Risk sensitive control of discrete time partially observed Markov processes with infinite horizon

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Abstract

In this paper existence of solutions to the Bellman equation corresponding to risk sensitive control of partially observed discrete time Markov processes is shown; this in turn leads to the existence of optimal strategies. The method used in the paper is based on discounted risk sensitive approximation.

1 Introduction

On a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ consider a controlled Markov process (x_n) taking values on a complete separable metric space E endowed with the Borel σ-algebra $\mathcal{E}$. Assume that x_n has a controlled transition operator $P^{a_n}(x_n, \cdot)$, where a_n is the control at time n taking values in a compact metric space U and adapted to the observation σ-field $Y^n = \sigma\{y_1, \dots, y_n\}$. The observation process (y_n) takes its values either in R^d or in a countable set D and for $n = 1, 2, \dots$ is given by the formula

$$y_n = h(x_n, w_n) \tag{1}$$

where (w_n) is a sequence of i.i.d. R^d or D valued respectively random variables with strictly positive density (distribution) g. In the case when (y_n) is R^d valued we assume that for each $x \in E$, the function $h(x, \cdot)$ is a C^1 diffeomorphism of R^d, while when (y_n) is D-valued the function $h(x, \cdot)$ is a $1-1$ mapping of D.
Letting $X^n = \sigma\{x_o, x_1, \dots, x_n\}$ one can easily show (see section 1.3 of [7] that

$$P\{y_{n+1} \in A \mid X^{n+1}, Y^n\} = \int_A r(x_{n+1}, y) dy \tag{2}$$

for a Borel measurable function $r : E \times R^d \to R^+$, $A \in \beta(R^d)$ with R^d replaced by D and the integral by summation when y_n is D-valued.
In what follows we shall assume that

(A1)	$\forall_{\varphi \in C(E)}$	the mapping
	$E \times U \ni (x, a) \mapsto P^a\varphi(x)$	$:= \int_E \varphi(z) P^a(x, dz)$ is continuous
(B1)	$r \in C(E \times R^d)$	or $C(E \times D)$
(B2)	$\forall_{C(R^d)}$	the mapping
	$E \ni x \mapsto R_\varphi(x)$	$:= \int_{R^d} \varphi(y) r(x, y) dy$
	or $\forall_{\varphi \in C(D)}$	the mapping
	$E \ni x \mapsto R_\varphi(x)$	$:= \sum_{y \in D} \varphi(y) r(x, y)$ is continous

Let $c : E \times U \to R^+$ be a continuous bounded function. Our aim is to minimize the following exponential ergodic performance criterion (called risk sensitive ergodic cost functional)

$$J_\mu((a_n)) = \limsup_{n\to\infty} \frac{1}{4} log E_\mu \left\{ exp \left\{ \sum_{i=0}^{n-1} c(x_i, a_i) \right\} \right\} \tag{3}$$

over all Y^n adapted controls a_n, where $\mu \in \mathcal{P}(E)$ is the initial law of (x_n). Completely observed discrete time risk sensitive ergodic control problems were studied in [2] - [4]. Partially observed discrete time risk sensitive control was studied in the case of finite time horizon in [5]. In the paper we show the existence of optimal strategies and solutions to the Bellman equation corresponding to the cost functional (3) with partial observation. Our method is based on a suitable measure transformation, and on asymptotic analysis for discount factor β going to 1 of the risk sensitive discounted cost functional

$$J_\mu^{\beta,\gamma}((a_n)) = E_\mu \left\{ \exp\left\{ \sum_{i=0}^{\infty} \gamma \beta^i c(x_i, a_i) \right\} \right\} \tag{4}$$

with $0 < \beta < 1$ and $\mu > 0$.
The proof of the existence of solutions to the Bellman equations corresponding to the cost functional (1) is similar (in particular Propositions 1 and 2) to that used for ergodic partially observed problem in [8]. To complete the introduction we define a measure transformation that enables us to formulate Bellman equations for risk sensitive control problems. Consider a new probability measure P^oon Ω such that the restrictions P_n^o, P_n of P^o and P respectively to the σ-field $F_n = \sigma\{X^n, Y^n\}$ satisfy

$$P_n(d\omega) = \prod_{i=1}^{n} \frac{r(x_i, y_i)}{g(y_i)} P_n^o(d\omega) \tag{5}$$

Recalling Lemma 1.8 of [7] we have

Lemma 1 *Under the measure P^o, the y_n are i.i.d. with common density g, independent of x_j for $j \leq n$ and (x_n) is a controlled Markov process having for n and Y^n - adapted a_n the same transition probability $P^{a_n}(x_n, dy)$.*

2 Discounted risk sensitive cost

Let (σ_n) be a stochastic process with values in $M^+(E)$, the space of nontrivial finite measures on E defined as follows

$$\begin{aligned} \sigma_0(\varphi) &= \mu(\varphi) \\ \sigma_n\varphi &= E^o\left\{ \exp\left\{ \sum_{i=0}^{n-1} \beta^i c(x_i, a_i) \right\} \varphi(x_n) \prod_{i=1}^{n} \frac{r(x_i, y_i)}{g(y_i)} \mid Y^n \right\} \end{aligned} \tag{6}$$

for $\varphi \in b\mathcal{E}$ the space of bounded Borel functions on E, where μ is the initial law of (x_i). By direct calculation, taking into account that under P^o random variables y_i are i.i.d. with density g we obtain that for $n \geq 1$, $\varphi \in b\mathcal{E}$

$$\sigma_n(\varphi) = \sigma_{n-1}(\Phi^{\beta^{n-1}}(y_n, a_{n-1})\varphi) = (\Phi^{\beta^{n-1}})^*(y_n, a_{n-1})\sigma_{n-1}(\varphi) \quad P \;\; a.e. \tag{7}$$

with

$$\Phi^\gamma(y,a)\varphi(x) = \int_E \varphi(z)\frac{r(z,y)}{g(y)}P^a(x,dz)e^{\gamma c(x,a)} \tag{8}$$

and $(\Phi^\gamma)^*$ being an adjoint operator defined on $M^+(E)$ by the formula

$$(\Phi^\gamma)^*(y,a)\eta(B) = \int_E\int_E \chi_B(z)\frac{r(z,y)}{g(y)}P^a(x,dz)e^{\gamma c(x,a)}\eta(dx) \tag{9}$$

for $\eta \in M^+(E)$.

Let $\mathcal{P}(E)$ be the space of probability measures on E endowed with the weak convergence topology. Given a bounded Borel measurable function $f : \mathcal{P}(E) \to R^+$ we can extend it to a bounded Borel measurable function on $M^+(E)$ using an operator $S : b\mathcal{B}(\mathcal{P}(E)) \to b\mathcal{B}(M^+(E))$ defined as follows

$$Sf(\eta) = \eta(E)f(\frac{\eta}{\eta(E)}) \tag{10}$$

for $\eta \in M^+(E)$.
We have

Lemma 2 *If $f \in b\mathcal{B}(\mathcal{P}(E))$ is concave i.e.*

$$f(\alpha\nu_1 + (1-\alpha)\nu_2) \geq \alpha f(\nu_1) + (1-\alpha)f(\nu_2)$$

for $\nu_1, \nu_2 \in \mathcal{P}(E)$, $\alpha \in [0,1]$, then $Sf \in b\beta(M^+(E))$ is also concave.

Proof. For $\eta_1, \eta_2 \in M^+(E)$, $\alpha \in [0,1]$ using the concavity of f we have

$$\begin{aligned}
&Sf(\alpha\eta_1 + (1-\alpha)\eta_2) = (\alpha\eta_1 + (1-\alpha)\eta_2)(E) \cdot f(\frac{\alpha\eta_1 + (1-\alpha)\eta_2}{\alpha}\eta_1(E) + (1-\alpha)\eta_2(E) \\
= \;& (\alpha\eta_1 + (1-\alpha)\eta_2)(E)f(\frac{\alpha\eta_1(E)}{\alpha\eta_1(E) + (1-\alpha)\eta_2(E)}\frac{\eta_1}{\eta_1(E)} \\
+ \;& \frac{(1-\alpha)\eta_2(E)}{\alpha\eta_1(E) + (1-\alpha)\eta_2(E)} \cdot \frac{\eta_2}{\eta_2(E)}) \\
\geq \;& (\alpha\eta_1 + (1-\alpha)\eta_2)(E)\frac{\alpha\eta_1(E)}{\alpha\eta_1(E) + (1-\alpha)\eta_2(E)} \\
\cdot \;& f(\frac{\eta_1}{\eta_1(E)}) + (\alpha\eta_1 + (1-\alpha)\eta_2)(E)\frac{(1-\alpha)\eta_2(E)}{\alpha\eta_1(E) + (1-\alpha)\eta_2(E)} f\left(\frac{\eta_2}{\eta_2(E)}\right) \\
= \;& \alpha Sf(\eta_1) + (1-\alpha)Sf(\eta_2).
\end{aligned}$$

$\square$

Fix $L > 1$. Let for $\beta \in (0,1)$, $\mu \in \mathcal{P}(E)$, and $\nu \in [0,L]$,

$$w_\beta(\mu,\gamma) = inf_{(a_n)} E_\mu \left\{ \exp\left\{ \sum_{i=0}^{\infty} \gamma\beta^i c(x_i,a_i) \right\} \right\} \tag{11}$$

with infimum taken over all U-valued Y^n adapted sequences (a_n).

In the next theorem 1 and in the rest of the paper we shall use the notation corresponding to R^d - valued observation process (y_n). Formulas and proofs provided hereafter hold also for the case of a countable observation space D substituting integrations in R^d with summations in D.

Theorem 1 *Under (A_1), (B_1) and (B_2) w_β is the unique in $C(\mathcal{P}(E) \times [0,L])$ solution to the following Bellman equation*

$$w_\beta(\mu,\gamma) = inf_{\alpha \in U}[\int_R^d Sw_\beta((\Phi^\gamma)^*(y,a)\mu, \gamma\beta)g(y)dy] \tag{12}$$

satisfying $1 \leq w_\beta(\cdot,\mu) \leq \exp\left\{\frac{\gamma\|c\|}{1-\beta}\right\}$.
Furthermore for fixed μ, $w_\beta(\cdot,\mu)$ is a concave function.
Finally there exists a measurable function $u : \mathcal{P}(E) \times [0,L] \mapsto U$ for which the infimum on the right hand side of (12) is attained and the control $\hat{a}_n = u(\frac{\sigma_n}{\sigma_n(E)}; \gamma\beta^n)$ is optimal.

Proof. For $f \in C(\mathcal{P}(E) \times [0,L])$ define the operator

$$T_\beta f(\mu,\gamma) = inf_{a \in U}[\int_{R^d} Sf((\Phi^\mu)^*(y,a)\mu, \gamma\beta)g(y)dy] \tag{13}$$

We shall show that w_β is a fixed point of T_β. The proof consists of several steps.

Step 1. The operator T_β transforms $C(\mathcal{P}(E) \times [0,L])$ into itself. For this purpose it suffices to show that for $f \in C(\mathcal{P}(E) \times [0,L])$ the continuity of the mapping

$$\begin{aligned} \mathcal{P}(E) \times [0,L] \times U \ni (\mu,\gamma,a) \mapsto & \int_R^d Sf((\Phi^\gamma)^*(y,a)\mu, \gamma\beta)g(y)dy \qquad (14) \\ = & \int_{R^d} (\Phi^\gamma)^*(y,a)\mu(E) \cdot f(\frac{(\Phi^\gamma)^*(y,a)\mu}{(\Phi^\gamma)^*(y,a)\mu(E)}, \gamma\beta)g(y)dy \\ = & \int_{R^d}\int_E\int_E r(z,y)P^a(x,dz)e^{\gamma c(x,a)}\mu(dx)f(N(y,\mu,\gamma,a),\gamma\beta)dy \end{aligned}$$

where for $\varphi \in b\mathcal{E}$

$$N(y,\mu,\gamma,a)(\varphi) = \frac{\int_E \int_E \varphi(z) r(z,y) P^a(x,dz) e^{\gamma c(x,a)} \mu(dx)}{\int_E \int_E r(z,y) P^a(x,dz) e^{\gamma c(x,a)} \mu(dx)}$$

Notice now that by (A_1), (B_1) and (B_2) using Prokhorov theorem (see Thm. 6.1 and 6.2 of [1]), for $\varphi \in C(E)$ the mapping

$$\begin{aligned} &R^d \times \mathcal{P}(E) \times [0,L] \times U \ni (y,\mu,\gamma,a) \\ &\quad \mapsto \int_E \int_E \varphi(z) r(z,y) P^a(x,dz) e^{\gamma c(x,a)} \mu(dx) \end{aligned}$$

is continuous.
Consequently the mapping

$$R^d \times \mathcal{P}(E) \times [0,L] \times U \ni (y,\mu,\gamma,a) \mapsto N(y,\mu,\gamma,a) \in \mathcal{P}(E)$$

is continuous and similarly the mapping (14) is also continuous.

Step 2. For $\underline{\mathrm{f}}(\mu,\gamma) \equiv 1$ and $\bar{f}(\mu,\gamma) = e^{\gamma \frac{\|c\|}{1-\beta}}$ for $\mu \in \mathcal{P}(E)$ and $\gamma \in [0,L]$ we have $T_\beta \underline{\mathrm{f}} \geq \underline{\mathrm{f}}$ and $T_\beta \bar{f} \leq \bar{f}$.

Therefore

$$\underline{\mathrm{f}} \leq T_\beta \underline{\mathrm{f}} \leq \ldots \leq T_\beta^n \underline{\mathrm{f}} \leq T_\beta^n \bar{f} \leq \ldots \leq T_\beta \bar{f} \leq \bar{f} \tag{15}$$

Step 3. The iterations $T_\beta^n \underline{\mathrm{f}}$ and $T_\beta^n \bar{f}$ have the following interpretations

$$\begin{aligned} &T_\beta^n \bar{f}(\mu,\gamma) \qquad (16) \\ &= inf_{(a_n)} E_\mu^0 \{ e^{\frac{\gamma\beta\|c\|}{1-\beta}} e^{\gamma \sum_{i=0}^{n-1} \beta^i c(x_i,a_i)} \prod_{i=1}^{n-1} \frac{r(x_i,y_i)}{g(y_i)} \} \\ &= \inf_{(a_n)} E_\mu \{ e^{\frac{\gamma\beta^n\|c\|}{1-\beta}} e^{\gamma \sum_{i=0}^{n-1} \beta^i c(x_i,a_i)} \} \end{aligned}$$

and

$$\begin{aligned} &T_\beta^n \underline{\mathrm{f}}(\mu,\gamma) \qquad (17) \\ &= \inf_{(a_n)} E_\mu^o \{ e^{\gamma \sum_{i=0}^{n-1} \beta^i c(x_i,a_i)} \prod_{i=1}^{n-1} \frac{r(x_i,y_i)}{g(y_i)} \} \\ &= \inf_{a_n} E_\mu \{ e^{\gamma \sum_{i=0}^{n-1} \beta^i c(x_i,a_i)} \} \end{aligned}$$

Hence

$$T_\beta^n \underline{f}(\mu,\gamma) \le w_\beta(\mu,\gamma) \le T_\beta^n \underline{f}(\mu,\gamma) \tag{18}$$

and

$$\begin{aligned} 0 &\le T_\beta^n \bar{f}(\mu,\gamma) - T_\beta^n \underline{f}(\mu,\gamma) \\ &\le \sup_{(a_n)} (e^{\frac{x\beta^n\|c\|}{1-\beta}} - 1) \cdot E_\mu\{e^{\gamma \sum_{i=0}^{n-1} \beta^i c(x_i,a_i)}\} \\ &\le e^{\frac{\gamma\|c\|}{1-\beta}} \cdot (e\frac{\gamma\beta^n\|c\|}{1-\beta} - 1) \to 0 \text{ as } n \to \infty \end{aligned} \tag{19}$$

Consequently $w_\beta(\mu,\gamma) = lim_{n\to\infty} T_\beta^n \bar{f}(\mu,\gamma) = lim_{n\to\infty} T_\beta^n \underline{f}(\mu,\gamma)$ and $w_{\underline{\beta}}(\mu,\gamma)$ as a limit of increasing sequence $T_\beta^n \underline{f}(\mu,\gamma)$ and decreasing sequence $T_\beta^n \bar{f}(\mu,\gamma)$ of continuous functions is also continuous.
Since by Lemma 2 if $f(\cdot,\gamma)$ is concave them $T_\beta f(\cdot,\gamma)$ is also a concave function as a limit of concave functions. The existence of a measurable selector u follows from the continuity of the right hand side (12) in μ and a (see step 1). The optimality of $\hat{a}_n$ can be shown using step 3 in a standard way.

□

3 Main result

Let

(A2) $\lambda := \inf_{a,a' \in U}$ $\inf_{x,x' \in E}$ $\inf_{B \in E, P^{a'}(x',B) > O} \frac{P^a(x,B)}{P^{a'}(x',B)} > 0$

We have

Proposition 1. *Under* $(A1),(A2),(B1)$ *and* $(B2)$ *the following inequalities hold: for* μ, $\nu \in \mathcal{P}(E)$, $\beta \in (0,1)$, $\gamma \in [0,L]$

$$\lambda e^{-\gamma\|c\|} \le \frac{w_\beta(\mu,\gamma)}{w_\beta(\nu,\gamma)} \le \lambda^{-1} e^{\gamma\|c\|} \tag{20}$$

Proof. For $B \in \mathcal{E}$ we have

$$\int_E \chi_B(z)\frac{r(z,y)}{g(y)}P^a(x,dz)e^{\gamma c(x,a)} \geq$$
$$\lambda \int_E \chi_B(z)\frac{r(z,y)}{g(y)}P^{a'}(x',dz)e^{\gamma c(x',a')-\gamma\|c\|}$$

and consequently for $a, a' \in U$

$$(\Phi^\gamma)^*(y,a)\mu(B) \geq \lambda e^{-\gamma\|c\|}(\Phi^\gamma)^*(y,a')\nu(B) \tag{21}$$

Let

$$m^\gamma(y,a,a') = (\Phi^\gamma)^*(y,a)\mu - \lambda e^{-\gamma\|c\|}(\Phi^\gamma)^*(y,a')\nu \tag{22}$$

By Theorem 1 there exist $a, a' \in U$ such that

$$\frac{w_\beta(\mu,\gamma)}{w_\beta(\nu,\gamma)} = \frac{\int_{R^d} Sw_\beta((\Phi^\gamma)^*(y,a)\mu,\gamma\beta)g(y)dy}{\int_{R^d} Sw_\beta((\Phi^\gamma)^*(y,a)\nu,\gamma\beta)g(y)dy} \tag{23}$$

Consider the measure $m^\gamma(y,a,a')$ defined in (22). If $m^\gamma(y,a,a')(E) = 0$ by the definition of operator S we have

$$Sw_\beta((\Phi^\gamma)^*(y,a)\mu,\gamma\beta)) = \lambda e^{-\gamma\|c\|}Sw_\beta((\Phi^\gamma)^*(y,a)\nu,\gamma\beta)$$

In the case when $m^\gamma(y,a,a') \in M^+(E)$ by Lemma 2 we obtain

$$\begin{aligned} & Sw_\beta((\Phi^\gamma)^*(y,a)\mu,\gamma\beta) \geq \lambda e^{-\gamma\|c\|}Sw_\beta((\Phi^\gamma)^*(y,a')\nu,\gamma\beta) \\ + \quad & (1-\lambda e^{-\gamma\|c\|})Sw_\beta(\frac{1}{1-\lambda e^{-\gamma\|c\|}}m^\gamma(y,a,a'),\gamma\beta) \end{aligned}$$

Therefore, using nonnegativity of w_β, from (23) we have

$$\frac{w_\beta(\mu,\gamma)}{w_\beta(\nu,\gamma)} \geq \lambda e^{-\gamma\|c\|}$$

and interchanging μ with ν we obtain also the second ineqality in (20).

□

Assume now that

$$\text{(A3)} \qquad \text{if} \quad \mathcal{P}(E) \ni \mu_n \Rightarrow \mu \in \mathcal{P}(E) \quad \text{then}$$

$$\sup_{a\in U} |VertP^a(\mu_n,\cdot) - P^a(\mu,\cdot)\|_{\mathrm{var}} \to 0$$

where $\Rightarrow$ stands for weak convergence of probability measures and $\|\|_{\mathrm{var}}$ is the variation norm.
Let

$$\lambda(\nu,\mu) := \inf_{a\in U} \inf_{B\in\mathcal{E},\mathcal{P}^{\dashv}(\mu,\mathcal{B})>\prime} \inf_{\gamma\in[0,L]} \frac{\int_E e^{\gamma c(x,a)} P^a(x,B)\nu(dx)}{\int_E e^{\gamma c(x,a)} P^a(x,B)\mu(dx)} \tag{24}$$

Notice that ander (A_2) and (A_3)

$$\nu \Rightarrow \mu \text{ implies that } \lambda(\nu,\mu) \to 1 \text{ and } \lambda(\mu,\nu) \to 1 \tag{25}$$

Furthermore

Proposition 2. *Under (A_1)-(A_2),(B_1),(B_2) for $\mu,\nu \in \mathcal{P}(E)$ and $\beta \in (0,1)$ we have*

$$\lambda(\nu,\mu) \le \frac{w_\beta(\nu,\gamma)}{w_\beta(\mu,\gamma)} \le [\lambda(\mu,\nu)]^{-1} \tag{26}$$

Proof. For $B \in \mathcal{E}$, $y \in R^d$, $a \in U$

$$(\Phi^\gamma)^*(y,a)\mu(B) = \int_E \int_E \chi_B(z) \frac{r(z,y)}{g(y)} P^a(x,dz) e^{\gamma c(x,a)} \mu(dx) \tag{27}$$

$$\le \frac{1}{\lambda(\nu,\mu)} \int_E \int_E \chi_B(z) \frac{r(z,y)}{g(y)} P^a(x,dz) e^{\gamma c(x,a)} \nu(dx) =$$

$$= \frac{1}{\lambda(\nu,\mu)} (\Phi^\gamma)^*(y,a)\nu(B)$$

where we used the fact that under $(A2)$, $\lambda(\nu,\mu) > 0$. Using the definition (24) of $\lambda(\nu,\mu)$ we clearly have that $\lambda(\nu,\mu) \le 1$. If $\lambda(\nu,\mu) = 1$ then for $a \in U$ $P^a(\nu,\cdot) = P^a(\mu,\cdot)$ and therefore $w_\beta(\nu,\gamma) = w_\beta(\mu,\gamma)$ and (26) holds.
Consider the case $\lambda(\nu,\mu) < 1$. By (27) for $a \in U$

$$m^\gamma(y,a)(\mu,\nu) = (\Phi^\gamma)^*(y,a)\nu - \lambda(\nu,\mu)(\Phi^\gamma)^*(y,a)\mu \in M^+(E)$$

Let now $a \in U$ be such that

$$w_\beta(\nu,\gamma) = \int_{R^d} Sw_\beta((\Phi^\gamma)^*(y,a)\nu,\beta) g(y) dy$$

From Lemma 2 we have

$$\begin{aligned}
&\frac{w_\beta(\nu,\gamma)}{w_\beta(\mu,\gamma)} \geq \frac{\int_{R^d} Sw_\beta((\Phi^\gamma)^*(y,a)\nu,\gamma\beta)g(y)dy}{\int_{R^d} Sw_\beta((\Phi^\gamma)^*(y,a)\mu,\gamma\beta)g(y)dy} \geq \\
\geq\ & \frac{\lambda(\nu,\mu)\int_{R^d} Sw_\beta((\Phi^\gamma)^*(y,a)\mu,\gamma\beta)g(y)dy}{\int_{R^d} Sw_\beta((\Phi^\gamma)^*(y,a)\mu,\gamma\beta)g(y)dy} \\
+\ & \frac{(1-\lambda(\nu,\mu))\int_{R^d} Sw_\beta(\frac{1}{1-\lambda(\nu,\mu)}(\Phi^\gamma)^*(y,a)\mu,\gamma\beta)g(y)dy}{\int_{R^d} Sw_\beta((\Phi^\gamma)^*(y,a)\mu,\gamma\beta)g(y)dy} \\
\geq\ & \lambda(\nu,\mu)
\end{aligned}$$

Interchanging the roles of ν and μ we immediately obtain the second inequality in (26).

□

Fix $\overline{\nu} \in \mathcal{P}(E)$ and define

$$h_\beta(\mu,\gamma) := \frac{w_\beta(\mu,\gamma)}{w_\beta(\overline{\nu},\gamma)} \tag{28}$$

$$\lambda_\beta(\gamma) := \frac{w_\beta(\overline{\nu},\gamma)}{w_\beta(\overline{\nu},\gamma\beta)} \tag{29}$$

From (12) we obtain that

$$h_\beta(\mu,\gamma)\cdot\lambda_\beta(\gamma) = \inf_{a\in U}[\int_{R^d} Sh_\beta((\Phi^\gamma)^*(y,a)\mu,\gamma\beta)g(y)dy] \tag{30}$$

By Proposition 1, for $\mu \in \mathcal{P}(E)$, $\beta \in (1,0)$, $\gamma \in [0,L]$

$$\lambda e^{-\gamma\|c\|} \leq h_\beta(\mu,\gamma) \leq \lambda^{-1}e^{\gamma\|c\|} \tag{31}$$

and

$$\lambda^2 e^{-2\gamma\|c\|} \leq \lambda_\beta(\gamma) \leq \lambda^{-2}e^{2\gamma\|c\|} \tag{32}$$

Since by Proposition 2, for $\gamma \in [0,L]$

$$\lambda(\nu,\mu) \leq h_\beta(\mu,\gamma) \leq [\lambda(\mu,\nu)]^{-1} \tag{33}$$

the family $\{h_\beta(\cdot,\gamma),\ \beta \in (0,1),\ \gamma \in [0,L]\}$ is equicontinuous and uniformly bounded. Therefore by Ascoli theorem (Thm.33 of Chapter 9 in [6]) using diagonal procedure one can find a subsequence $\beta_n \uparrow 1$, a sequence of numbers

$\lambda_0(\gamma)$, $\lambda_1(\gamma)$, ..., $\lambda_n(\gamma)$, ..., and a sequence of functions $\overline{w}_0(\cdot,\gamma)$, $\overline{w}_1(\cdot,\gamma)$, ..., $\overline{w}_n(\cdot,\gamma)$, ... $\in C(\mathcal{P}(E))$ such that for $m = 0, 1, \ldots$,

$$\lambda_{\beta_n}(\gamma\beta^m) \to \lambda_m(\gamma)$$

and

$$h_{\beta_n}(\cdot,\gamma\beta^m) \to \overline{w}_m(\cdot,\gamma)$$

as $n \to \infty$ uniformly on compact subsets of $\mathcal{P}(E)$. Moreover from (31) and (32) we have for $m = 0, 1, 2, \ldots$, $\mu \in \mathcal{P}(E)$

$$\lambda e^{-\gamma\|c\|} \leq \overline{w}_m(\mu,\gamma) \leq \lambda^{-1} e^{\gamma\|c\|} \tag{34}$$

and

$$\lambda^2 e^{-2\gamma\|c\|} \leq \lambda_m(\gamma) \leq \lambda^{-2} e^{2\gamma\|c\|} \tag{35}$$

Furthermore the functions $\overline{w}_m(\cdot,\gamma)$ for $m = 0, 1, \ldots$ being limits of concave functions are concave, and

Lemma 3 *The functions $\overline{w}_m$ and constants $\lambda_m(\gamma)$ are solutions to the following equations*

$$\overline{w}_m(\mu,\gamma)\lambda_m(\gamma) = \inf_{a\in U}[\int_{R^d} S\overline{w}_{m+1}((\Phi^\gamma)^*(y,a)\mu,\gamma)g(y)dy] \tag{36}$$

and

$$\lambda(\nu,\mu) \leq \frac{\overline{w}_m(\nu,\gamma)}{\overline{w}_m(\mu,\gamma)} \leq (\lambda(\mu,\nu))^{-1} \tag{37}$$

Proof. By (30) to prove (36) it suffices to show that for $\mu \in \mathcal{P}(E)$

$$\begin{aligned} &\sup\nolimits_{a\in U} | \textstyle\int_R^d (Sh_{\beta_N}((\Phi^\gamma)^*(y,a)\mu,\gamma\beta_n^{m+1}) \\ &-S\overline{w}_{m+1}((\Phi^\gamma)^*(y,a)\mu,\gamma)g(y)dy| \to 0 \end{aligned} \tag{38}$$

$$as \tag{39}$$

$$n \to \infty \tag{40}$$

By $(A1)$ and $(B2)$ using Prokhorov Theorem (see [1]) for a given $\epsilon > 0$ one can find compact subsets K_1, $K_2 \subset E$ and $\Gamma \subset R^d$ such that

$$\begin{aligned} &\int_{R^d\times E\times E\setminus\Gamma\times K_1\times K_2} r(z,y)P^a(x,dz)e^{\gamma c(x,a)}\mu(dx) \\ &|h_{\beta_n}(N(y,\mu,\gamma,a),\gamma\beta_n^{m+1}) - \overline{w}_{m+1}(N(y,\mu,\gamma,a),\gamma)|dy \leq \epsilon \end{aligned}$$

with $N(y,\mu,\gamma,a)$ defined as in the step 1 of the proof of Theorem 1. Since by continuity of the mapping $(y,a)\to N(y,\mu,\gamma,a)$ the family $\{N(y,\mu,\gamma,a),\ y\in \Gamma,\ a\in U\}$ is compact in $\mathcal{P}(E)$ and (compare to (14))

$$\begin{aligned}
&\int_{R^d} S\overline{w}_{m+1}((\Phi^\gamma)^*(y,a)\mu,\gamma)g(y)dy \\
= &\int_{R^d}\int_E\int_E r(z,y)P^a(x,dz)e^{\gamma c(x,a)}\mu(dx)\overline{w}_{m+1}(N(y,\mu,\gamma,a),\gamma)dy
\end{aligned}$$

and

$$\begin{aligned}
&\int_{R^d} Sh_{\beta_n}((\Phi^\gamma)^*(y,a)\mu,\gamma\beta)g(y)dy \\
= &\int_{R^d}\int_E\int_E r(z,y)P^a(x,dz)e^{\gamma c(x,a)}\mu(dx)h_{\beta_n}(N(y,\mu,\gamma,a),\gamma\beta_n)dy
\end{aligned}$$

we obtain (38), and consequently (36). The inequalities (37) follows from arguments similar to those as in the proof of Proposition 2 using the from of the equation (36), as well as concavity and nonnegativity of $\overline{w}_m(\cdot,\gamma)$.

□

Consequently by (34) and (37) the family $\overline{w}_m(\cdot,\gamma), m=0,1,2,\ldots$ is equicontinuous and uniformly bounded. Therefore using Ascoli theorem (thm. 33 Chapter 9 of [6]) again we can choose a subsequence m_k, a constant $\lambda(\gamma)$ and a function $w(\cdot,\gamma)\in C(\mathcal{P}(E))$ such that

$$\lambda_{m_k}(\gamma)\to\lambda(\gamma)$$

and

$$\overline{w}_{m_k}(\mu,\gamma)\to w(\mu,\gamma)$$

uniformly in μ from compact subsets of $\mathcal{P}(E)$. Using a limit procedure to equation (36) similar to the one used in the proof of Lemma 3 for the equation (30) we finally obtain that $w(\cdot,\gamma)$ and $\lambda(\gamma)$ satisfy the equation

$$w(\mu,\gamma)\lambda(\gamma)=\inf_{a\in U}[\int_{R^d} Sw((\Phi^\gamma)^*(y,a)\mu,\gamma)g(y)dy] \tag{41}$$

To obtain a probabilistic interpretation of (39) we introduce the following $M^+(E)$ valued process (ζ_n)

$$\zeta_o(\phi) = \mu(\phi) \tag{42}$$

$$\zeta_n(\phi) = E^o\left\{\exp\left\{\sum_{i=0}^{n-1}\gamma c(x_i,a_i)\right\}\frac{1}{[\lambda(\gamma)]^n}\phi(x_n)\prod_{i=1}^{n}\frac{r(x_i,y_i)}{g(y_i)}\ |Y^n\right\}$$

for $\phi \in b\mathcal{E}$. Similarly to (7) - (9) one can show that for $n \geq 1$

$$\zeta(\phi) = \zeta_{n-1}(\frac{1}{\lambda(\gamma)}\Phi^\gamma(y_n,a_{n-1})\phi) = \tag{43}$$

$$= \frac{1}{\lambda(\gamma)}(\Phi^\gamma)^*(y_n,a_{n-1}\zeta_{n-1}(\phi) \quad P$$

$$a.e. \tag{44}$$

with Φ^γ defined in (8).
We can now formulate our main result

Theorem 2 *Under $(A1)$ - $(A3)$, $(B1)$, $(B2)$ for $\gamma \in [0,L]$ there is a constant $\lambda(\gamma)$ a function $w(\cdot,\gamma) \in C(\mathcal{P}(E))$ for which (39) holds. Moreover for $\mu \in \mathcal{P}(E)$*

$$\lambda e^{-\gamma||c||} \leq w(\mu,\lambda) \leq \lambda^{-1}e^{\gamma||c||} \tag{45}$$

and the function $w(\cdot,\lambda)$ is concave.
Finally

$$log\lambda(\gamma) = \inf_{a_n}\limsup_{n\to\infty}\frac{1}{n}logE_x\left\{exp\left\{\sum^{n-1} i=0\gamma c(x_i,a_i)\right\}\right\} \tag{46}$$

and there exists a measurable function $u_\gamma : \mathcal{P}(E) \to U$ for which the infimum on the right hand side of (39) is attained and the control strategy

$$\hat{a}_n = u_\gamma(\frac{\zeta_n}{\zeta_n(E)}) \tag{47}$$

is optimal.

Proof. The existence of $\lambda(\gamma)$ and $w(\cdot,\gamma)$ satisfying (39) has been already proved. The bounds (42) follow from (34), while the concavity of $w(\cdot,\gamma)$ is a consequence of the concavity of $\overline{w}_m(\cdot,\gamma)$.
Similarly to step 1 in the proof of Theorem 1 we can show (compare to (14)) that the mapping

$$\mathcal{P}(E)\times U \ni (\mu,a) \to \int_{R^d} Sw((\Phi^\gamma)^*(y,a)\mu,\gamma)g(y)dy$$

is continuous, so that there is a measurable selector $u\gamma : \mathcal{P}(E) \to U$ for the infimum in (39).
Iterating (39) for control strategy $\hat{a}_n$ we obtain for $n = 1, 2, \ldots$

$$w(\mu,\gamma) = E^o\left\{\zeta_n(E)w(\frac{\zeta_n}{\zeta_n(E)},\gamma)\right\}$$

with inequality " $\leq$ " for any other strategy (a_n).
Therefore using the bounds (42) we have that

$$\begin{aligned} 0 &= lim_{n\to\infty}\frac{1}{n}logE^o\left\{\zeta_n(E)\right\} \\ &= lim_{n\to\infty}\frac{1}{n}logE\left\{exp\left\{\sum_{i=0}^{n-1}\gamma c(x_i,\hat{a}_i)\right\}\right\} - log\lambda(\gamma) \end{aligned}$$

for the strategy $\hat{a}_n$ and

$$\begin{aligned} 0 &\leq limsup_{n\to\infty}\frac{1}{n}logE^o\left\{\zeta_n(E)\right\} \\ &= \limsup_{n\to\infty}\frac{1}{4}logE\left\{exp\left\{\sum_{i=0}^{n-1}\gamma c(x_i,a_i)\right\}\right\} - log\lambda(\gamma) \end{aligned}$$

for any other strategy (a_n). Consequently the strategy $\hat{a}_n$ given by (44) is optimal and (43) holds.

$\square$

Remark 1. *Substituting* $(A2)$ *with the assumption*
(A3') if $\mathcal{P}(E) \ni \mu_n \Rightarrow \mu \in \mathcal{P}(E)$ and $U \ni a_n \to a \in U$ the measures $P^{a_n}(\mu_n,\cdot)$ converge to $P^a(\mu,\cdot)$ in variation norm

we have that $(A1)$ *and* $(A3)$ *are satisfied.*

Remark 2. *Notice that all the assumptions imposed in the paper hold for the discrete time model in* R^d *considered in [5]*

$$\begin{aligned} x_{n+1} &= A(x_n, a_n) + B(x_n)w_n \\ y_{n+1} &= C(x_{n+1}) + D(x_n)v_n \end{aligned}$$

where w_n,v_n *are i.i.d. d-dimensional Gaussian vectors and the matrix functions A, B, C, D,* B^{-1}, D^{-1} *(inverse matrices to B and D respectively) are bounded and continuous.*

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