Average Cost per Unit Time Control of Stochastic Manufacturing Systems: Revisited*

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Abstract

An optimal production planning for a stochastic manufacturing system is considered. The system consists of a single, failure-prone machine that produces a finite number of different products. The objective is to determine a rate of production that minimizes an average cost per unit time criterion where the demand is random. The results given in this paper are based on some large deviation estimates and the Hamilton-Jacobi-Bellman equations for convex functions.

Keywords: dynamic programming, stochastic manufacturing systems, large deviations, average cost per unit time.

1 Introduction

The control of stochastic manufacturing systems has been a topic for active research during the last two decades (cf. [10] and the references therein). The control problems have usually been formulated with a discounted cost criterion. An average cost per unit time problem was initially investigated in [1] where the problem is formulated for a machine with a single product, a piecewise linear cost function and a constant demand rate. Subsequently, the average cost per unit time problem was considered in a number of papers, but these approaches usually contained some heuristic arguments. However, some rigorous approaches have appeared (e.g., [9], [10]).

In this paper, a single machine, multiproduct model is considered with a more general convex cost function than has been treated in previous works and a random demand. In Section 2, the stochastic manufacturing system is formulated as a finite state Markov process with exponential random times between the changes of states. The inventory rate equation is the difference of a production rate and a random demand rate. The cost per unit time arises as the integral of a convex function of the inventory. In Section 3, a discounted

*Research supported by NSF Grant DMS 9971790 and KBN Grant 2P03A 015 15.

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cost control problem is formulated and solved. The convexity of the optimal cost is verified and the optimal cost is shown to be the unique solution of an equation. In Section 4, the boundedness of the solutions for the discounted cost control problem is considered. It is shown that the first entrance time of a state has an exponential moment using some large deviations estimates. The study of stronger exponential estimates allows us to overcome a gap from the papers [9] and [10]. In Section 5, the average cost per unit time solution is obtained by a vanishing discount approach. It is shown that the optimal cost satisfies a Hamilton-Jacobi-Bellman equation for convex functions that is motivated by [7]. In Section 6, an example of a one product machine is given where an optimal control is explicitly described.

2 Preliminaries

An m-product manufacturing system with a random production capacity, that is described by the real-valued process $(\alpha(t),\, t \geq 0)$, and with a random demand rate, that is described by the $\mathbb{R}^m$-valued process $(d(t),\, t \geq 0)$, is considered. The processes $(\alpha(t),\, t \geq 0)$ and $(d(t),\, t \geq 0)$ are independent and Markov with finite state spaces $A \subset \mathbb{R}_+$ and $D \subset \mathbb{R}_+^m$ respectively. It is assumed that the processes $(\alpha(t),\, t \geq 0)$ and $(d(t),\, t \geq 0)$ remain in states $\alpha \in A$ and $d \in D$ for random times that are exponential random variables with parameters $a_1(\alpha)$ and $a_2(d)$ respectively where $a_1 : A \to (0, \infty)$ and $a_2 : D \to (0, \infty)$ and then the processes make transitions to the states $\alpha' \in A$ and $d' \in D$ with the transition probabilities $P_1(\alpha, \alpha')$ and $P_2(d, d')$ respectively. The Markov processes $(\alpha(t),\, t \geq 0)$ and $(d(t),\, t \geq 0)$ have infinitesimal generators $Q_1 = (q^1(\alpha, \alpha');\, \alpha, \alpha' \in A)$ and $Q_2 = (q^2(d, d');\, d, d' \in D)$ that are given by the following expressions

$$q^1(\alpha, \alpha') = \begin{cases} a_1(\alpha)P_1(\alpha, \alpha') & \text{if } \alpha \neq \alpha' \\ -a_1(\alpha)(1 - P_1(\alpha, \alpha')) & \text{if } \alpha = \alpha' \end{cases}$$

and

$$q^2(\alpha, \alpha') = \begin{cases} a_2(d)P_2(d, d') & \text{if } d \neq d' \\ -a_2(d)(1 - P_2(d, d')) & \text{if } d = d'. \end{cases}$$

It is assumed that the Markov processes $(\alpha(t),\, t \geq 0)$ and $(d(t),\, t \geq 0)$ are recurrent and have unique invariant measures denoted by μ and Φ respectively.

Let $X_i(t)$ be the amount of inventory of the ith product at time $t \in \mathbb{R}_+$. The values of the process $(X_i(t),\, t \geq 0)$ are allowed to be negative to represent a backlog of the ith product. The random variable $u_i(t)$ is the production rate control for the ith product at time $t \in \mathbb{R}$. The control vector $u(t) = (u_1(t), \ldots, u_m(t))^{\mathrm{T}}$ is an admissible control if the process $(u_i(t),\, t \geq 0)$ is nonnegative, adapted to $\mathcal{F}_t = \sigma(\alpha(s), d(s),\, 0 \leq s \leq t)$ for each $i \in \{1, \ldots, m\}$, $\sum_{i=1}^m u_i(t) \leq \alpha(t)$ and $u_i(\cdot, \omega) : \mathbb{R}_+ \to \mathbb{R}_+$ is Lebesgue measurable for almost all $\omega \in \Omega$ and $i \in \{1, \ldots, m\}$. The inventory process $(X(t),\, t \geq 0)$ is an $\mathbb{R}^m$-valued process that is a solution of the stochastic ordinary differential equation

$$\begin{aligned}\frac{dX}{dt} &= u(t) - d(t) \\ X(0) &= x.\end{aligned} \tag{1}$$

It is assumed that

$$\sum_{\alpha \in A} \alpha\mu(\alpha) > \sum_{d \in D} (\sum_{i=1}^{m} d_i)\Phi(d) > 0 \tag{2}$$

where $d = (d_1, \dots, d_m)^{\mathrm{T}}$. The inequality (2) implies that for a sufficiently long (random) time horizon there is a surplus of production compared to the demand.

Let $\hat{C}$ be the following family of continuous functions

$$\hat{C} = \left\{ f \in C(\mathbb{R}^m, \mathbb{R}) : \lim_{|x| \to \infty} f(x)e^{-\gamma|x|} = 0 \text{ for each } \gamma > 0 \right\}.$$

For each admissible control $(u(t),\, t \geq 0)$ and $X(0) = x$, $\alpha(0) = \alpha$, $d(0) = d$ the average cost per unit time criterion is

$$J(x, \alpha, d, (u(t),\, t \geq 0)) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_{x,\alpha,d}^{(u(\cdot))} \left[\int_0^T h(X(s))ds \right] \tag{3}$$

where $h : \mathbb{R}^m \to \mathbb{R}_+$ is convex, $h \in \hat{C}$, $h(0) = 0$ and $h(x) \to \infty$ as $|x| \to \infty$.

3 Discounted Control Problem

To solve the average cost per unit time control problem (1), (3), initially some results are established for a discounted control problem. For a fixed $\beta > 0$, the discounted cost criterion is

$$J_\beta(x, \alpha, d, (u(t),\, t \geq 0)) = \mathbb{E}_{x,\alpha,d}^{(u(\cdot))} \left[\int_0^\infty e^{-\beta s} h(X(s))ds \right] \tag{4}$$

where x, α, d, $u(\cdot)$ and h are the same as for the average cost per unit time criterion (3). Let

$$w^\beta(x, \alpha, d) = \inf_{(u(\cdot))} J_\beta(x, \alpha, d, (u(t),\, t \geq 0)) \tag{5}$$

where the infimum is taken over all of the admissible controls.

Let $\partial w_z^\beta(x, \alpha, d)$ be the directional derivative of w^β in the direction $z \in \mathbb{R}^m$ for which the following limit exists

$$\partial w_z^\beta(x, \alpha, d) := \lim_{\delta \downarrow 0} \frac{w^\beta(x + \delta z, \alpha, d) - w^\beta(x, \alpha, d)}{\delta}.$$

Remark 1 It follows, for example, from the proof of Lemma 7.3 [10] that a convex function $f : \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz and, even more, if $|x - x'| < 1$ then

$$f(x) - f(x') \leq |x - x'| \left(f\left(x' + \frac{x - x'}{|x - x'|} \right) - f(x') \right). \tag{6}$$

Thus one can show (see also section 23 of [8]) that ∂f_z exists.

The following result verifies the convexity of w^β and thereby the existence of ∂w_z^β for each $z \in \mathbb{R}^m$ by Remark 1.

Theorem 1 *For each $\beta > 0$, $\alpha \in A$ and $d \in D$ the map*

$$w^\beta(\cdot, \alpha, d) : \mathbb{R}^m \to \mathbb{R} \tag{7}$$

is convex and is an element of $\hat{C}$. If h is strictly convex, then $w^\beta(\cdot, \alpha, d)$ is strictly convex. Furthermore, in the family of functions $f : \mathbb{R}^m \times A \times D \to \mathbb{R}$ and $f(\cdot, \alpha, d) \in \hat{C}$ for $\alpha \in A$ and $d \in D$, w^β is the unique solution of the following equations

$$\begin{aligned} w^\beta(x, \alpha, d) = \inf_{(u(\cdot))} \Bigg[\int_0^\infty & e^{-(\beta + a_1(\alpha) + a_2(d))s} \\ & \times \Bigg(h(X'(s)) + a_1(\alpha) \sum_{\alpha' \in A} w^\beta(X'(s), \alpha', d) P_1(\alpha, \alpha') \\ & \quad + a_2(d) \sum_{d' \in D} w^\beta(X'(s), \alpha, d') P_2(d, d') \Bigg) ds \Bigg] \end{aligned} \tag{8}$$

and

$$\begin{aligned} w^\beta(x, \alpha, d) = \inf_{(u(\cdot))} \Bigg[\int_0^T & e^{-(\beta + a_1(\alpha) + a_2(d))s} \\ & \times \Bigg(h(X'(s)) + a_1(\alpha) \sum_{\alpha' \in A} w^\beta(X'(s), \alpha', d) P_1(\alpha, \alpha') \\ & \quad + a_2(\alpha) \sum_{d' \in D} w^\beta(X'(s), \alpha, d') P_2(d, d') \Bigg) ds \\ & + e^{-(\beta + a_1(\alpha) + a_2(d))T} w^\beta(X'(T), \alpha, d) \Bigg] \end{aligned} \tag{9}$$

for each $T \geq 0$ with the infimum over all (deterministic) Lebesgue measurable functions $u(\cdot) \in U(\alpha)$, where for $\alpha \in A$

$$U(\alpha) := \left\{ v \in \mathbb{R}_+^m : \sum_{i=1}^m v_i \leq \alpha \right\} \tag{10}$$

where

$$X'(s) = x + \int_0^s u(t)dt - sd.$$

Furthermore, for each $x \in \mathbb{R}^m$, $\alpha \in A$ and $d \in D$

$$\begin{aligned}\inf_{v \in U(\alpha)} \left[\partial w^{\beta}_{v-\alpha}(x,\alpha,d)\right] - (\beta + a_1(\alpha) + a_2(\alpha))w^{\beta}(x,\alpha,d) \\ + h(x) + a_1(\alpha)\sum_{\alpha' \in A} w^{\beta}(x,\alpha',d)q^1(\alpha,\alpha') \\ + a_2(d)\sum_{d' \in D} w^{\beta}(x,\alpha,d')q^2(d,d') = 0. \qquad (11)\end{aligned}$$

Proof. The convexity or strong convexity (if h is strongly convex) of the map (7) follows from Lemma 3.1(i) [11]. By Remark 1 the map (7) is continuous.

Let

$$M := \max\{|\alpha - d^i| : \alpha \in A,\, d \in D,\, i = 1,2,\dots,m\}. \qquad (12)$$

For each $\gamma > 0$ and $\delta \in (0,\gamma)$ with $\delta < \beta/mM$ there is a $K > 0$ such that

$$h(X(s)) \le Ke^{\delta(|x|+smM)}$$

for each admissible control $(u(t),\, t \ge 0)$ and $s \in \mathbb{R}_+$ and

$$\begin{aligned}e^{-\gamma|x|}w^{\beta}(x,\alpha,d) &\le e^{-\gamma|x|}\mathbb{E}_{x,\alpha,d}\left[\int_0^{\infty} e^{-\beta s}h(X(s))ds\right] \\ &\le Ke^{-\gamma|x|}\int_0^{\infty} e^{-\beta s}e^{\delta(|x|+smM)}ds \\ &\le Ke^{-(\gamma-\delta)|x|}\frac{1}{\beta - \delta mM}.\end{aligned}$$

This inequality verifies that $w^{\beta}(\cdot,\alpha,d) \in \hat{C}$.
Let τ be the random time

$$\tau := \inf\{s \ge 0 : \alpha(s) \ne \alpha \text{ or } d(s) \ne d\}.$$

The equality (8) can be evaluated at τ to obtain

$$w^{\beta}(x,\alpha,d) = \inf_{(u(\cdot))}\left[\int_0^{\tau} e^{-\beta s}h(X'(s))ds + e^{-\beta\tau}w^{\beta}(X'(\tau),\alpha(\tau),d(\tau))\right]. \qquad (13)$$

For $\gamma > 0$ fixed let $\hat{C}_{\gamma}$ be given by

$$\begin{aligned}\hat{C}_{\gamma} = \Big\{ f : f : \mathbb{R}^m \times A \times D \to \mathbb{R},\ f(\cdot,\alpha,d) : \mathbb{R}^m \to \mathbb{R} \text{ is continuous for each} \\ \alpha \in A, \text{ and } d \in D, \text{ and } \|f\|_{\gamma} := \sup_{x \in \mathbb{R}^m}\max_{\alpha \in A}\max_{d \in D} e^{-\gamma|x|}|f(x,\alpha,d)| < \infty \Big\}.\end{aligned}$$

For $f \in \hat{C}_\gamma$ let S_β be the operator given by

$$S_\beta f(x, \alpha, d) = \inf_{(u(\cdot))} \left[\int_0^\tau e^{-\beta s} h(X'(s))ds + e^{-\beta\tau} f(X'(\tau), \alpha(\tau), d(\tau)) \right] \quad (14)$$

where the infimum is taken over all of the admissible controls. If $\gamma m M < \beta$ then $\|S_\beta f\|_\gamma < \infty$. If $f_1, f_2 \in \hat{C}_\gamma$ and $\gamma < \beta/mM$ then

$$\begin{aligned}
&\|S_\beta f_1 - S_\beta f_2\|_\gamma \le \\
&\qquad \sup_{x\in\mathbb{R}^m} \max_{\alpha\in A} \max_{d\in D} \sup_{(u(\cdot))} e^{-\gamma|x|} \mathbb{E}\left[e^{-\beta\tau} |(f_1 - f_2)(X'(\tau), \alpha(\tau), d(\tau))|\right] \\
&\qquad \le \sup_{x\in\mathbb{R}^m} \max_{\alpha\in A} \max_{d\in D} \sup_{(u(\cdot))} e^{-\gamma|x|} \mathbb{E}\left[e^{-\beta\tau} \|f_1 - f_2\|_\gamma e^{\gamma|X'(\tau)|}\right] \\
&\qquad \le \max_{\alpha\in A} \max_{d\in D} \|f_1 - f_2\|_\gamma \mathbb{E}_{\alpha,d}\left[e^{-\beta\tau} e^{\gamma m M \tau}\right].
\end{aligned}$$

Since $a_1 > 0$ and $a_2 > 0$ it follows that

$$\Lambda_\beta := \max_{\alpha\in A} \max_{d\in D} \mathbb{E}_{\alpha,d}\left[e^{-(\beta-\gamma m M)\tau}\right] < 1$$

and thus

$$\|S_\beta f_1 - S_\beta f_2\|_\gamma \le \Lambda_\beta \|f_1 - f_2\|_\gamma. \quad (15)$$

Therefore the equation (13) has a unique solution in $\hat{C}_\gamma$ for $\gamma < \beta/mM$. Let $f_0 \equiv 0$ and form the sequence $(S^n_\beta f_0,\ n \in \mathbb{N})$. Since h is nonnegative, the sequence $(S^n_\beta f_0(x, \alpha, d),\ n \in \mathbb{N})$ of real numbers is increasing for each $x \in \mathbb{R}^m$, $\alpha \in A$ and $d \in D$. Moreover $S^n_\beta f_0(x, \alpha, d)$ coincides with the optimal value of the discounted cost criterion restricted to the first n transitions of the pair $(\alpha(t), d(t))$ i.e.

$$S^n_\beta f_0(x, \alpha, d) = \inf_{((u\cdot))} \mathbb{E}^{(u(\cdot))}_{x,\alpha,d} \left[\int_0^{\tau_n} e^{-\beta s} h(X(s))ds \right],$$

where $\tau_0 := \tau$ and $\tau_n := \tau_{n-1} + \tau \circ \Theta_{\tau_{n-1}}$ for $n = 1, 2, ...$, with the shift operator Θ corresponding to the pair $(\alpha(t), d(t))$. Letting $n \to \infty$, $S^n_\beta f_0(x, \alpha, d)$ converges (since $a_1 > 0$ and $a_2 > 0$) to $w^\beta(x, \alpha, d)$ defined by (5). Consequently

$$w^\beta(x, \alpha, d) = \lim_{n\to\infty} S^n_\beta f_0(x, \alpha, d) \quad (16)$$

is the unique solution to (8) and (13) in $\hat{C}_\gamma$.

Using the methods of proof of Theorem 3.2.1 [5] (or the proof of Proposition 2 [7]), there is a $U(\alpha)$-valued Lebesgue measurable function $(\hat{u}(t),\ t \ge 0)$ for which the infimum on the right hand side of (8) is attained. By dynamic programming (the principle of optimality), w^β is a solution to (9) for each $T > 0$ with the infimum achieved by $(\hat{u}(t),\ t \ge 0)$. If $w^\beta(\cdot, \alpha, d) \in \hat{C}$ for each $\alpha \in A$ and $d \in D$ is a solution to (9) for each $T > 0$, then letting $T \to \infty$ in (9) gives (8).

If $u(s) = v \in U(\alpha)$ for all $s \in [0, T]$ then it follows that

$$\begin{aligned} w^\beta(x,\alpha,d) \le \int_0^T & e^{-(\beta+a_1(\alpha)+a_1(d))s} \\ & \times \Bigg(h(X'(s)) + a_1(\alpha) \sum_{\alpha' \in A} w^\beta(X'(s),\alpha',d) P_1(\alpha,\alpha') \\ & \quad + a_2(d) \sum_{d' \in D} w^\beta(X'(s),\alpha,d') P_2(d,d') \Bigg) ds \\ & + e^{-(\beta+a_1(\alpha)+a_2(d))T} w^\beta(X'(T),\alpha,d) \end{aligned} \tag{17}$$

and

$$\lim_{T \downarrow 0} \frac{w^\beta(X'(T),\alpha,d) - w^\beta(x,\alpha,d)}{T} = \partial w^\beta_{v-d}(x,\alpha,d).$$

Thus

$$\begin{aligned} \partial w^\beta_{v-d}(x,\alpha,d) & - (\beta + a_1(\alpha) + a_2(d)) w^\beta(x,\alpha,d) \\ & + h(x) + a_1(\alpha) \sum_{\alpha' \in A} w^\beta(x,\alpha',d) q^1(\alpha,\alpha') \\ & + a_2(d) \sum_{d' \in D} w^\beta(x,\alpha,d') q^2(d,d') \ge 0. \end{aligned} \tag{18}$$

Let $(\hat{u}(\cdot))$ be an optimal strategy such that there is equality in (17) for each $T \ge 0$ (the existence of which follows from Theorem 3.2.1 [5] or the proof of Proposition 2 [7]). Let $(T_n,\, n \in \mathbb{N})$ be a sequence of real numbers that decreases to zero and let $\hat{v}$ be given by

$$\lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} \hat{u}(s) ds = \hat{v}. \tag{19}$$

It is clear that $\hat{v} \in U(\alpha)$ and by (6)

$$\begin{aligned} \left| \frac{w^\beta(X'(T_n),\alpha,d) - w^\beta(x,\alpha,d)}{T_n} - \frac{w^\beta(x + T_n \hat{v},\alpha,d) - w^\beta(x,\alpha,d)}{T_n} \right| & \\ \le C \left| \frac{1}{T_n} \int_0^{T_n} \hat{u}(s) ds - \hat{v} \right|. & \end{aligned} \tag{20}$$

The right hand side of this inequality tends to 0 as $n \to \infty$ by (19). Replacing T by T_n and $u(\cdot)$ by $\hat{u}(\cdot)$ in (9), it follows that

$$\begin{aligned} \partial w^\beta_{\hat{v}-d}(x,\alpha,d) & - (\beta + a_1(\alpha) + a_2(d)) w^\beta(x,\alpha,d) \\ & + h(x) + a_1(\alpha) \sum_{\alpha' \in A} w^\beta(x,\alpha',d) q^1(\alpha,\alpha') \end{aligned}$$

$$+ a_2(d) \sum_{d' \in D} w^\beta(x, \alpha, d') q^2(d, d') = 0$$

so equation (11) is verified. □

Remark 2 Note that if $f : \mathbb{R}^m \to \mathbb{R}$ is convex then the directional derivative $\partial_\cdot f(x) : \mathbb{R}^m \to \mathbb{R}$ is convex for each $x \in \mathbb{R}^m$. Since $U(\alpha)$ is convex the infimum in (11) is attained. The proof of Theorem 1 verified this fact in another way.

Remark 3 Notice that at each point of the continuity of $u(t)$ the directional derivative in $(u(t) - d)$ is an infinitesimal operator (see [4]) corresponding to the process $X'(t)$.

Remark 4 In the definition of the directional derivative $\partial_z f(x)$, it is allowed that $z = 0$ and $\partial_0 f(x) = 0$. Let $x^\beta_{\alpha,d}$ be a point of minimum of $w^\beta(\cdot, \alpha, d)$ for $\alpha \in A$ and $d \in D$. It follows that

$$\partial w^\beta_z(x^\beta_{\alpha,d}, \alpha, d) \geq 0 \tag{21}$$

for each $z \in \mathbb{R}^m$. If $d \in U(\alpha)$ then

$$\inf_{v \in U(\alpha)} \left[\partial w^\beta_{v-d}(x, \alpha, d) \right] = 0. \tag{22}$$

If $m = 1$, then the point $x^\beta_{\alpha,d}$ for $\alpha \in A$, $d \in D$ defines an optimal strategy. Specifically by (11), if $\alpha(t) = \alpha$ and $d(t) = d$ then choose the control $\tilde{u}(t) = \alpha$ for $x(t) < x^\beta_{\alpha,d}$, $\tilde{u}(t) = \min(\alpha, d)$ for $x(t) = x^\beta_{\alpha,d}$ and $\tilde{u}(t) = 0$ for $x(t) > x^\beta_{\alpha,d}$. If h is not strictly convex then the points $x^\beta_{\alpha,d}$ may not be uniquely determined so in the following it is assumed that $x^\beta_{\alpha,d}$ is chosen so that it is the largest value in $\mathbb{R}$ for the infimum of $w^\beta(\cdot, \alpha, d)$.

4 An Analysis of the Boundedness of the Discounted Value Functions

To solve the average cost per unit time control problem by letting the discount rate approach 0 it is important to verify some boundedness properties of the discounted value functions. Let $\bar{\alpha} \in A$ and $\bar{d} \in D$ be fixed. Define $\bar{w}^\beta$ as follows

$$\bar{w}^\beta(x, \alpha, d) := w^\beta(x, \alpha, d) - w^\beta(0, \bar{\alpha}, \bar{d}). \tag{23}$$

In this section it is shown that the family $(\bar{w}^\beta(\cdot, \alpha, d),\ \beta > 0)$ is locally bounded. It is assumed that

$$\sum_{d \in D} d_i \Phi(d) > 0 \tag{24}$$

for each $i \in \{1, \ldots, m\}$. By the recurrency of the elements of D, the inequality (24) may not be satisfied for some i_0 only if all of the elements of D have the

i_0th coordinate equal to 0. This means that there is no demand for the i_0th product so that only products that satisfy (24) need to be considered.

To obtain an upper bound on an exponential moment of the first entrance time, the following large deviations estimates are important.

Proposition 1 *For each $\varepsilon > 0$ such that*

$$\varepsilon < \min\left\{\sum_{d\in D}(\sum_{i=1}^{m} d_i)\Phi(d) - \sum_{\alpha\in A}\alpha\mu(\alpha),\ \sum_{d\in D} d_i\Phi(d),\ i = 1, 2, \dots, m\right\}$$

there are constants $\rho > 0$ and $t_0 \geq 0$ such that for the processes $(y(t))$ and $(z_i(t), i \in \{1, \dots, m\})$ given by

$$y(t) = \int_0^t \left(\alpha(s) - \sum_{i=1}^{m} d_i(s)\right) ds \tag{25}$$

and

$$z_i(t) = \int_0^t d_i(s)ds \tag{26}$$

the following inequalities are satisfied for $t \geq t_0$

$$\max_{\alpha\in A}\max_{d\in D} P_{\alpha,d}\left(\frac{1}{t}y(t) \leq \varepsilon\right) \leq e^{-\rho t} \tag{27}$$

and

$$\max_{i=1,\dots,m}\max_{d\in D} P_d\left(\frac{1}{t}z_i(t) \leq \varepsilon\right) \leq e^{-\rho t}. \tag{28}$$

The verification of the inequalities (27) and (28) follows from the upper estimates for the empirical distributions (Theorem 3, [2]) and the assumptions (2) and (24). The methods of the proof of Theorem 3 in [3] can also be used.

The following result is an upper bound for an exponential moment of a first entrance time.

Proposition 2 *If $(x, \alpha, d) \in \mathbb{R}^m \times A \times D$ and $x' \in \mathbb{R}^m$ then there are an admissible control $(\bar{u}(\cdot))$, and constants $C > 0$, $\delta_0 \in (0, \rho)$ where ρ is given in Proposition 1 such that for $\delta \in (0, \delta_0)$*

$$\max_{\alpha\in A}\max_{d\in D}\mathbb{E}_{x,\alpha,d}[e^{\delta\tau}] \leq 1 + \delta C(1 + |x - x'|)e^{\delta C(1+|x-x'|)} \tag{29}$$

where $X(0) = x$ and

$$\tau = \inf\{s \geq 0 : X(s) = x'\}. \tag{30}$$

Proof. Initially assume that $x_i \leq x'_i$ for $i \in \{1, \dots, m\}$. Let $(\bar{u}(t) = \bar{u}_1(t), \dots, \bar{u}_m(t))$, $t \geq 0)$ be defined by

$$\bar{u}_i(t) = \begin{cases} \dfrac{x'_i - X_i(t)}{\displaystyle\sum_{j=1}^{m}(x'_j - X_j(t))} & \text{if } \alpha(t) < \displaystyle\sum_{i=1}^{m} d_i(t) \\ \dfrac{\left(\alpha(t) - \displaystyle\sum_{i=1}^{m} d_i(t)\right)(x'_i - X_i(t))}{\displaystyle\sum_{j=1}^{m}(x'_j - X_j(t))} + d_i & \text{if } \alpha(t) \geq \displaystyle\sum_{i=1}^{m} d_i(t) \end{cases} \tag{31}$$

for $t \leq \tau$ and $i \in \{1, \dots, m\}$.

For the control $\bar{u}$ given by (31) the random time τ can be expressed as

$$\tau = \inf \left\{ s \geq 0 : \mathbb{X}(s) = \sum_{i=1}^{m} x'_i \right\}$$

where $\mathbb{X}(s) = \sum_{i=1}^{m} X_i(s)$. For $s \leq \tau$ there is the random differential equation

$$\frac{d\mathbb{X}}{ds} = \alpha(s) - \sum_{i=1}^{m} d_i(s)$$

so that $\mathbb{X}(s) = \mathbb{X}(0) + y(s)$ where $y(s)$ is defined in (25). If $t \geq t_0$ and $\sum_{i=1}^{m}(x'_i - x_i) \leq \varepsilon t$ then by (27)

$$P(\tau \geq t) \leq P\left(y(t) \leq \sum_{i=1}^{m}(x'_i - x_i)\right) \leq e^{-\rho t}.$$

There is a $C > 0$ that does not depend on x, α, d, x' or δ such that

$$\begin{aligned} \mathbb{E}_{x,\alpha,d}[e^{\delta\tau}] &= 1 + \delta \int_0^\infty e^{\delta t} P(\tau \geq t) dt \\ &\leq 1 + \delta \left[C(1 + |x - x'|) e^{\delta C(1+|x-x'|)} + \frac{1}{\rho - \delta} \right]. \end{aligned} \tag{32}$$

Now assume that $x_i \leq x'_i$ is not satisfied for all $i \in \{1, \dots, m\}$. Let $I = \{i \in \{1, \dots, m\} : x_i > x'_i\}$ and $\bar{u}(t) = 0$ for $t \leq \tau'$ where

$$\tau' = \inf\{s \geq 0 : X_i(s) \leq x'_i \text{ for each } i \in \{1, \dots, m\}\}.$$

Using (28) it follows that

$$\begin{aligned} P(\tau' \geq t) &\leq P(X_i(t) \geq x'_i \text{ for each } i \in \{1, \dots, m\}) \\ &= P(z_i(t) \leq x_i - x'_i \text{ for each } i \in \{1, \dots, m\}) \end{aligned}$$

$$\leq \sum_{i\in I} P\left(\frac{1}{t}z_i(t) \leq \frac{|x_i - x_i'|}{t}\right) \leq me^{-\rho t} \tag{33}$$

for all $t \geq t_0$ and $|x - x'| \leq \varepsilon t$. Thus for some $C > 0$ that does not depend on x, α, d, x' and each $\gamma \in (0, \rho)$ the following inequality is satisfied

$$\begin{aligned}\mathbb{E}_{x,\alpha,d}[e^{\gamma\tau'}] &= 1 + \gamma \int_0^\infty e^{\gamma t} P(\tau \geq t)dt \\ &\leq 1 + \gamma \left[C(1 + |x - x'|)e^{\gamma C(1+|x-x'|)} + \frac{m}{\rho - \gamma}\right].\end{aligned} \tag{34}$$

Note that $|X(\tau') - x| \leq \tau' M\sqrt{m}$ so that $|X(\tau') - x'| \leq \tau M\sqrt{m} + |x - x'|$. Since $X_i(\tau') \leq x_i'$ for each $i \in \{1, \dots, m\}$, the control $\bar{u}$ defined by (31) is used for $t \geq \tau'$ and by (32) it follows that

$$\begin{aligned}&\mathbb{E}_{X(\tau'),\alpha(\tau'),d(\tau')}[e^{\delta\tau}] \\ &\quad \leq 1 + \delta \left[C(1 + \tau' M\sqrt{m} + |x - x'|)e^{\delta C(1+\tau' M\sqrt{m}+|x-x'|)} + \frac{1}{\rho - \delta}\right].\end{aligned}$$

If $\bar{\tau} = \tau' + \tau \circ \theta_{\tau'}$, where $\theta_{\tau'}$ is the translation of the paths by τ' then

$$\begin{aligned}\mathbb{E}_{x,\alpha,d}\left\{e^{\delta\bar{\tau}}\right\} &\leq \mathbb{E}_{x,\alpha,d}\Big\{e^{\delta\tau'} \\ &\quad + \delta\left[C(1 + \tau' M\sqrt{m} + |x - x'|)e^{\delta C(1+|x-x'|)}e^{\delta(CM\sqrt{m}+1)\tau'} + \frac{1}{\rho - \delta}e^{\delta\tau'}\right]\Big\}.\end{aligned}$$

For $\gamma \in (0, \rho)$ it is clear that

$$\mathbb{E}_{x,\alpha,d}[\tau' e^{\gamma\tau'}] = \int_0^\infty e^{\gamma s} P(\tau' \geq s)ds$$

and using (33) and (34) it follows that for $\delta \in (0, \rho/(CM\sqrt{m} + 1))$

$$\mathbb{E}_{x,\alpha,d}[e^{\delta\bar{\tau}}] \leq 1 + \delta[C'(1 + |x - x'|)e^{\delta C'(1+|x-x'|)}]. \qquad \square$$

The following result provides an upper bound on an exponential moment for a random time that is a refinement of the one in Proposition 2. Analogous results for an upper bound on the rth moment of this random time given in [9], [10] contain a gap.

Proposition 3 *For $m > 1$ let all of the entries of the transition matrices P_1 and P_2 be strictly positive. For each pair of points (x, α, d) and (x', α', d') in $\mathbb{R}^m \times A \times D$ there is an admissible control $\bar{u}$ and constants $C > 0$ and $\delta_0 \in (0, \rho)$ such that for $\delta \in (0, \delta_0)$*

$$\mathbb{E}_{x,\alpha,d}[e^{\delta\tau}] \leq 1 + \delta C(1 + |x - x'|)e^{\delta C(1+|x-x'|)} \tag{35}$$

where

$$\tau = \inf\{s \geq 0 : X(s) = x',\, \alpha(s) = \alpha',\, d(s) = d'\} \tag{36}$$

Proof. Initially let $m = 1$. By the ergodicity of $(\alpha(t),\, t \geq 0)$ and $(d(t),\, t \geq 0)$, there are constants $t_1 > 0$ and $\varepsilon > 0$ such that for $t \geq t_1$

$$\min_{\alpha,\alpha' \in A} \min_{d,d' \in D} P_{\alpha,d}(\alpha(t) = \alpha', d(t) = d') \geq \varepsilon. \tag{37}$$

Let τ_1 and σ_1 be the random times given by

$$\begin{aligned} \tau_1 &= \inf\{s \geq 0 : X(s) = x'\} \\ \sigma_1 &= \inf\{s \geq \tau_1 : X(s) = x' + t_1 M\} \end{aligned}$$

where M is given in (12). By induction, define τ_{n+1} and σ_{n+1} for $n \in \mathbb{N}$ as

$$\begin{aligned} \tau_{n+1} &= \inf\{s \geq \sigma_n : X(s) = x'\} \\ \sigma_{n+1} &= \inf\{s \geq \tau_{n+1} : X(s) = x' + t_1 M\}. \end{aligned}$$

The control $\bar{u}(t)$ for $t \leq \tau_1$ coincides with the control used in Proposition 2 and for $n \in \mathbb{N}$

$$\bar{u}(t) = \begin{cases} \alpha(t) & \text{if } t \in (\tau_n, \sigma_n] \\ 0 & \text{if } t \in (\sigma_n, \tau_{n+1}]. \end{cases}$$

Let $\tilde{\tau}$ be the random time given by

$$\tilde{\tau} = \inf\{\tau_i : \alpha(\tau_i) = \alpha',\, d(\tau_i) = d'\}. \tag{38}$$

By (37), the probability that $\alpha(\tau_{i+1}) = \alpha'$ and $d(\tau_{i+1}) = d'$ regardles of $\alpha(\sigma_i)$ and $d(\sigma_i)$ is greater than some $\varepsilon' > 0$, so by iteration as in [10] it follows that

$$P(\tilde{\tau} > \tau_n) \leq (1 - \varepsilon')^{n-1}$$

for $n \in \mathbb{N}$.

Now let $m > 1$. Use the control $\bar{u}$ described in the proof of Proposition 2 for $t \in [0, \tau_1]$ where $\tau_1 = \inf\{s \geq 0 : X(s) = x'\}$ and then let $\bar{u}(t) = 0$ for $t \in (\tau_1, \sigma_1]$ where $\sigma_1 = \inf\{s \geq \tau_1 : |X(s) - x'| = 1\}$. Starting from $(X(\sigma_1), \alpha(\sigma_1), d(\sigma_1))$ again use $\bar{u}$ for $t \in (\sigma_1, \tau_2]$ where $\tau_2 = \inf\{s \geq \sigma_1 : X(s) = x'\}$. For $n \in \mathbb{N}$ let

$$\begin{aligned} \tau_{n+1} &= \inf\{s \geq \sigma_n : X(s) = x'\} \\ \sigma_n &= \inf\{s > \tau_n : |X(s) - x'| = 1\}. \end{aligned}$$

Let $\bar{u}(t) = 0$ for $t \in (\tau_n, \sigma_n]$ and let $\bar{u}(t)$ be as in the proof of Proposition 2 for $t \in (\sigma_n, \tau_{n+1}]$.

For $n \in \mathbb{N}$ define τ'_n as

$$\tau'_n = \inf\{s \geq \tau_n : (\alpha(s), d(s)) \neq (\alpha(\tau_n), d(\tau_n))\}$$

and define $\tilde{\tau}$ as

$$\tilde{\tau} = \inf\{\tau'_i : \alpha(\tau'_i) = \alpha',\, d(\tau'_i) = d'\}. \tag{39}$$

Since all of the entries of the transition matrices P_1 and P_2 are strictly positive, there is an $\varepsilon' > 0$ such that

$$P(\tilde{\tau} > \tau'_n) \leq (1 - \varepsilon')^{n-1}.$$

By the definitions of τ and $\tilde{\tau}$ in (38) and (39) it is clear that $\tau < \tilde{\tau}$. To complete the proof it suffices to show that for some $\delta > 0$ the inequality (35) is satisfied for τ replaced by $\tilde{\tau}$.
The following inequality is straightforward

$$\begin{aligned}
\mathbb{E}_{x,\alpha,d}[e^{\delta\tilde{\tau}}] &= 1 + \delta \int_0^\infty e^{\delta s} P(\tilde{\tau} \geq s) ds \\
&= 1 + \delta \sum_{k=1}^\infty \mathbb{E}_{x,\alpha,d} \left[\int_{\tau_{n-1}}^{\tau_n} e^{\delta s} P(\tilde{\tau} \geq s) ds \right] \\
&\leq \mathbb{E}_{x,\alpha,d}[e^{\delta\tau_2}] + \delta \sum_{n=3}^\infty \mathbb{E}_{x,\alpha,d} \left[\int_{\tau_{n-1}}^{\tau_n} e^{\delta s} P(\tilde{\tau} \geq \hat{\tau}_{n-2}) ds \right] \\
&\leq \mathbb{E}_{x,\alpha,d}[e^{\delta\tau_2}] \\
&\quad + \sum_{n=3}^\infty \mathbb{E}_{x,\alpha,d} \Big[e^{\delta\tau_{n-1}} \mathbb{E}_{X(\tau_{n-1}),\alpha(\tau_{n-1}),d(\tau_{n-1})}[e^{\delta(\tau_n - \tau_{n-1})} - 1] \\
&\qquad\qquad \times (1 - \varepsilon')^{n-3} \Big] \qquad (40)
\end{aligned}$$

where $\tau_0 = 0$ and

$$\hat{\tau}_n = \begin{cases} \tau_n & \text{if } m = 1 \\ \tau'_n & \text{if } m > 1. \end{cases}$$

By Proposition 2, it is clear that it suffices to verify the convergence of the series on the right hand side of the inequality (40).
By Proposition 2, there is a constant $C > 0$ such that

$$\max_{\tilde{\alpha}\in A} \max_{\tilde{d}\in D} \mathbb{E}_{x',\tilde{\alpha},\tilde{d}}[e^{\delta\bar{\tau}_1} - 1] \leq \delta C e^{\delta C}$$

where $\bar{\tau}_1 = \inf\{s \geq \sigma_1 : X(s) = x'\}$ and

$$\sigma_1 = \begin{cases} \inf\{s \geq 0 : X(s) = x' + t_1 M\} & \text{if } m = 1 \\ \inf\{s \geq 0 : |X(s) - x'| = 1\} & \text{if } m > 1. \end{cases}$$

By Proposition 2

$$\begin{aligned}
\mathbb{E}_{x,\alpha,d}[e^{\delta\tau_{n+2}}] &= \mathbb{E}_{x,\alpha,d} \left[e^{\delta\tau_{n+1}} \mathbb{E}_{X(\tau_{n+1}),\alpha(\tau_{n+1}),d(\tau_{n+1})}[e^{\delta(\tau_{n+1}-\tau_n)}] \right] \\
&\leq \mathbb{E}_{x,\alpha,d}[e^{\delta\tau_{n+1}}](1 + \delta C e^{\delta C}) \\
&\leq \mathbb{E}_{x,\alpha,d}[e^{\delta\tau_1}](1 + \delta C e^{\delta C})^{n+1}.
\end{aligned}$$

If $(1+\delta Ce^{\delta C})(1-\varepsilon') < 1$, then the series on the right hand side of the inequality (40) conveges and the inequality (35) is verified. □

Remark 5 Note that it is necessary to choose $\delta > 0$ sufficiently small to obtain the inequality (35). In [9] and [10], the authors consider a bound on $\mathbb{E}[\tau^r]$ but their verifications need to be modified. It seems that to obtain this latter bound it is necessary to obtain a better approximation for τ such as given in the proof of Proposition 3.

Remark 6 It is clear from the proof of Proposition 3 that if the demand process $(D(t),\, t \geq 0)$ is a constant, then the requirement that the entries of the transition matrices P_1 and P_2 are positive can be removed. In this case, let $\sigma_n = \inf\{s \geq \tau_n : X(s) = x' + t_1 d\}$ and using no control there is the equality $X(\tau_{n+1}) = x'$ where $\tau_{n+1} = \sigma_n + t_1$. Then the methods in [9], [10] can be used.

By Remark 6 for $m > 1$ it can be assumed that either the entries of P_1 and P_2 are positive or the demand is a constant.
Now a boundedness property of the family of optimal discounted costs is given.

Theorem 2 *There is a $\beta_0 > 0$ such that if $\beta \in (0, \beta_0)$ then $(\beta w^\beta(0, \bar{\alpha}, \bar{d},\, \beta > 0)$ is bounded. Furthermore, for $c_1 > 0$ there is a $c_2 > 0$ such that for each $\beta > 0$*

$$\sup_{|x| \leq c_1} \max_{\alpha \in A} \max_{d \in D} |\bar{w}^\beta(x, \alpha, d)| \leq c_2 \tag{41}$$

where $\bar{w}^\beta(x, \alpha, d) = w^\beta(x, \alpha, d) - w^\beta(0, \bar{\alpha}, \bar{d})$.

Proof. Let τ_0 be the random time given by

$$\tau_0 = \inf\{s > 0 : X(s) = 0,\, \alpha(s) = \bar{\alpha}, d(s) = \bar{d}\}.$$

By Proposition 3, there is a control $\bar{u}(\cdot)$ such that

$$\mathbb{E}_{0,\bar{\alpha},\bar{d}}[e^{\delta \tau_0}] < \infty$$

for $\delta > 0$ sufficiently small. By dynamic programming, it follows that

$$w^\beta(0, \bar{\alpha}, \bar{d}) \leq \mathbb{E}_{0,\bar{\alpha},\bar{d}} \left[\int_0^{\tau_0} e^{\beta s} h(X(s)) ds + e^{-\beta \tau_0} w^\beta(0, \bar{\alpha}, \bar{d}) \right].$$

Since

$$h(X(s)) \leq \|h\|_\delta e^{\gamma s M \sqrt{m}}$$

with $\| \cdot \|_\gamma$ and M given in the proof of Theorem 1, it follows that for $\gamma < \delta(M\sqrt{m})^{-1}$

$$0 \leq w^\beta(0, \bar{\alpha}, \bar{d}) \mathbb{E}_{0,\bar{\alpha},\bar{d}}(1 - e^{-\beta \tau_0}) \leq C$$

where $C > 0$ does not depend on β. Thus the boundedness of $(\beta w^\beta(0, \bar{\alpha}, \bar{d}),\, \beta > 0)$ can be verified as in the proof of Theorem 3.2 [10].
By Proposition 3, there is a control $\bar{u}(\cdot)$ such that

$$\mathbb{E}_{x,\alpha,d}[e^{\delta \tau_0}] \leq 1 + \delta C(1 + |x|) e^{\delta C(1+|x|)}.$$

Moreover

$$w^\beta(x, \alpha, d) \leq \mathbb{E}_{x,\alpha,d} \left[\int_0^{\tau_0} h(X(s)) ds + w^\beta(0, \bar{\alpha}, \bar{d}) \right].$$

Since

$$h(X(s)) \leq \|h\|_\gamma e^{\gamma(|x|+sM\sqrt{m})}$$

with $\gamma \leq \delta(M\sqrt{m})^{-1}$, it follows that

$$\bar{w}^\beta(x,\alpha,d) \leq C'(1+|x|)e^{C'(1+|x|)}.$$

By a similar procedure with $\tau = \inf\{s \geq 0 : X(s) = x,\, \alpha(s) = \alpha,\, d(s) = d\}$ it follows that

$$w^\beta(0,\bar{\alpha},\bar{d}) - w^\beta(x,\alpha,d) \leq C'(1+|x|)e^{C'(1+|x|)}$$

and the verification of (41). □

5 A Vanishing Discount Approach

From equation (9) it follows that for each $T > 0$

$$\begin{aligned}
\bar{w}^\beta(x,\alpha,d) = \inf_{(u(\cdot))} \Bigg[\int_0^T & e^{-(\beta+a_1(\alpha)+a_1(d))s} \\
& \times \Bigg(h(X'(s)) - \beta w^\beta(0,\bar{\alpha},\bar{d}) \\
& \qquad + a_1(\alpha) \sum_{\alpha' \in A} \bar{w}^\beta(X'(s),\alpha',d) P_1(\alpha,\alpha') \\
& \qquad + a_2(d) \sum_{d' \in D} \bar{w}^\beta(X'(s),\alpha,d') P_2(d,d') \Bigg) ds \\
& + e^{-(\beta+a_1(\alpha)+a_2(d))T} \bar{w}^\beta(X'(T),\alpha,d) \Bigg]
\end{aligned} \tag{42}$$

By Theorem 2 and Remark 1, the family $(\bar{w}^\beta(x,\alpha,d),\, \beta > 0)$ is bounded and equicontinuous on each bounded set so by the Arzela-Ascoli Theorem there are a function $w(x,\alpha,d)$ and a sequence $(\beta_n,\, n \in \mathbb{N})$ decreasing to 0 such that

$$\lim_{n\to\infty} \bar{w}^{\beta_n}(x,\alpha,d) = w(x,\alpha,d)$$

where the convergence is uniform on bounded sets. By the boundedness of $(\beta w^\beta(0,\bar{\alpha},\bar{d}),\, \beta > 0)$ there is a sequence $(\beta_n,\, n \in \mathbb{N})$ that decreases to 0 such that

$$\lim_{n\to\infty} \beta_n w^{\beta_n}(0,\bar{\alpha},\bar{d}) = \lambda$$

and by (41) for each $(x,\alpha,d) \in \mathbb{R}^m \times A \times D$

$$\lim_{n\to\infty} \beta_n w^\beta(x,\alpha,d) = \lambda.$$

Letting $(\beta_n,\, n \in \mathbb{N})$ be the above sequence it follows that

$$w(x,\alpha,d) = \inf_{(u(\cdot))} \Bigg[\int_0^T e^{-(a_1(\alpha)+a_1(d))s} \times \Bigg(h(X'(s)) - \lambda + a_1(\alpha) \sum_{\alpha' \in A} w(X'(s),\alpha',d) P_1(\alpha,\alpha') + a_2(d) \sum_{d' \in D} w(X'(s),\alpha,d') P_2(d,d') \Bigg) ds + e^{-(a_1(\alpha)+a_2(d))T} w(X'(T),\alpha,d) \Bigg]. \tag{43}$$

The function $w(\cdot,\alpha,d)$ is a limit of the convex functions so it is convex and thereby continuous.

The following theorem shows that (λ, w) satisfies a Hamilton-Jacobi-Bellman equation.

Theorem 3 *The pair (λ, w) given above satisfies the following equation*

$$\inf_{v \in U(\alpha)} \Bigg[\partial w_{v-d}(x,\alpha,d) - (a_1(\alpha)+a_1(d)) w(x,\alpha,d) + h(x) - \lambda + a_1(\alpha) \sum_{\alpha' \in A} w(x,\alpha',d) q^1(\alpha,\alpha') + a_2(d) \sum_{d' \in D} w(x,\alpha,d') q^2(d,d') = 0 \Bigg]. \tag{44}$$

Forthermore, for each $(x,\alpha,d) \in \mathbb{R}^m \times A \times D$ and admissible control $(u(\cdot))$ the following inequality is satisfied

$$\lambda \leq J(x,\alpha,d,(u(t),\, t \geq 0)). \tag{45}$$

If an admissible control $(\hat{u}(\cdot))$ satisfies for almost all $t \geq 0$ the following equation

$$\partial w_{\hat{u}(t)-d(t)}(X(t),\alpha(t),d(t)) - (a_1(\alpha(t))+a_2(d(t))) w(X(t),\alpha(t),d(t)) + h(X(t)) - \lambda + a_1(\alpha(t)) \sum_{\alpha' \in A} w(X(t),\alpha',d(t)) q^1(\alpha(t),\alpha') + a_2(d) \sum_{d' \in D} w(X(t),\alpha(t),d') q^2(d(t),d') = 0. \tag{46}$$

and

$$\limsup_{T\to\infty} \frac{1}{T} \mathbb{E}^{(\hat{u}(\cdot))}_{x,\alpha,d} [w(X(T),\alpha(T),d(T))] \geq 0 \tag{47}$$

then

$$\lambda = J(x,\alpha,d,(\hat{u}(t),\, t \geq 0)).$$

Proof. It follows from the proof of Theorem 3.2.1 [5] that there is an optimal deterministic control $(\hat{u}(\cdot))$ such that the infimum on the right hand side of (43) is attained. By the principle of optimality this control does not depend on T. Proceeding as in the proof of Theorem 1 it follows that the equality (44) is satisfied. Using a Tauberian theorem as in the proof of Theorem 4.3.(ii) [9] it follows that the inequality (45) is satisfied. For a control $(\hat{u}(\cdot))$ satisfying (46) it follows by Dynkin's formula (see Remark 3) that

$$w(x,\alpha,d) = \mathbb{E}^{\hat{u}(\cdot))}_{x,\alpha,d}\left[\int_0^T (h(X(s)) - \lambda)ds + w(X(T),\alpha(T),d(T))\right].$$

By the inequality (47) it follows that

$$\lambda \geq J(x,\alpha,d,(\hat{u}(t),\, t \geq 0))$$

and by the inequality (45) the optimality of $(\hat{u}(\cdot))$ is verified. □

Remark 7 The inequality (47) is difficult to verify if $m > 1$. If $m = 1$ then there is an explicit form for an optimal control as in Lemmas 5.1 and 5.2 [10]. In the next section an alternative approach for determining an optimal control is given using some methods of [6].

6 Optimal Controls for a One Product Machine

By Remark 4, an optimal control for the discounted cost J_β is completely determined by the points $x^\beta_{\alpha,d}$ for $\alpha \in A$ and $d \in D$. The asymptotics of this family of points for $\beta \downarrow 0$ is now investigated. Initially it is shown that this family is bounded. This can be compared to Proposition 1 and Lemma 1 in [6].

Proposition 4 *There is a $Q > 0$ such that*

$$x^\beta_{\alpha,d} \in [0,Q] \qquad \text{for all } \alpha \in A,\ d \in D \text{ and } \beta \in (0,1).$$

Proof. If $x^\beta_{\alpha,d} < 0$ then the optimal control is modified by letting $u(t) = 0$ for $x(t) \geq 0$, $u(t) = \min(\alpha,d)$ for $x(t) = 0$ and $u(t) = \alpha$ for $x(t) < 0$ if $\alpha(t) = \alpha$ and $d(t) = d$. Since $h(0) = 0$ is the minimal value of h, the value of J_β for this modified cost is not greater than the cost for the optimal control. Therefore $x^\beta_{\alpha,d}$ is not the largest infimum point of $w^\beta(\cdot,\alpha,d)$ which is assumed in Remark 4. Thus, $x^\beta_{\alpha,d} \geq 0$ for $\alpha \in A$, $d \in D$ and $\beta \in (0,1)$.

Let $y(t)$ and $K_{\alpha,d}$ be given by

$$y(t) = \int_0^t (\alpha(s) - d(s))ds$$

and

$$K_{\alpha,d}(N) = \mathbb{E}\left[\int_0^\tau h((y(s)+N)^-)ds\right]$$

for $\tau = \inf\{s \geq 0 : y(s) \geq 0\}$. By (29) (cf. Lemma 3 [6])

$$\lim_{N\to\infty} K_{\alpha,d}(N) = 0. \tag{48}$$

Now assume that $x^{\beta_n}_{\hat{\alpha},\hat{d}} = \max\{x^{\beta_n}_{\alpha,d};\ \alpha \in A,\ d \in D\} \to \infty$ as $\beta_n \downarrow 0$. Let $N^{\beta_n} = x^{\beta_n}_{\hat{\alpha},\hat{d}} - L$ for some $L \in (0, x^{\beta_n}_{\hat{\alpha},\hat{d}})$.

Each time the optimality controlled surplus, production capacity-demand, reaches $(N^{\beta_n}, \hat{\alpha}, \hat{d})$ with probability at least $\delta > 0$, the path $X(\cdot)$ reaches the point $x^{\beta_n}_{\hat{\alpha},\hat{d}}$ before returning to N^{β_n}. Let $n_0 \in \mathbb{N}$ be such that for $n \geq n_0$ (using (48)) there is the inequality

$$\max_{\alpha\in A}\max_{d\in D} K_{\alpha,d}(N^{\beta_n})[(1-\delta)+(1-\delta)^2+\cdots] < h(L) \leq h(x^{\beta_n}_{\hat{\alpha},\hat{d}}) - h(N^{\beta_n}).$$

Then modify the optimal control in the following way: Each time the process $(X(t), \alpha(t), d(t),\ t \geq 0)$ reaches $(N^{\beta_n}, \hat{\alpha}, \hat{d})$ it is stopped at $(N^{\beta_n}, \cdot, \cdot)$. When $X(\cdot)$ falls below N^{β_n}, the full production capacity is used to achieve N^{β_n} and when $X(\cdot)$ meets the path of the optimally controlled surplus, $X(\cdot)$ is controlled in the optimal way. This modified control is better than the assumed optimal control. Thus

$$\sup_{\beta\in(0,1)} \max\{x^{\beta}_{\alpha,d} : \alpha \in A, d \in D\} < \infty. \qquad \square$$

By Proposition 4 there exist $x_{\alpha,d}$ for $\alpha \in A$ and $d \in D$ such that for the sequence $(\beta_n,\ n \in \mathbb{N})$ decreasing to 0,

$$\lim_{n\to\infty} x^{\beta_n}_{\alpha,d} = x_{\alpha,d}$$

for $\alpha \in A$ and $d \in D$. Clearly $x_{\alpha,d} \in [0, Q]$.

Let

$$\tilde{w}^{\beta}(x,\alpha,d) := w^{\beta}(x,\alpha,d) - \inf_{x'\in\mathbb{R}}\min_{\alpha'\in A}\min_{d\in D} w^{\beta}(x',\alpha',d'). \tag{49}$$

Since $x_{\alpha,d} \in [0, Q]$, by Proposition 3 there is an analogue of Theorem 2 for $\tilde{w}^{\beta}$, that is, for some $\beta_0 > 0$

$$\beta \inf_{x'\in\mathbb{R}}\min_{\alpha'\in A}\min_{d'\in D} \tilde{w}^{\beta}(x',\alpha',d')$$

is bounded for $\beta \in (0, \beta_0)$ and the family $(\tilde{w}^{\beta}(\cdot,\alpha,d),\ \beta > 0)$ is locally bounded. By the convexity of $\tilde{w}^{\beta}(\cdot,\alpha,d)$, there is local equicontinuity. Thus there are $w(x,\alpha,d)$, λ and a subsequence of $(\beta_n,\ n \in \mathbb{N})$ (that is denoted as $(\beta_n,\ n \in \mathbb{N})$ for notational simplicity) such that

$$\lim_{n\to\infty} \beta_n \inf_{x'\in\mathbb{R}}\min_{\alpha'\in A}\min_{d'\in D} \tilde{w}^{\beta_n}(x',\alpha',d') = \lambda \tag{50}$$

and

$$\lim_{n\to\infty} \tilde{w}^{\beta_n}(x,\alpha,d) = w(x,\alpha,d) \tag{51}$$

where the convergence is uniform on bounded sets. By Proposition 3 and the fact that $x_{\alpha,d} \in [0, Q]$ for $\alpha \in A$ and $d \in D$ it follows that

$$\lim_{n\to\infty} \beta_n w^{\beta_n}(x, \alpha, d) = \lambda \tag{52}$$

for each $(x, \alpha, d) \in \mathbb{R} \times A \times D$.

Thus for (λ, w) defined by (50) and (51) all of the results of Theorem 3 are satisfied. By the definition of $\tilde{w}^\beta$, $w(x, \alpha, d) \geq 0$ so the inequality (47) is trivially satisfied and $x_{\alpha,d}$ is a minimum of $w(\cdot, \alpha, d)$ for each $(\alpha, d) \in A \times D$. By Theorem 3, the following result is verified.

Theorem 4 *Let $m = 1$. The control*

$$\hat{u}(t) = \begin{cases} 0 & \text{if } X(t) > x_{\alpha(t),d(t)} \\ \alpha(t) & \text{if } X(t) < x_{\alpha(t),d(t)} \\ \min(\alpha(t), d(t)) & \text{if } X(t) = x_{\alpha(t),d(t)} \end{cases} \tag{53}$$

is optimal for J given by (3) *and for each $(x, \alpha, d) \in \mathbb{R}^m \times A \times D$*

$$J(x, \alpha, d, (\hat{u}(t),\, t \geq 0)) = \lambda. \tag{54}$$

Remark 8 For $m \geq 1$ if it can be verified that

$$\sup_{\beta\in(0,1)} \max_{\alpha\in A} \max_{d\in D} |x^\beta_{\alpha,d}| < \infty$$

then applying the above approach it follows that for any control $(\hat{u}(\cdot))$ satisfying (46) with w defined by (51)

$$\lambda = J(x, \alpha, d, (\hat{u}(t),\, t \geq 0)).$$

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