Risk Sensitive Portfolio Optimization

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Abstract

In the paper discrete time portfolio selection with maximization of the risk sensitized growth rate with and without transaction costs is considered.

Keywords: risk sensitive stochastic control, portfolio selection, Bellman equation, transaction costs

1991 Mathematical Subject Classification: 90A09, 93E20

1 Introduction

Assume we are given a market consisting of m securitites and k factors. The set of securities include stocks, bonds, cash and derivative securities. The set of factors (see e.g. [1], [9]) may include dividend yields, price - earning rations, short - term interest rates, rate of inflation. Let (Ω, F, F_t, P) be a probability space and $W(t) = (w_1(t), \ldots, w_{k+m}(t))^T$ be an R^{k+m} valued standard Wiener process. Denote by $x^i(n)$ the level of the i-th factor at time n, for $n = 0, 1, \ldots,$. We shall assume that the factors change at discrete moments for simplicity say at times $n = 0, 1, \ldots$ and the vector vector $x(n) = (x^1(n), \ldots, x^k(n))^T$ is a solution to the following recurrence system of equations

$$x^i(n+1) = b_i(x(n))dt + \sum_{j=1}^{k+m} \lambda_{ij}(w_j(n+1) - w_j(n)) := g_i(x(n), W_n(1)) \tag{1}$$

for $i = 1, \ldots, k$ and $n = 0, 1, \ldots$, where $W_n(s) = W(s+n) - W(n)$ and we implicitly defined a function g_i. Denote by $S_i(t)$ the price of the i-th security at time t. Following [1] and [2] we shall assume that the dynamics of the security prices $S_i(t)$ in the time interval $[n, n+1)$ is given by the following stochastic differential equation

$$dS_i(t) = a_i(x(n))S_i(t)dt + \sum_{j=1}^{k+m} \sigma_{ij} S_i(t) dw_j(t) \tag{2}$$

*The paper supported by KBN grant no. 2 P03A 015 15

for $i = 1, \ldots, m$.
In what follows we shall assume that the functions b_i of (1) and a_i of (2) are bounded and b_i is continuous. Consequently by 5.6D of [8] for $t \geq n$ we have that

$$S_i(t) = S_i(n) \exp\left\{a_i(x(n))(t-n) - \frac{1}{2}\sum_{j=1}^{k+m} \sigma_{ij}^2 (t-n) + \sum_{j=1}^{k+m} \sigma_{ij}(w_j(t) - w_j(n))\right\} \quad (3)$$

Assume we are given an initial capital $V(0) = v$ which we invest in the securities. We shall assume furthermore that we can change our portfolio only at discrete times $n = 0, 1, \ldots$ and we are allowed to buy any nonnegative number of the securities (i.e. also parts) we can afford. We do not admitt short selling nor short borrowing. Denote by $N_i(t)$ the number of the i-th securities we have in our portfolio and by $V(t)$ our capital at time t after possible transactions. Let $N_i^-(t)$ and $V^-(t)$ be the number of i-th security and the value of our capital before transactions at time t. To avoid construction problems shall not allow multiple transactions at the same time. The process $N_i(t)$ is piecewise constant, right continuous process changing its values at positive integer times only. Therefore we have

$$V(n) = \sum_{i=1}^{m} N_i(n) S_i(n) \quad (4)$$

and for $n \leq t < n+1$

$$V(t) = \sum_{i=1}^{m} N_i(n) S_i(t) \quad (5)$$

Denote by $h_i(t)$ the part of the capital invested at time t in the i-th security.
Clearly vector $(h_1(t), \ldots, h_m(t))^T \in H := \left\{(h_1, \ldots, h_m)^T \in [0,1]^m : \sum_{i=1}^m h_i = 1\right\}$ and $h_i(t) = \frac{N_i(t)S_i(t)}{V(t)}$. By Itô rule for $n \leq t < n+1$ we have

$$\begin{aligned} dV(t) &= \sum_{i=1}^{m} N_i(n) dS_i(t) \\ &= \sum_{i=1}^{m} a_i(x(n)) S_i(t) N_i(n) dt + \sum_{i=1}^{m} N_i(n) \sum_{j=1}^{k+m} \sigma_{ij} S_i(t) dw_j(t) \\ &= V(t) \sum_{i=1}^{m} a_i(x(n)) h_i(t) dt + V(t) \sum_{i=1}^{m} h_i(t) \sum_{j=1}^{k+m} \sigma_{ij} dw_j(t) \end{aligned} \quad (6)$$

Using 5.6D of [8] again, for $n \leq t < n+1$ we obtain

$$\begin{aligned} V(t) = V(n) \exp\Bigg\{ &\sum_{i=1}^{m} \int_n^t a_i(x(n)) h_i(s) ds - \frac{1}{2} \int_n^t \sum_{j=1}^{k+m} \left(\sum_{i=1}^{m} h_i(s) \sigma_{ij}\right)^2 ds \\ &+ \int_n^t \sum_{i=1}^{m} h_i(s) \sum_{j=1}^{k+m} \sigma_{ij} dw_j(s) \Bigg\} \end{aligned} \quad (7)$$

Since for $n \le t < n+1$

$$h_i(t) = \frac{h_i(n)V(n)}{S_i(n)} \frac{S_i(t)}{V(t)} \tag{8}$$

we have that

$$\begin{aligned} h_i(t) = h_i(n) \exp \Bigg\{ & a_i(x(n))(t-n) - \frac{1}{2} \sum_{j=1}^{k+m} \sigma_{ij}^2 (t-n) \\ & - \sum_{l=1}^{m} \int_n^t a_l(x(n)) h_l(s) ds + \frac{1}{2} \int_n^t \sum_{j=1}^{k+m} \left(\sum_{l=1}^{m} h_l(s) \sigma_{lj} \right)^2 ds \\ & + \sum_{j=1}^{k+m} \sigma_{ij} (w_j(t) - w_j(n)) - \int_n^t \sum_{l=1}^{m} h_l(s) \sum_{j=1}^{k+m} \sigma_{lj} dw_j(s) \Bigg\} \end{aligned} \tag{9}$$

By Itô rule (compare to (6.34) of [8]) we obtain

$$\begin{aligned} dh_i(t) = h_i(t) \Bigg\{ & a_i(x(n)) - \sum_{l=1}^{m} a_l(x(n)) h_l(t) + \sum_{j=1}^{k+m} \left(\sum_{l=1}^{m} h_l(t) \sigma_{lj} \right)^2 \\ & - \sum_{j=1}^{k+m} \sigma_{ij} \sum_{l=1}^{m} h_l(t) \sigma_{lj} \Bigg\} dt + h_i(t) \sum_{j=1}^{k+m} \left(\sigma_{ij} - \sum_{l=1}^{m} h_l(t) \sigma_{lj} \right) dw_j(t) \end{aligned} \tag{10}$$

Clearly h_i is a unique solution to (10) within the class of bounded processes. One can write then that for $n \le t < n+1$, $h_i(t) = \Gamma_t^i(x(n), h(n), W_n(\cdot))$, where $W_n(\cdot)$ is the trajectory of $W(s+n) - W(n)$ for $0 \le s \le 1$ and $\Gamma_t^i : R^k \times [0,1]^m \times C_0([0,1], R^{k+m}) \mapsto R$ is a measurable function and $C_0([0,1], R^{k+m})$ is the space of continuous trajectories on the time interval $[0,1]$, starting from 0 and taking values in R^{k+m}. Consequently

$$\frac{V^-(n+1)}{V(n)} = \exp \{ F(x(n), h(n), W_n(\cdot)) \} \tag{11}$$

where $h(n) = (h_1(n), \ldots, h_m(n))^T$ and

$$\begin{aligned} & F(x(n), h(n), W_n(\cdot)) = \sum_{i=1}^{m} \int_n^{n+1} a_i(x(n)) h_i(s) ds - \\ & \frac{1}{2} \sum_{j=1}^{k+m} \int_n^{n+1} \left(\sum_{i=1}^{m} h_i(s) \sigma_{ij} \right)^2 ds + \int_n^{n+1} \sum_{i=1}^{m} h_i(s) \sum_{j=1}^{k+m} \sigma_{ij} dw_j(s) \end{aligned} \tag{12}$$

Our purpose is to maximize for fixed $\Theta \in (0,1)$ the risk sensitized growth rate of our capital process given by the following cost functional

$$\begin{aligned} & \liminf_{t \to \infty} \frac{-2}{\Theta} t^{-1} \ln E \left\{ V(t)^{\frac{-\Theta}{2}} \right\} \\ & = \liminf_{t \to \infty} \frac{-2}{\Theta} t^{-1} \ln E \left\{ \exp \left\{ \frac{-\Theta}{2} \ln V(t) \right\} \right\} \end{aligned} \tag{13}$$

By Taylor expansion about $\Theta = 0$ of $\ln E\left\{\exp\left\{\frac{-\Theta}{2}\ln V(t)\right\}\right\}$ we obtain that

$$\frac{-2}{\Theta}\ln E\left\{\exp\left\{\frac{-\Theta}{2}\ln V(t)\right\}\right\} = E\left\{lnV(t)\right\} - \frac{\Theta}{4}Var\left(\ln V(t)\right) + O(\Theta^2) \tag{14}$$

so that the cost functional (13) can be interpreted as the long run growth rate of $V(t)$ diminished by the value of the risk, the measure of which is variance (Var) of $\ln V(t)$ multiplied by $\frac{\Theta}{2}$.
For a fixed positive Θ, maximization of the cost functional (13) is equivalent to minimization of

$$\liminf_{t\to\infty} t^{-1}\ln E\left\{\exp\left\{\frac{-\Theta}{2}\ln V(t)\right\}\right\} \tag{15}$$

In the paper we shall study both the case with and without transaction costs. As will be clear later, we show that optimal strategies maximizing the cost functional (13) depend on the levels of the factors $x(t)$ and in the case of proportional transaction costs depend additionally on the values of the process $h(t) = (h_1(t), \dots, h_m(t))^T$ at transaction times only.
The paper in some sense generalizes papers [1] and [2]. We allow more general factor dynamics comparing to [1], however to use discrete time risk sensitive Bellman equation we have to restrict ourselves to discrete time decisions. In the case of transaction costs the authors of [2] concentrate on the construction of the model skipping the proofs of the existence and uniqueness of so called risk sensitive quasivariational inequalities. In our paper using for discrete decision times and imposing a technical assumption we provide complete mathematical proofs. Our proofs are based on the local contraction property in the span norm shown in [4] and extended in [5] and [10].

2 The case without transaction costs

Notice first that if the are no transaction costs for positive integer n we have $V^-(n) = V(n)$. Consequently using (11) and (12), also (7) and boundedness of a_i for the cost functional (15) we have

$$\liminf_{t\to\infty} t^{-1}\ln E\left\{\exp\left\{\frac{-\Theta}{2}\ln V(t)\right\}\right\} \tag{16}$$

$$\liminf_{n\to\infty} n^{-1}\ln E\left\{\exp\left\{\frac{-\Theta}{2}\sum_{l=0}^{n-1}F(x(l), h(l), W_l(\cdot))\right\}\right\} \tag{17}$$

One can consider the following Bellman equation

$$e^{v(x)+\lambda} = \inf_h E\left\{e^{\frac{-\Theta}{2}F(x,h,W_0(\cdot))+v(g(x,W_0(1)))}\right\} \tag{18}$$

with
$g(x, W_0(\cdot)) = (g_1(x, W_0(1)), \dots, g_k(x, W_0(1)))^T$
and infimum considered over all $h = (h_1, \dots, h_m)^T \in H$. For $f \in C(R^k)$ define the following operator

$$Tf(x) = \inf_h \ln E\left\{e^{\frac{-\Theta}{2}F(x,h,W_0(\cdot))+f(g(x,W_0(1))}\right\} \tag{19}$$

By the proof of Lemma 2.3 of [3] one can notice that when

$$E\left\{|\frac{-\Theta}{2}F(x,h,W_0(\cdot)) + f(g(x,W_0(1))|e^{\frac{-\Theta}{2}F(x,h,W_0(\cdot))+f(g(x,W_0(1)))}\right\} < \infty \tag{20}$$

we have that

$$\begin{aligned} Tf(x) = \inf_h & \sup_{\mu \in P(C_0([0,1],R^{k+m}))} \left(\int_{C_0([0,1],R^{k+m})} \left\{\frac{-\Theta}{2}F(x,h,z(\cdot))\right.\right. \\ & \left.\left. + f(g(x,z(1))\right\} \mu(dz) - I(\mu,W)\right) \end{aligned} \tag{21}$$

where W denotes an $k+m$-Wiener measure on $C_0([0,1],R^{k+m})$ and for $\mu,\nu \in P(C_0([0,1],$ $R^{k+m}))$, we have $I(\mu,\nu) = \int_{C_0([0,1],R^{k+m})} \log \frac{d\nu}{d\mu} d\nu$, provided $\nu << \mu$ and $\log \frac{d\nu}{d\mu}$ is in $L^1(\nu)$, and $I(\mu,\nu) = +\infty$ in the other cases. Moreover the supremum in (21) is attained for the measure $\mu_{x,h,f}$ defined as follows

$$\mu_{x,h,f}(dz) = \frac{e^{\frac{-\Theta}{2}F(x,h,z(\cdot))+f(g(x,z(1)))}W(dz)}{E\left\{e^{\frac{-\Theta}{2}F(x,h,W_0(\cdot))+f(g(x,W_0(1)))}\right\}} \tag{22}$$

Let for $A \in \mathcal{B}(R^k)$

$$\bar{\mu}_{x,h,f}(A) = \frac{E\left\{1_A(g(x,W_0(1))e^{\frac{-\Theta}{2}F(x,h,W_0(\cdot))+f(g(x,W_0(1)))}\right\}}{E\left\{e^{\frac{-\Theta}{2}F(x,h,W_0(\cdot))+f(g(x,W_0(1)))}\right\}} \tag{23}$$

Denote by $\|\mu\|_{var}$ variation norm of a measure μ and by $\|f\|_{sp} := \sup_x f(x) - \inf_y f(y)$ the span norm of a function f.
We have the following auxiliary

Lemma 1 *Assume rank*$(\Lambda) = k$, *where* $\Lambda = (\lambda_{ij})$. *For a given* $M > 0$ *there is an* $0 < L(M) < 1$ *such that for* f_1, $f_2 \in C(R^k)$, x_1, $x_2 \in R^k$, h^1, $h^2 \in H$, *if* $\|f_1\|_{sp} \le M$ *and* $\|f_2\|_{sp} \le M$ *we have*

$$\|\bar{\mu}_{x_1,h^1,f_1} - \bar{\mu}_{x_2,h^2,f_2}\|_{var} \le 2L(M) \tag{24}$$

Proof. Assume that for sequences f_1^n, $f_2^n \in C(R^k)$, x_1^n, $x_2^n \in R^k$, $h^1(n)$, $h^2(n) \in H$, $A_n \in \mathcal{B}(R^k)$ we have $\|f_1^n\|_{sp} \le M$ and $\|f_2^n\|_{sp} \le M$ and

$$\bar{\mu}_{x_1^n,h^1(n),f_1^n}(A_n) - \bar{\mu}_{x_2^n,h^2(n),f_2^n}(A_n) \to 1 \tag{25}$$

Then $\bar{\mu}_{x_1^n,h^1(n),f_1^n}(A_n^c) \to 0$ and $\bar{\mu}_{x_2^n,h^2(n),f_2^n}(A_n) \to 0$. Using Schwarz inequality we have that

$$\begin{aligned} \bar{\mu}_{x,h,f}(A) \ge e^{-\|f\|_{sp}} \frac{E\left\{1_A(g(x,W_0(1))e^{\frac{-\Theta}{2}F(x,h,W_0(\cdot))}\right\}}{E\left\{e^{\frac{-\Theta}{2}F(x,h,W_0(\cdot))}\right\}} \ge e^{-\|f\|_{sp}} \\ \left(E\left\{1_A(g(x,W_0(1))\right\}\right)^2 \left(E\left\{e^{\frac{-\Theta}{2}F(x,h,W_0(\cdot))}\right\}\right)^{-1} \left(E\left\{e^{\frac{\Theta}{2}F(x,h,W_0(\cdot))}\right\}\right)^{-1} \end{aligned} \tag{26}$$

Since by boundedness of a

$$E\left\{\exp\left\{\frac{\pm\Theta}{2}F(x,h,W_0(\cdot))\right\}\right\} \le e^{\frac{\Theta}{2}\|a\|+(\frac{\Theta}{4}+\frac{\Theta^2}{8})\|\sigma\|^2(k+m)} \tag{27}$$

where $\|a\| = \sup_{x\in R^k}\sup_{i=1,\dots,m}|a_i(x)|$, and $\|\sigma\| = \sup_{i=1,\dots,m}\sup_{j=1,\dots,k+m}|\sigma_{ij}|$, from (25) and (26) we obtain that $\left(E\left\{1_{A_n^c}(g(x_1^n,W_0(1)))\right\}\right)\to 0$ and $\left(E\left\{1_{A_n}(g(x_2^n,W_0(1)))\right\}\right)\to 0$ which contradicts the assumption that b is bounded and the rank of the matrix Λ is equal to k. Consequently (24) holds.

□

Remark 1 *Notice that in te proof of Lemma 1 we are exploiting so called uniform ergodicity of the Markov process $x(n)$ which means that*

$$\sup_{A\in\mathcal{B}(R^k)}\sup_{x,x'\in R^k}|P(x,A)-P(x',A)|<1$$

where $P(x,\cdot)$ is a transition probability of the process $(x(n))$. There are various nice ergodic properties of uniformly ergodic processes, for details see section 3.3 of [7].

The main result of this section can be formulated as follows

Theorem 1 *For $\Theta < 1-L(M)$, where $M = \|a\| + (\frac{3}{4})\|\sigma\|^2(k+m)$, if $rank(\Lambda) = k$, there is a unique up to additive constant function $v\in C(R^k)$ and a constant λ which are solutions to the equation (18). The constant λ is the optimal value of the cost functional (15). Moreover an optimal strategy for (15) is to have $\frac{h_i^*(x(n))V(n)}{S_i(n)}$ i-th securities at time n, where $h_i^* : R^k \mapsto [0,1]$, for $i=1,\dots,m$ are functions for which infimum in (18) is attained.*

Proof. Notice first that by (27) the operator T transforms $C(R^k)$ into itself. Moreover as we shall show below T is a local contraction in $C(R^k)$ endowed with the span norm $\|\cdot\|_{sp}$. Assume for $f_1,\ f_2\in C(R^k)$, $x_1,\ x_2\in R^k$ we choose $h^1,\ h^2\in H$ such that for $l=1,2$

$$Tf_l(x_l) = \ln E\left\{e^{\frac{-\Theta}{2}F(x_l,h^l,W_0(\cdot))+f_l(g(x_l,W_0(1))}\right\} \tag{28}$$

Similarly as in the proof of Proposition 2 of [4] using (21) (by boundedness of a (20) is clearly satisfied) and (24)-(26) we obtain

$$\begin{aligned}
&Tf_1(x_2)-Tf_2(x_2)-(Tf_1(x_1)-Tf_2(x_1))\le\\
&\int_{C_0([0,1],R^{k+m})}[f_1(g(x_2,z(\cdot)))-f_2(g(x_2,z(\cdot)))]\mu_{x_2,h^2,f_1}(dz)\\
&-\int_{C_0([0,1],R^{k+m})}[f_1(g(x_1,z(\cdot)))-f_2(g(x_1,z(\cdot)))]\mu_{x_1,h^1,f_2}(dz)\\
&\le\int_{R^k}(f_1(y)-f_2(y))(\bar{\mu}_{x_2,h^2,f_1}-\bar{\mu}_{x_1,h^1,f_2})(dy)\le\\
&\frac{1}{2}\|f_1-f_2\|_{sp}\|\bar{\mu}_{x_2,h^2,f_1}-\bar{\mu}_{x_1,h^1,f_2}\|_{var}.
\end{aligned} \tag{29}$$

Consequently if $\|f_1\|_{sp} \leq M$ and $\|f_2\|_{sp} \leq M$, by Lemma 1 we have

$$\|Tf_1 - Tf_2\|_{sp} \leq L(M)\|f_1 - f_2\|. \tag{30}$$

Therefore there is at most one fixed point of the operator T in $C(R^k)$ endowed with the span norm $\|\cdot\|_{sp}$. Notice now that by (27) $\|T0\|_{sp} \leq \Theta M$. Since by (30) $\|T(T0) - T0\|_{sp} \leq L(M)\Theta M$ we have $\|T^2 0\|_{sp} \leq \Theta M(1 + L(M))$, and therefore by induction

$$\|T^n 0\|_{sp} \leq \Theta M(1 + L(M) + \ldots + (L(M))^{n-1}) \leq M \tag{31}$$

and $\|T^{n+1}0 - T^n 0\|_{sp} \leq (L(M))^n \|T0\|_{sp}$.
Finally, by (29) and (30) there is a unique (up to additive constant) function $v \in C(R^k)$ and a unique constant λ which are solutions to the equation (18). By Proposition 1 of [4] or Theorem 2.1 of [6] λ is the optimal value of the cost functional (18) and the portfolio strategy defined in the statement of the theorem is optimal.

□

Remark 2 *The above theorem has an important economic interpretation: the optimal portfolio strategy at time n depends only on the value of the factor levels $x(n)$. If $a_i =$ const. for $i = 1, \ldots, m$ the optimal strategy is to have at each time n the same fixed part of capital h_i^* invested in the i -th security, for $i = 1, \ldots, m$. The dependence of optimal portfolio strategy on the levels of the factors only, was shown in [1] for linear models, under the assumption about the lack of correlation between residuals in the security prices and factor equations.*

3 The case with proportional transaction costs

Assume now that there are proportional transaction costs. Let $h_i^-(t)$ be the part of the capital $V^-(t)$ which is invested at time t in the i -th security for $i = 1, \ldots, m$. If at time t we change the portfolio and invest $h_i(t)$ of $V^-(t)$ in the i -th security, for $i = 1, \ldots, m$, we have to pay proportional transaction costs of the form $d(h(t) - h^-(t))V^-(t)$, where $d : [-1, 1]^m \mapsto (0, 1)$ is a given continuous function. Consequently $V(t) = V^-(t) - d(h(t) - h^-(t))V^-(t)$. We shall assume as before that we change our portfolio only at times $n = 0, 1, \ldots,$ and we start with an initial portfolio $h_i^-(0) = h \in H$. Denote by τ_l, $l = 1, \ldots,$ random moments of succesive portfolio changes. Letting $c = -\ln(1 - d)$ we have

$$V(\tau_l) = V^-(\tau_l) e^{-c(h(\tau_l) - h^-(\tau_l))}. \tag{32}$$

Therefore by (11)

$$V^-(\tau_l) = V(\tau_{l-1}) \exp\left\{ \sum_{j=\tau_{l-1}}^{\tau_l - 1} F(x(j), h(j), W_j(\cdot)) \right\} \tag{33}$$

and the cost functional (15) takes the form

$$\liminf_{n\to\infty} n^{-1} \ln E\left\{\exp\left\{\frac{-\Theta}{2}\sum_{j=0}^{n-1} F(x(j),h(j),W_j(\cdot)) + \frac{\Theta}{2}\sum_{l=1}^{n} 1_{\tau_l\le n} c(h(\tau_l)-h^-(\tau_l))\right\}\right\} \tag{34}$$

Bellman equation corresponding to the above optimization problem is of the form: find $v \in C(R^k \times H)$ and a constant λ such that for $x \in R^k$ and $h \in H$

$$e^{v(x,h)+\lambda} = \min\left\{\ln E\left\{e^{\frac{-\Theta}{2}F(x,h,W_0(\cdot))+v(g(x,W_0(1)),\Gamma_1(x,h,W_0(\cdot))}\right\},\right.$$
$$\left.\inf_{h'} \ln E\left\{e^{\frac{-\Theta}{2}(F(x,h',W_0(\cdot))-c(h'-h))+v(g(x,W_0(1)),\Gamma_1(x,h',W_0(\cdot))}\right\}\right\} \tag{35}$$

with $\Gamma_1 = (\Gamma_1^1,\ldots,\Gamma_1^m)^T$ and Γ_t^i defined following the formula (10). Let $U = \{0,1\}\times H$, and for $a = (a_0,a_1)$ put $C(x,h,a,W_0(\cdot)) = F(x,h,W_0(\cdot))$ when $a_0 = 0$ and $C(x,h,a,W_0(\cdot)) = (F(x,a_1,W_0(\cdot)) - c(a_1 - h))$ when $a_0 = 1$. Then we can rewrite (35) as follows

$$e^{v(x,h)+\lambda} = \inf_{a\in U} E\left\{\exp\left\{\frac{-\Theta}{2}C(x,h,a,W_0(\cdot)) + 1_{a_0=0}v(g(x,W_0(1)),\Gamma_1(x,h,W_0(\cdot)))\right.\right.$$
$$\left.\left.+1_{a_0=1}v(g(x,W_0(1)),\Gamma_1(x,a_1,W_0(\cdot)))\right\}\right\}. \tag{36}$$

For $f \in C(R^k \times H)$ define now the following operator

$$\bar{T}f(x,h) = \inf_{a\in U} E\left\{\exp\left\{\frac{-\Theta}{2}C(x,h,a,W_0(\cdot)) + 1_{a_0=0}f(g(x,W_0(1)),\Gamma_1(x,h,W_0(\cdot)))\right.\right.$$
$$\left.\left.+1_{a_0=1}f(g(x,W_0(1)),\Gamma_1(x,a_1,W_0(\cdot)))\right\}\right\}. \tag{37}$$

By similar arguments as in (20) we can write that

$$\bar{T}f(x,h) = \inf_{a\in U} \sup_{\mu\in P(C_0([0,1],R^{k+m}))} \left(\int_{C_0([0,1],R^{k+m})} \left\{\frac{-\Theta}{2}C(x,h,a,z(\cdot))\right.\right.$$
$$+1_{a_0=0}f(g(x,z(1)),\Gamma_1(x,h,z(\cdot)))$$
$$\left.\left.+1_{a_0=1}f(g(x,z(1)),\Gamma_1(x,a_1,z(\cdot)))\right\}\mu(dz) - I(\mu,W)\right) \tag{38}$$

with W and $I(\mu,W)$ as in (21). The supremum is attained by the measure $\mu_{x,h,a,f}$ defined as follows

$$\mu_{x,h,a,f}(dz) = \left(\exp\left\{\frac{-\Theta}{2}C(x,h,a,z(\cdot)) + 1_{a_0=0}f(g(x,z(1)),\Gamma_1(x,h,z(\cdot)))\right.\right.$$
$$\left.\left.+1_{a_0=1}f(g(x,z(1)),\Gamma_1(x,a_1,z(\cdot)))\right\}W(dz)\right)$$
$$\left(E\left\{\exp\left\{\frac{-\Theta}{2}C(x,h,a,W_0(\cdot)) + 1_{a_0=0}f(g(x,W_0(1)),\Gamma_1(x,h,W_0(\cdot)))\right.\right.\right.$$
$$\left.\left.\left.+1_{a_0=1}f(g(x,W_0(1)),\Gamma_1(x,a_1,W_0(\cdot)))\right\}\right\}\right)^{-1}. \tag{39}$$

For $A \in \mathcal{B}(R^k \times H)$ define

$$\bar{\mu}_{x,h,a,f}(A) = E\left\{1_A(g(x, W_0(1)), 1_{a_0=0}\Gamma_1(x, h, W_0(\cdot)) + 1_{a_0=1}\Gamma_1(x, a_1, W_0(\cdot)))\right.$$
$$(\exp\left\{\frac{-\Theta}{2}C(x, h, a, W_0(\cdot)) + 1_{a_0=0}f(g(x, W_0(1)), \Gamma_1(x, h, W_0(\cdot)))\right.$$
$$\left.+1_{a_0=1}f(g(x, W_0(1)), \Gamma_1(x, a_1, W_0(\cdot)))\right\})$$
$$\left(E\left\{\exp\left\{\frac{-\Theta}{2}C(x, h, a, W_0(\cdot)) + 1_{a_0=0}f(g(x, W_0(1)), \Gamma_1(x, h, W_0(\cdot)))\right.\right.\right.$$
$$\left.\left.\left.+1_{a_0=1}f(g(x, W_0(1)), \Gamma_1(x, a_1, W_0(\cdot)))\right\}\right\}\right)^{-1}. \tag{40}$$

In what follows we would like to obtain an analog of Lemma 1 for measures $\bar{\mu}_{x,h,a,f}$. Unfortunately, since the process $h(s)$ does not have nice ergodic properties (i.e. if $h_i(0) = 0$ then until we change portfolio $h_i^-(s) = 0$) to continue our approach we have to impose a technical assumption

(T) *there is $\delta > 0$ such that at each time $n = 0, 1, \ldots$ we have that $h(n) \in H_\delta$*

where $H_\delta = \{h \in H : h_i \geq \delta \text{ for } i = 1, \ldots, m\}$. This means that at each possible decision time n, whenever $h^-(n) \notin H_\delta$ we are pressed to change portfolio choosing $h(n) \in H_\delta$.
Let

$$\Xi := \begin{pmatrix} \Lambda \\ \Sigma \end{pmatrix}$$

where $\Sigma := (\sigma_{ij})$ and let

$$U_\delta := \{a = (a_0, a_1) \in U : a_1 \in H_\delta\} \tag{41}$$

We have the following

Lemma 2 *Assume that $\mathrm{rank}(\Xi) = m + k$. Then, for a given $\delta < \frac{1}{m}$ and $M > 0$ there is an $0 < \bar{L}_\delta(M) < 1$ such that for f_1, $f_2 \in C(R^k \times H)$, x_1, $x_2 \in R^k$, h^1, $h^2 \in H_\delta$, a^1, $a^2 \in U_\delta$, if $\|f_1\|_{sp} \leq M$ and $\|f_2\|_{sp} \leq M$ we have*

$$\|\bar{\mu}_{x_1,h^1,a^1,f_1} - \bar{\mu}_{x_2,h^2,a^2,f_2}\|_{var} \leq 2\bar{L}_\delta(M) \tag{42}$$

Proof. We follow the arguments of the proof of Lemma 1 and therefore have to show that for a sequence $A_n \in \mathcal{B}(R^k \times H)$ and sequences x_1^n, $x_2^n \in R^k$, $h^1(n)$, $h^2(n) \in H_\delta$ the convergences

$$E\left\{1_{A_n^c}(g(x_1^n, W_0(1)), \Gamma_1(x_1^n, h^1(n), W_0(\cdot)))\right\} \to 0 \tag{43}$$

and

$$E\left\{1_{A_n}(g(x_2^n, W_0(1)), \Gamma_1(x_2^n, h^2(n), W_0(\cdot)))\right\} \to 0 \tag{44}$$

lead to a contradiction.
Let for $i = 1, \ldots, m$

$$\Psi_1^i(x,h,W_0(\cdot)) := \ln h_i + a_i(x) - \frac{1}{2}\sum_{j=1}^{k+m}\sigma_{ij}^2 + \sum_{j=1}^{k+m}\sigma_{ij}(w_j(1)-w_j(0)). \tag{45}$$

One can notice that by boundedness of a, b and $\ln h_i$ for $h \in H_\delta$ and nonsingularity of the matrix Ξ the convergences

$$E\left\{1_{D_n^c}(g(x_1^n, W_0(1)), \Psi_1(x_1^n, h^1(n), W_0(\cdot)))\right\} \to 0$$

and

$$E\left\{1_{D_n}(g(x_2^n, W_0(1)), \Psi_1(x_2^n, h^2(n), W_0(\cdot)))\right\} \to 0$$

where $\Psi_1 := (\Psi_1^1, \Psi_1^2, \ldots, \Psi_1^m)^T$, for any sequence $D_n \in \mathcal{B}(R^{k+m})$ are not allowed. The same holds therefore for g and projection of exponent of Ψ on H and suitable projections A_n of exponents of the last m coordinates of D_n on H. Consequently (44) and (45) contradict nonsingularity of Ξ and (42) holds.

$\square$

Remark 3 *Notice that if h^1 and $h^2 \in H$ only, (42) may not be satisfied. In that case $\ln h_i^1$ or $\ln h_i^2$ in (45) is not bounded.*

Assuming (T) for some $\delta > 0$ we shall show below the existence of the solutions to the Bellman equation of the form

$$e^{v(x,h)+\lambda} = \inf_{a\in U_\delta} E\left\{\exp\left\{\frac{-\Theta}{2}C(x,h,a,W_0(\cdot)) + 1_{a_0=0}v(g(x,W_0(1)),\Gamma_1(x,h,W_0(\cdot)))\right.\right.$$
$$\left.\left. +1_{a_0=1}v(g(x,W_0(1)),\Gamma_1(x,a_1,W_0(\cdot)))\right\}\right\}. \tag{46}$$

An analog of the operator $\bar{T}$ of (37) will be of the form

$$\bar{T}f(x,h) = \inf_{a\in U_\delta} E\left\{\exp\left\{\frac{-\Theta}{2}C(x,h,a,W_0(\cdot)) + 1_{a_0=0}f(g(x,W_0(1)),\Gamma_1(x,h,W_0(\cdot)))\right.\right.$$
$$\left.\left. +1_{a_0=1}f(g(x,W_0(1)),\Gamma_1(x,a_1,W_0(\cdot)))\right\}\right\}. \tag{47}$$

for $f \in C(R^k \times H)$.
We have the following

Theorem 2 *Assume (T) and that $rank(\Xi) = m+k$. Then for $\Theta < 1 - \bar{L}_\delta(M)$, where δ is of (T) and $M = \|c\|_{sp} + \|a\| + (\frac{3}{4})\|\sigma\|^2(k+m)$ there is a unique up to additive constant function $v \in C(R^k \times H)$ and a constant λ which are solutions to the equation (46). The constant λ is optimal value of the cost functional (34). Moreover an optimal strategy for (34) is to have $\frac{h_i^*(x(n),h^-(x(n)))V(n)}{S_i(n)}$ i-th securities at time n, where $h_i^* : R^k \times H \mapsto [0,1]$ for $i = 1,\ldots,m$ are functions for which the infimum in (46) is attained.*

Proof. We follow the arguments of the proof of theorem 1. Namely, by Lemma 2 $\bar{T}$ is a local contraction in $C(R^k \times H)$. Then it remains to notice that $\|\bar{T}^n 0\|_{sp} \leq M$. The optimality of λ and h_i^*, for $i = 1, \ldots, m$ can be shown using arguments of Theorem 2.1 of [6].

$\square$

Remark 4 *Economical interpretation of the Theorem 2 says that the optimal portfolio strategy under proportional transaction costs at time* n *depends only on the value of factors* $x(n)$ *and the value* $h^-(n)$ *of portfolio before transaction. Although we were able to show this under a technical assumption (T), we expect that the above property is true in general.*

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