0 & 0 \\
0 & \frac{4x-8}{(x-1)x^2} & 0 & \frac{4-5x+7x^2}{(x-1)x^2} & \frac{4x^2+2}{x(x-1)} & 0 & \frac{4}{(x-1)x^2} & \frac{1+2x}{x(x-1)} \\
0 & \frac{3-x}{(x-1)x^2} & \frac{2x-4}{(x-1)x^2} & -\frac{4}{(x-1)x^2} & \frac{-3x-1+3x^2}{(x-1)x^2} & \frac{2x^2+1}{x(x-1)} & 0 & \frac{4}{(x-1)x^2} \\
0 & 0 & \frac{6-2x}{(x-1)x^2} & 0 & \frac{-8}{(x-1)x^2} & \frac{-6-x-x^2}{(x-1)x^2} & 0 & 0 \\
0 & 0 & 0 & \frac{6x-12}{(x-1)x^2} & 0 & 0 & \frac{12-9x+9x^2}{(x-1)x^2} & \frac{6x^2+3}{x(x-1)} \\
0 & 0 & 0 & \frac{3-x}{(x-1)x^2} & \frac{4x-8}{(x-1)x^2} & 0 & -\frac{4}{(x-1)x^2} & \frac{7-7x+5x^2}{(x-1)x^2} \\
0 & 0 & 0 & 0 & \frac{6-2x}{(x-1)x^2} & \frac{2x-4}{(x-1)x^2} & 0 & \frac{-8}{(x-1)x^2} \\
0 & 0 & 0 & 0 & 0 & \frac{9-3x}{(x-1)x^2} & 0 & 0
\end{matrix}
$$

Our program finds the solution

$$e^{\int 3x^{-2}+2x^{-1}-3(x-1)^{-1}} \begin{pmatrix} -x^4 \\ -x^4 \\ x^3 \\ -x^4 \\ x^3 \\ -x^2 \\ -x^4 \\ x^3 \\ -x^2 \\ x \end{pmatrix}$$

Factorization Using Eigenring

Definitions

A system $[A]$ $Y' = AY$, $A \in M_n(\mathbb{C}(x))$ is called:

- reducible, if it is equivalent (over $\mathbb{C}(x)$) to a system of the form

$$Z' = \begin{pmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,2} \end{pmatrix} Z. \tag{9}$$

- decomposable if $[A]$ is equivalent to a system of the form (9) with $A_{2,1} = 0$.

- irreducible (indecomposable) if it is not reducible (decomposable).

- completely reducible, if it is equivalent to a block–diagonal system

$$T[A] = \mathrm{diag}(A_{1,1}, \ldots, A_{s,s})$$

where each system $[A_{i,i}]$, $1 \leq i \leq s$, is irreducible.

The Eigenring Method

This method was introduced by M. Singer (1996) for factoring scalar differential operators over $K = \mathbb{C}(x)$.

Definition: The eigenring $\mathcal{E}(A)$ of a system $[A]$ is the finite dimensional $\mathbb{C}-$ *algebra* of all the matrices $T \in M_n(\mathbb{C}(x))$ satisfying the matrix equation

$$T' = AT - TA.$$

• A simple way to compute $\mathcal{E}(A)$ is to convert the above equation into a n^2–dimensional first order linear differential system and search for rational solutions of this system.

Some Properties

- Elements of $\mathcal{E}(A)$ map a solution of $[A]$ to a solution of $[A]$.
- If $T \in \mathcal{E}(A)$ then all its eigenvalues are constant.
- If two systems $[A]$ and $[B]$ are equivalent, their eigenrings $\mathcal{E}(A)$ and $\mathcal{E}(B)$ are isomorphic as $\mathbb{C}-$algebras. In particular, one has $dim_{\mathbb{C}}\mathcal{E}(A) = dim_{\mathbb{C}}\mathcal{E}(B)$

  More precisely If $B = P^{-1}AP - P^{-1}P'$ with $P \in GL(n, \mathbb{C}(x))$ then

$$\mathcal{E}(A) = P^{-1}\mathcal{E}(B)P := \{P^{-1}TP \mid T \in \mathcal{E}(B)\}.$$

- If $[A]$ is decomposable then $\dim_{\mathbb{C}} \mathcal{E}(A) > 1$.

Factorization of Systems with Nontrivial Eigenring

Theorem 1 If $\dim_{\mathbb{C}} \mathcal{E}(A) > 1$ then $[A]$ is reducible and the reduction can be carried out by a matrix $P \in GL(n, K)$ that can be computed explicitly.

Cor. Given a system $[A]$ one can construct an equivalent matrix equation $[B]$ having a block-triangular form

$$\begin{pmatrix} B_{1,1} & 0 & & 0 \\ B_{2,1} & B_{2,2} & & \\ \vdots & & \ddots & 0 \\ B_{s,1} & \dots & & B_{s,s} \end{pmatrix}$$

where s is the maximal possible, i.e. for each $1 \leq i \leq s$, the eigenring of $[B_{i,i}]$ is trivial (having dimension 1).

Proof of Theorem1

Suppose $\dim \mathcal{E}(A) > 1$. Then there is $T \in \mathcal{E}(A)$ with rank $r < n$. One can compute $P \in GL(n, K)$ such that

$$S := P^{-1}TP = \begin{pmatrix} S_{1,1} & 0 \\ S_{2,1} & 0 \end{pmatrix},$$

where $S_{1,1}$ is an $r \times r$ matrix and $\begin{pmatrix} S_{1,1} \\ S_{2,1} \end{pmatrix}$ has rank r.

Let $B = P^{-1}(AP + P')$ then $S \in \mathcal{E}(B)$, Decompose B in the same form as S

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}.$$

The equation $SB - BS = S'$ implies

$$\begin{pmatrix} S_{1,1} \\ S_{2,1} \end{pmatrix} B_{1,2} = 0.$$

Since $\begin{pmatrix} S_{1,1} \\ S_{2,1} \end{pmatrix}$ is of full rank, then $B_{1,2} = 0$.

Factorization of Decomposable Systems

Proposition Suppose that $\mathcal{E}(A)$ contains an element T which has $s \geq 2$ distinct eigenvalues $\lambda_1, \ldots, \lambda_s \in \mathbb{C}$ then $[A]$ is decomposable.

Moreover, if $P \in GL(n, K)$ is such that

$$J = P^{-1} T P = \bigoplus_{i=1}^{s} J_i \qquad \text{with} \;\; spec(J_i) = \lambda_i$$

Then the matrix $B = P[A] = P^{-1}(AP + P')$ has the form

$$P[A] = \bigoplus_{i=1}^{s} B_i .$$

Example

$$A = \begin{bmatrix} 9 & -6\,x^{-2} & 0 & 6\,x^{-2} & 6\,x^{-2} \\ \frac{1-x}{x^2} & \frac{4\,x^2-9\,x+4}{x^2-x^3} & 6\,\frac{x-1}{x^2} & \frac{-3+3\,x-4\,x^2+4\,x^3}{x^4} & 4\,\frac{x-1}{x^2} \\ 0 & 5\,\left(x^4-x^3\right)^{-1} & \frac{5-x}{x^2} & -3\,x^{-2} & 5\,x^{-3} \\ 0 & (1-x)^{-1} & 0 & 3\,x^{-3} & -1 \\ x^{-2} & \frac{x^2+5\,x-4}{x^3-x^2} & 6\,\frac{x-1}{x^2} & -\frac{3+4\,x^2}{x^4} & \frac{x^2-4}{x^2} \end{bmatrix}$$

A basis of $\mathcal{E}(A)$ is (I_5, T) where

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -x^{-1} & 0 & 1/2\,\frac{-2\,x+2}{x} & 1/2\,\frac{-2\,x+2}{x} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -x^{-1} & 0 & x^{-1} & 1/4\,\frac{-4\,x+4}{x} \end{bmatrix}$$

Let

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ \frac{x-1}{x} & 0 & \frac{1+x}{x} & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ -x^{-1} & 0 & x^{-1} & 0 & 0 \end{bmatrix}$$

Then

$$J := P^{-1}TP = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

As expected

$$P[A] = B_1 \oplus B_2$$

where

$$B_1 = \begin{bmatrix} -5\,x^{-1} & x^{-1} \\ -\frac{9\,x^2+5\,x-6}{x^2} & \frac{9\,x+1}{x} \end{bmatrix}$$

and

$$B_2 = \begin{bmatrix} \frac{-10\,x^3+8\,x^4+6\,x^2-3\,x+3}{x^3(x-1)} & -\frac{x^4+x^3+3-3\,x}{x^3(x-1)} & 6\,\frac{x-1}{x} \\ 2\,\frac{3-3\,x+3\,x^2-4\,x^3+4\,x^4}{x^3(x-1)} & -\frac{2\,x^3-6\,x+6+x^4}{x^3(x-1)} & 6\,\frac{x-1}{x} \\ -\frac{-7-x+3\,x^2-9\,x^3+8\,x^4}{x^3(x-1)} & \frac{x^3+3\,x^2-6\,x-2+x^4}{x^3(x-1)} & -\frac{-5\,x-5+6\,x^2}{x^2} \end{bmatrix}$$

More Recent Developments

- Modular Algorithms for Linear Differential Equations: PhD Thesis of Thomas Cluzeau'2004
- Algorithms for solving directly systems of higher order differential equations: PhD Thesis of Carole El Bacha'2011
- Reduced Forms of Linear Differential Systems and Applications to Integrability of Hamiltonian Systems. Specific Reduction Algorithms for Hamiltonian Systems : PhD Thesis of Ainhoa Aparicio'2010
- Formal reduction of pfaffian systems: PhD Thesis of Nicolas LeRoux'2006
- Singularly-perturbed linear differential systems, completely integrable Pfaffian systems : PhD Thesis of Suzy S. Maddah'2015

Implementation

Packages

Computer Algebra team, University of Limoges

- M. Barkatou, E. Pfluegel: ISOLDE (late 90s-2012), a package written in Maple with algorithms for global solutions (polynomial, rational, exponentiel solutions, factorization, etc) as well as local analysis (singularities, formal solutions, formal reduction, . . .) for both differential and (q-) difference equations
- Higher order linear differential equations: Carole El-Bacha'2011
- IntegrableConnections : *RationalSolutions* (& *Eigenring*), *HyperexponentialSolutions* : Thomas Cluzeau
- Linear ordinary first-order differential systems with singularities, Singularly-perturbed linear differential systems, completely integrable Pfaffian systems, Apparent and Removable Singularities : 2013 -2015: Mathemagix: Lindalg; Maple: miniISOLDE, ParamInt, PfaffInt, AppSing : Suzy S. Maddah.

On Completely Integrable Pfaffian systems with normal crossings

Completely Integrable Pfaffian systems with normal crossings

For more details see PhD Thesis of Suzy S. Maddah'2015

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} Y = \begin{bmatrix} x_1^3 + x_1^2 + x_2 & x_2^2 \\ -1 & x_1^3 + x_1^2 - x_2 \end{bmatrix} Y \\ \\ x_2^3 \frac{\partial}{\partial x_2} Y = \begin{bmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} Y \end{cases} .$$

Required: Compute solutions in a neighborhood of $(x, y) = (0, 0)$.

General form

$$[\underline{A}] \qquad \begin{cases} x_1^{p_1+1} \, \frac{\partial}{\partial x_1} Y &= A_{(1)}(x_1, x_2, \dots, x_m) \; Y \\ x_2^{p_2+1} \, \frac{\partial}{\partial x_2} Y &= A_{(2)}(x_1, x_2, \dots, x_m) \; Y \\ \vdots \\ x_m^{p_m+1} \, \frac{\partial}{\partial x_m} Y &= A_{(m)}(x_1, x_2, \dots, x_m) \; Y \end{cases}$$

For $i, j \in \{1, \dots, m\}$,

- p_i is an integer and $\underline{p} = (p_1, \dots, p_m)$ is called Poincaré rank
- $A_{(i)} \in \mathrm{R} = \mathbb{C}[[x_1, \dots, x_m]]$ (i^{th}-component), and
- Integrability conditions:

$$x_i^{p_i+1} \, \frac{\partial}{\partial x_i} A_{(j)}(x) - x_j^{p_j+1} \, \frac{\partial}{\partial x_j} A_{(i)}(x) \; = A_{(i)}(x) \; A_{(j)}(x) - A_{(j)}(x) \; A_{(i)}(x$$

Fundamental matrix of formal solutions

$$\Phi(x_1^{1/s_1}, \ldots, x_m^{1/s_m}) \prod_{i=1}^{m} x_i^{C_i} \prod_{i=1}^{m} \exp(Q_i(x_i^{-1/s_i}))$$

- Φ is an invertible matrix whose entries are meromorphic series in $(x_1^{1/s_1}, \ldots, x_m^{1/s_m})$ over $\mathbb{C}$;
- $Q_i(x_i^{-1/s_i})$ is a diagonal matrix of polynomials in x_i^{-1/s_i} over $\mathbb{C}$ without contant terms.
- C_i is a constant matrix which commutes with $Q_i(x_i^{-1/s_i})$.

- H. Charrière, P. Deligne, R. Gérard, A. H. M. Levelt, Y. Sibuya, A. van den Essen, ... (70's and 80's)
- Algorithms: Reduction of Poincaré rank, Constructing Solutions of regular systems - Barkatou and LeRoux (2006), Closed Form Solutions of Integrable Connections: Barkatou-Cluzeau-ElBacha-Weil'2012

Example

Poincaré Rank Reduction, Barkatou-LeRoux'2006

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} Y = A_{(1)}(x_1, x_2)\ Y = \left(\begin{bmatrix} x_1^3 + x_2 & x_2^2 \\ -1 & -x_2 + x_1^3 \end{bmatrix} \right) Y \\ x_2^2 \frac{\partial}{\partial x_2} F = A_{(2)}(x_1, x_2)\ Y = \left(\begin{bmatrix} x_2 & x_2^2 \\ -2 & -3x_2 \end{bmatrix} \right) Y \end{cases}$$

$$\downarrow \quad Y = \left(\begin{bmatrix} x_1^3 & -x_2^2 \\ 0 & x_2 \end{bmatrix} \right) G$$

$$\begin{cases} x_1 x_2 \frac{\partial}{\partial x_1} G = \tilde{A}_{(1)}(x_1, x_2)\ G = \left(\begin{bmatrix} -2x_2 & 0 \\ -1 & x_2 \end{bmatrix} \right) G \\ x_2^3 \frac{\partial}{\partial x_2} G = \tilde{A}_{(2)}(x_1, x_2)\ G = \left(\begin{bmatrix} -x_2^2 & 0 \\ -2x_1^3 & -2x_2^2 \end{bmatrix} \right) G. \end{cases}$$

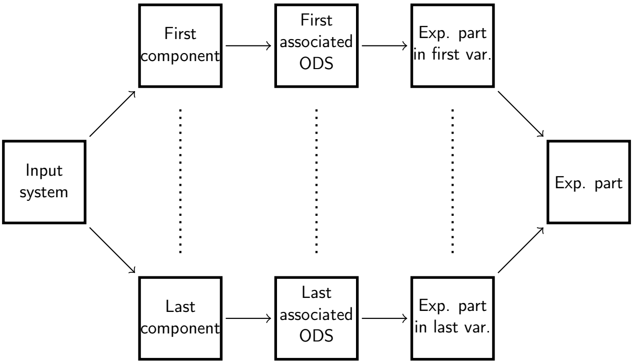


Figure: Computing the exponential part from associated ODS's

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} Y = \begin{bmatrix} x_1^3 + x_1^2 + x_2 & x_2^2 \\ -1 & x_1^3 + x_1^2 - x_2 \end{bmatrix} Y \\ \\ x_2^3 \frac{\partial}{\partial x_2} Y = \begin{bmatrix} x_2^2 - 2y - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} Y \end{cases}.$$

Associated system:

$$\begin{cases} x_1^4 \frac{d}{dx_1} \mathcal{Y} = \begin{bmatrix} x_1^3 + x_1^2 & 0 \\ -1 & x_1^3 + x_1^2 \end{bmatrix} \mathcal{Y} \\ \\ x_2^3 \frac{d}{dx_2} \mathcal{Y} = \begin{bmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} \mathcal{Y} \end{cases}.$$

$$\begin{cases} x_1^4 \frac{dY}{dx_1} & = \begin{bmatrix} x_1^3 + x_1^2 + x_2 & x_2^2 \\ -1 & x_1^3 + x_1^2 - x_2 \end{bmatrix} Y \\ \\ x_2^3 \frac{dY}{dx_2} & = \begin{bmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} Y \end{cases}.$$

With miniISOLDE or Lindalg we compute from the asscoiated system

$$\Phi(x_1, x_2)\, x_1^{C_1}\, x_2^{C_2} \exp\left(\begin{bmatrix} \frac{-1}{x_1} & 0 \\ 0 & \frac{-1}{x_1} \end{bmatrix}\right) \exp\left(\begin{bmatrix} \frac{3}{{x_2}^2} + \frac{2}{x_2} & 0 \\ 0 & \frac{3}{{x_2}^2} + \frac{2}{x_2} \end{bmatrix}\right).$$

Upon applying

$$Y = \exp\left(\frac{-1}{x_1}\right)\exp\left(\frac{3}{x_2^2}+\frac{2}{x_2}\right) G,$$

we have

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} G = \begin{bmatrix} x_1^3 + x_2 & x_2^2 \\ -1 & x_1^3 - x_2 \end{bmatrix} G \\ x_2^2 \frac{\partial}{\partial x_2} G = \begin{bmatrix} x_2 & x_2^2 \\ -2 & -3x_2 \end{bmatrix} G \end{cases}$$

And so, it is left to obtain:

$$G(x_1, x_2) = \Phi(x_1, x_2)\, x_1^{C_1}\, x_2^{C_2}.$$

For rank-reduction, we apply $G = T_1 H$ where

$$T_1 = \begin{bmatrix} x_2 x_1^3 & -x_2 \\ 0 & 1 \end{bmatrix}$$

which yields:

$$\begin{cases} x_1 \frac{\partial}{\partial x_1} H = \begin{bmatrix} -2 & 0 \\ -x_2 & 1 \end{bmatrix} H, \\ x_2 \frac{\partial}{\partial x_2} H = \begin{bmatrix} -2 & 0 \\ -2x_1^3 & -1 \end{bmatrix} H. \end{cases}$$

Finally, we compute

$$T_2 = \begin{bmatrix} 1 & 0 \\ \frac{x_2}{3} + 2x_1^3 & -1 \end{bmatrix}.$$

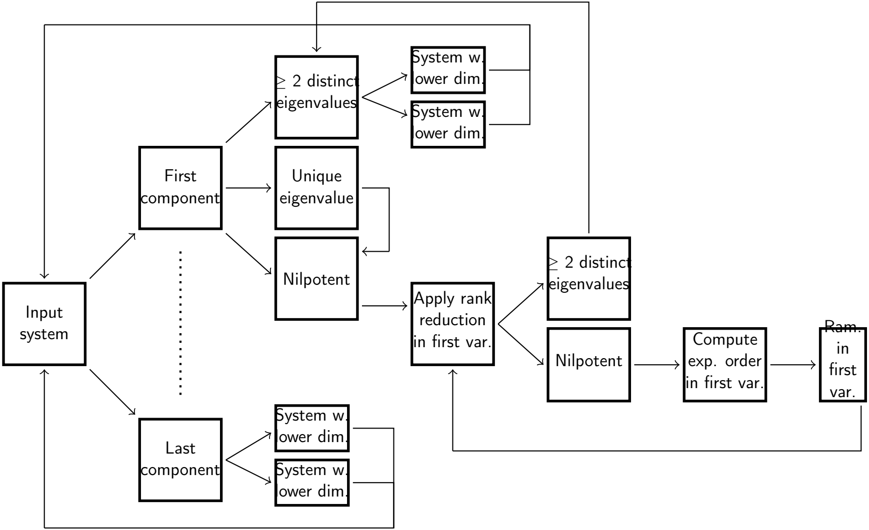
Then $H = T_2\, U$ yields

$$\begin{cases} x_1 \frac{\partial}{\partial x_1} U = C_1\, U = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} U, \\ x_2 \frac{\partial}{\partial x_2} U = C_2\, U = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} U. \end{cases}$$

$$\begin{cases} x_1^4 \frac{\partial}{\partial x_1} F = \begin{bmatrix} x_1^3 + x_1^2 + x_2 & x_2^2 \\ -1 & x_1^3 + x_1^2 - x_2 \end{bmatrix} F \\ \\ x_2^3 \frac{\partial}{\partial x_2} F = \begin{bmatrix} x_2^2 - 2x_2 - 6 & x_2^3 \\ -2x_2 & -3x_2^2 - 2x_2 - 6 \end{bmatrix} F \end{cases} .$$

A fundamental matrix of formal solutions is given by

$$T_1 \; T_2 \; x_1^{C_1} \; x_2^{C_2} \; \exp(\begin{bmatrix} \frac{-1}{x_1} & 0 \\ 0 & \frac{-1}{x_1} \end{bmatrix}) \; \exp(\begin{bmatrix} \frac{3}{{x_2}^2} + \frac{2}{x_2} & 0 \\ 0 & \frac{3}{{x_2}^2} + \frac{2}{x_2} \end{bmatrix}).$$


Input system
First component
Last component
≥ 2 distinct eigenvalues
Unique eigenvalue
Nilpotent
System w. lower dim.
System w. lower dim.
System w. lower dim.
System w. lower dim.
Apply rank reduction in first var.
≥ 2 distinct eigenvalues
Nilpotent
Compute exp. order in first var.
Ram. in first var.

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