Asymptotics of controlled finite memory filters

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Abstract. In the paper partially observed controlled problem with the memory consisting of the past k observations is considered. Controlled finite memory filters are approximated. It is shown that average cost per unit time cost functional is characterized by the value of an approximate filter and is continuous with respect to control functions.

Keywords: controlled Markov chain, ergodic cost, finite memory filter, Hilbert metric, stochastic control.

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1 Introduction

Given an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider a Markov process (x_n) on a compact metric space E with an initial distribution μ and a transition kernel $\mathbb{P}^a(x, dz)$, where a is the control parameter taking values in a compact metric space U.

The process (x_n) is observed through an observation process (y_n), $y_n \in \mathbb{R}^d$, which statistically depends on x_n via

$$y_n = h(x_n, \xi_n), \qquad n = 1, 2, \ldots,$$

where ξ_n are $\mathbb{R}^d$ valued i.i.d. with common, strictly positive density g, and for each n, ξ_n are independent of x_j for $j \leq n$. Furthermore, for each $x \in E$, $h(x, \cdot)$ is a C^1 diffeomorphism of $\mathbb{R}^d$ with inverse function $k(x, \cdot)$ and Jacobian of $k(x, y)$ is denoted by $\Delta(x, y)$. It is known (see [4]) that for $n = 0, 1, 2, \ldots$ we have

$$\mathbb{P}\{y_{n+1} \in B \,|\, x_0, x_1, \ldots, x_{n+1}, y_1, \ldots, y_n\} = \int\limits_B r(x_{n+1}, y) dy \tag{1.1}$$

$\mathbb{P}$-a.e., with $r(x, y) = g(k(x, y))|\Delta(x, y)|$, where $|\Delta(x, y)|$ denotes the determinant of Jacobian $\Delta(x, y)$. Fix some positive integer k and denote by $Y_k^n = \sigma\{y_{n-k+1}, \ldots, y_n\}$ for $n = k, k+1, \ldots$ and $Y_k^n = \sigma\{y_1, \ldots, y_k\}$ for $n = 1, 2, \ldots, k$.

We assume that the process (x_n) is controlled by a process $a = (a_n)$, $a_n \in U$ and a_n is Y_k^n-measurable such that $a_j = u_j(y_1, \ldots, y_j)$ for $j = 1, 2, \ldots, k-1$ and for $n \geq k$, a_n is of the form $a_n = u(y_{n-k+1}, \ldots, y_n)$, for some Borel measurable functions $u_j : \left(\mathbb{R}^d\right)^j \to U$ and $u : \left(\mathbb{R}^d\right)^k \to U$. The class of above defined control processes will be later denoted by $\mathcal{U}$. With an abuse of notation we shall write sometimes $u \in \mathcal{U}$ to point out that for $n \geq k$ the control a_n is of the form $a_n = u(y_{n-k+1}, \ldots, y_n)$. The control strategies introduced above depend on the last k observations only and are stationary i.e. the function u does not depend on time n. We are interested to minimize the following the average cost per unit time functional

$$J_\mu((a_n)) = \limsup_{n \to \infty} \frac{1}{T} \sum_{n=0}^{T-1} \mathbb{E}[c(x_n, a_n)] \tag{1.2}$$

over all controls $(a_n) \in \mathcal{U}$. In partially observed control problems with cost functional (1.2) the optimal strategies (see [4]) are ususally stationary functions of a filtering process which is a measure valued Markov process of conditional, with respect to the available observation probability distributions of (x_n). In the case when we have only finite memory observation consisting of the last k-th observations, the corresponding process of conditional probability distributions is no longer Markov and therefore we are not able to use Bellman equation approach. Due to the form of the cost functional it seems reasonable to consider stationary control functions. The advantage of our approach is that filtering process, unless we study a finite state problem, is infinite dimensional, practically impossible to calculate, whereas in the case studied in the above paper the control being a function of the last k-th observations is feasible. On the other hand the optimization problem with the cost functional (1.2) and finite memory observation is difficult and we are not able to use standard methods (see e.g. [4]). In the paper below we characterize the value of the cost functional (1.2) given $u \in \mathcal{U}$ (section 2). We show (using so called Hilbert metric approach) that finite memory controlled filtering process can be approximated by a process constructed similarly as filtering process but with arbitrary initial law. Then in section 3 sketch some continuity results which can be used to perform numerical approximation of nearly optimal control.

Definition 1.1. *Given $a \in \mathcal{U}$ a filter with the memory k is a measure valued stochastic process $\pi_n^{a,k}$ such that*

$$\pi_0^{a,k}(A) := \mu(A)$$

$$\pi_n^{a,k}(A) := \mathbb{E}[x_n \in A \,|\, Y_k^n] \quad for \qquad n = 1, 2, \ldots, \quad A \in \mathcal{B}(E), \tag{1.3}$$

where $\mathcal{B}(E)$ is the family of Borel subsets of E.

Define a new probability measure $\mathbb{P}^0$ on Ω such that the restrictions $\mathbb{P}_{|n}$, $\mathbb{P}^0_{|n}$ of $\mathbb{P}$ and $\mathbb{P}^0$ respectively to the σ-field $\mathcal{F}_n := \sigma\{(x_i, y_i),\ i = 1, \ldots, n\}$ for each $n = 1, 2, \ldots$ satisfy

$$\mathbb{P}^0_{|n}(d\omega) = \Lambda_n(\omega)\mathbb{P}_{|n}(d\omega) \tag{1.4}$$

with $\Lambda_n = \displaystyle\prod_{i=1}^{n} \frac{g(y_i)}{r(x_i, y_i)}$.

We have the following result from [4] (Lemma 1.8).

Lemma 1.1. *Under the measure $\mathbb{P}^0$, y_n are i.i.d. with the common density g, independent of x_j, $j \le n$ and (x_n) is a controlled Markov process with the same transition probability $\mathbb{P}^{a_n}(x_n, \cdot)$ at the generic period n as under $\mathbb{P}$.*

From Bayes' rule (see, for example, [5], p.230) we obtain

$$\pi_n^{a,k}(A) = \mathbb{E}[I_A(x_n) \,|\, Y_k^n] = \frac{\mathbb{E}^0\left[\prod_{i=1}^{n} \frac{r(x_i,y_i)}{g(y_i)} I_A(x_n) \,|\, Y_k^n\right]}{\mathbb{E}^0\left[\prod_{i=1}^{n} \frac{r(x_i,y_i)}{g(y_i)} \,|\, Y_k^n\right]}, \ \ n = 1, 2, \ldots, \, a \in \mathcal{U}, \, A \in \mathcal{B}(E),$$

where I_A denotes an indicator of the set A. We define a so called unnormalized controlled filter with the memory k as

$$\sigma_0^{a,k}(A) = \mu(A)$$

$$\sigma_n^{a,k}(A) = \mathbb{E}^0 \left[\prod_{i=1}^{n} \frac{r(x_i, y_i)}{g(y_i)} I_A(x_n) \,|\, Y_k^n \right], \qquad n = 1, 2, \ldots,\ a \in \mathcal{U},\ A \in \mathcal{B}(E). \quad (1.5)$$

Consequently,

$$\pi_n^{a,k}(A) = \frac{\sigma_n^{a,k}(A)}{\sigma_n^{a,k}(E)}, \qquad n = 1, 2, \ldots,\ a \in \mathcal{U},\ A \in \mathcal{B}(E), \qquad \mathbb{P}^0 - a.e. \quad (1.6)$$

By a standard calculation one can obtain the following
Lemma 1.2. *We have*

$$\sigma_n^{a,k}(A) = \int\limits_{\left(\mathbb{R}^d\right)^k \times E} \int\limits_{E^{k-1}} \int\limits_{A} \frac{r(x_n, y_n)}{g(y_n)} \mathbb{P}^{u(y_{n-k}, \ldots, y_{n-1})}(x_{n-1}, dx_n) \frac{r(x_{n-1}, y_{n-1})}{g(y_{n-1})} \times$$

$$\times \mathbb{P}^{u(y_{n-k-1}, \ldots, y_{n-2})}(x_{n-2}, dx_{n-1}) \frac{r(x_{n-2}, y_{n-2})}{g(y_{n-2})} \times \ldots \times \frac{r(x_{n-k+1}, y_{n-k+1})}{g(y_{n-k+1})} \times$$

$$\times \mathbb{P}^{u(y_{n-2k+1}, \ldots, y_{n-k})}(x_{n-k}, dx_{n-k+1}) F_{n-k}(dy_{n-2k+1}, \ldots, dy_{n-k}, dx_{n-k}) \quad (1.7)$$

for $n = 2k, 2k+1, \ldots$ and $A \in \mathcal{B}(E)$, with F_n are defined as follows:

$$F_k(C) = \int\limits_{(\mathbb{R}^d)^k} \int\limits_{E^{k+1}} I_C(y_1, \ldots, y_k, x_k) r(x_k, y_k) \mathbb{P}^{u_{k-1}(y_1, \ldots, y_{k-1})}(x_{k-1}, dx_k) r(x_{k-1}, y_{k-1}) \times$$

$$\times \mathbb{P}^{u_{k-2}(y_1, \ldots, y_{k-2})}(x_{k-2}, dx_{k-1}) \times \ldots \times r(x_2, y_2) \mathbb{P}^{u_1(y_1)}(x_1, dx_2) r(x_1, y_1) \mathbb{P}^{a_0}(x_0, dx_1) \mu(dx_0) \times$$

$$\times dy_k \times \ldots \times dy_1,$$

$$F_n(C) = \int\limits_{(\mathbb{R}^d)^k \times E} \int\limits_{E} \int\limits_{\mathbb{R}^d} I_C(y_{n-k+1}, \ldots, y_n, x_n) r(x_n, y_n) \mathbb{P}^{u(y_{n-k}, \ldots, y_{n-1})}(x_{n-1}, dx_n) dy_n \times$$

$$\times F_{n-1}(dy_{n-k}, \ldots, dy_{n-1}, dx_{n-1}) \qquad \text{for} \quad n = k+1, k+2, \ldots, \quad (1.8)$$

with $C \in \mathcal{B}\left((\mathbb{R}^d)^k \times E\right)$ *— the family of Borel subsets of* $\left((\mathbb{R}^d)^k \times E\right)$.

Remark 1.1. It is easy seen that $F_n(\cdot)$ for each $n = k, k+1, \ldots$ are probability measures on $\left(\mathbb{R}^d\right)^k \times E$.
Remark 1.2. An unnormalized filter with the memory k in the case of uncontrolled Markov chain (x_n) has the following form

$$\sigma_n^k(A) = \int_{\mathbb{R}^d} \int_A \frac{r(x_n, y_n)}{g(y_n)} \mathbb{P}(\sigma_{n-1}^k, dx_n) g(y_{n-k}) dy_{n-k}$$

for $n = k+1, k+2, \ldots$, $A \in \mathcal{B}(E)$.

2 Approximate controlled finite memory filter

We often do not know the initial law of (x_n). Therefore it makes sense to introduce a so called approximate controlled filter $(\tilde{\pi}_n^{a,k})$ with the memory k. Denote by $\mathcal{P}(S)$ (resp., $\mathcal{M}(S)$) the space of probability measures (resp., nonnegative measures) on some metric space S. Define

$$\tilde{\pi}_n^{a,k}(A) := \frac{\tilde{\sigma}_n^{a,k}(A)}{\tilde{\sigma}_n^{a,k}(E)}, \qquad n = k, k+1, \ldots, \ a \in \mathcal{U}, \ A \in \mathcal{B}(E), \qquad \mathbb{P}^0 - a.s., \tag{2.1}$$

where $\tilde{\sigma}_n^{a,k}(\cdot)$ is defined as $\sigma_n^{a,k}(\cdot)$ but with an initial law $\tilde{\mu}(\cdot)$ instead of $\mu(\cdot)$ in (1.7)-(1.8) for some $\tilde{\mu} \in \mathcal{P}(E)$. Another reason to study approximate finite memory filters is the fact that computation of the real filters as measure valued processes may be a difficult and time consuming computational problem.
Let for $\eta, \tilde{\eta} \in \mathcal{M}(S)$

$$\alpha(\eta, \tilde{\eta}) = \inf\{\lambda \in \mathbb{R}^+ \ : \ \lambda\eta \geq \tilde{\eta}\}. \tag{2.2}$$

and define the so called Hilbert metric ρ on $\mathcal{M}(S)$ (which is in fact a pseudometric) by

$$\rho(\eta, \tilde{\eta}) = \ln \alpha(\eta, \tilde{\eta}) + \ln \alpha(\tilde{\eta}, \eta), \qquad \eta, \tilde{\eta} \in \mathcal{M}(S). \tag{2.3}$$

Recalling Theorem 1.1 from [3] we have
Lemma 2.1. *Let* $T : \mathcal{M}(S) \to \mathcal{M}(S)$ *be a linear map. Then*

$$\rho(T\eta, T\tilde{\eta}) \leq \tanh\left(\frac{\Delta}{4}\right) \rho(\eta, \tilde{\eta}), \qquad \forall \eta, \tilde{\eta} \in \mathcal{M}(S)$$

($\tanh(\infty) \equiv 1$) *with* $\Delta = \sup_{\eta, \tilde{\eta} \in \mathcal{M}(S)} \rho(T\eta, T\tilde{\eta})$.
Notice that each linear operator on $\mathcal{M}(S)$ is nonexpansive in ρ pseudometric i.e.:

$$\rho(T\eta, T\tilde{\eta}) \leq \rho(\eta, \tilde{\eta}) \qquad \forall \eta, \tilde{\eta} \in \mathcal{M}(S)$$

and it may be a contraction for mutually absolutely continuous measures $T\eta$ and $T\tilde{\eta}$ (since $\rho(\eta,\tilde{\eta}) = +\infty$ if η and $\tilde{\eta}$ are not mutually absolutely continuous). Moreover the Hilbert metric does not change if η, $\tilde{\eta}$ in (2.3) are multiplied by positive scalars and can be written in the following equivalent form

$$\rho(\eta,\tilde{\eta}) = \ln\left[\sup \frac{\eta(A)\tilde{\eta}(B)}{\eta(B)\tilde{\eta}(A)}\right], \quad \eta,\tilde{\eta} \in \mathcal{M}(S) \tag{2.4}$$

where the supremum is over all $A, B \in \mathcal{B}(S)$ such that $\tilde{\eta}(A)\eta(B) > 0$.
For any $u \in \mathcal{U}$, $\nu \in \mathcal{M}\left(\left(\mathbb{R}^d\right)^k \times E\right)$ define the operator

$$T^u\nu(C) = \int\limits_{(\mathbb{R}^d)^k\times E} \int\limits_{E} \int\limits_{\mathbb{R}^d} I_C(y_2,\ldots,y_{k+1},x_{k+1}) r(x_{k+1},y_{k+1}) dy_{k+1} \mathbb{P}^{u(y_1,\ldots,y_k)}(x_k,dx_{k+1}) \times$$

$$\times \nu(dy_1,\ldots,dy_k,dx_k), \qquad C \in \mathcal{B}\left((\mathbb{R}^d)^k \times E\right), \tag{2.5}$$

where I_C is the characteristic operator of the set C.

Remark 2.1. From (2.5) and (1.8) we have

$$F_{n+1} = T^u F_n. \tag{2.6}$$

Assume that

$$(A1) \qquad \sup_{x,x'\in E} \sup_{v,v'\in U} \sup_{A\in\mathcal{B}(E)} \frac{\mathbb{P}^v(x,A)}{\mathbb{P}^{v'}(x',A)} =: M < \infty\ .$$

Lemma 2.2. *Under (A1) we have*

$$\sup_{u\in\mathcal{U}} \sup_{\nu,\tilde{\nu}\in\mathcal{M}\left(\left(\mathbb{R}^d\right)^k\times E\right)} \rho(T_k^u\nu, T_k^u\tilde{\nu}) \le 4k\ln M,$$

with T_k^u standing for the kth iteration of the operator T^u.
Proof. We have

$$T_k^u\nu(C) = \int\limits_{(\mathbb{R}^d)^k\times E} \int\limits_{\left(\mathbb{R}^d\right)^k} \int\limits_{E^k} I_C(y_{k+1},\ldots,y_{2k},x_{2k}) r(x_{2k},y_{2k}) \mathbb{P}^{u(y_k,\ldots,y_{2k-1})}(x_{2k-1},dx_{2k}) \times$$

$$\times r(x_{2k-1},y_{2k-1}) \mathbb{P}^{u(y_{k-1},\ldots,y_{2k-2})}(x_{2k-2},dx_{2k-1}) \times \ldots \times$$
$$\times r(x_{k+1},y_{k+1}) \mathbb{P}^{u(y_1,\ldots,y_k)}(x_k,dx_{k+1}) dy_{2k}\ldots dy_{k+1} \nu(dy_1,\ldots,dy_k,dx_k)$$

for any $C \in \mathcal{B}\left((R^d)^k \times E\right)$, $u \in \mathcal{U}$, $\nu \in \mathcal{M}\left(\left(\mathbb{R}^d\right)^k \times E\right)$. Then by (A1) for $x' \in E$

$$T_k^u \nu(C) \leq M^k \int\limits_{(\mathbb{R}^d)^k \times E} \int\limits_{\left(\mathbb{R}^d\right)^k} \int\limits_{E^k} I_C(y_{k+1}, \dots, y_{2k}, x_{2k}) r(x_{2k}, y_{2k}) \mathbb{P}^v(x', dx_{2k}) \times$$

$$\times r(x_{2k-1}, y_{2k-1}) \mathbb{P}^v(x', dx_{2k-1}) \times \ldots \times r(x_{k+1}, y_{k+1}) \mathbb{P}^v(x', dx_{k+1}) dy_{2k} \ldots dy_{k+1} \times$$

$$\times \nu(dy_1, \dots, dy_k, dx_k) = M^k \int\limits_{(\mathbb{R}^d)^k \times E} I_C(y_{k+1}, \dots, y_{2k}, x_{2k}) \times$$

$$\times \prod_{i=k+1}^{2k-1} \int\limits_E r(x_i, y_i) dy_i \mathbb{P}^v(x', dx_i) r(x_{2k}, y_{2k}) dy_{2k} \mathbb{P}^v(x', dx_{2k}) \nu(\mathbb{R}^d, \dots, \mathbb{R}^d, E).$$

In a similar way we obtain that

$$T_k^u \nu(C) \geq \frac{1}{M^k} \int\limits_{(\mathbb{R}^d)^k \times E} I_C(y_{k+1}, \dots, y_{2k}, x_{2k}) \times$$

$$\times \prod_{i=k+1}^{2k-1} \int\limits_E r(x_i, y_i) dy_i \mathbb{P}^v(x', dx_i) r(x_{2k}, y_{2k}) dy_{2k} \mathbb{P}^v(x', dx_{2k}) \nu(\mathbb{R}^d, \dots, \mathbb{R}^d, E).$$

Notice that $v \in U$ here is taken independently of $y_1, \dots, y_{2k}$. Consequently by the equivalent definition (2.4) of the Hilbert metric $\rho(\cdot, \cdot)$ we have

$$\sup_{u \in \mathcal{U}} \sup_{\nu, \tilde{\nu} \in \mathcal{M}\left(\left(\mathbb{R}^d\right)^k \times E\right)} \rho(T_k^u \nu, T_k^u \tilde{\nu}) \leq \ln M^{4k} = 4k \ln M.$$

■

Remark 2.2. Using the nonexpansive property of linear operators in the ρ pseudometric for $n \geq k$ we have

$$\sup_{u \in \mathcal{U}} \sup_{\nu, \tilde{\nu} \in \mathcal{M}\left(\left(\mathbb{R}^d\right)^k \times E\right)} \rho(T_n^u \nu, T_n^u \tilde{\nu}) \leq 4k \ln M. \tag{2.7}$$

Lemma 2.3. *Under (A1) for all $u \in \mathcal{U}$, $\nu, \tilde{\nu} \in \mathcal{M}\left(\left(\mathbb{R}^d\right)^k \times E\right)$*

$$\rho(T_k^u \nu, T_k^u \tilde{\nu}) \leq \delta \rho(\nu, \tilde{\nu}) \tag{2.8}$$

with $\delta = \tanh\left[k \ln M\right] < 1$ by (A1).
Proof. The claim of Lemma 2.3 is a straightforward from Lemma 2.1.

■

Following Lemma 2.2 in [1] we have

Lemma 2.4. *For* $\nu, \tilde{\nu} \in \mathcal{P}(S)$, $\|\nu - \tilde{\nu}\| \leq \frac{2}{\ln 2}\rho(\nu, \tilde{\nu})$, *where* $\|\cdot\|$ *is the total variation norm.*

Proposition 2.1. *Under (A1) for all* $u \in \mathcal{U}$ *and any* $\nu, \tilde{\nu} \in \mathcal{M}\left(\left(\mathbb{R}^d\right)^k \times E\right)$ *we have*

$$\rho(T_n^u \nu, T_n^u \tilde{\nu}) \leq \delta^{\left[\frac{n}{k}\right]-1} 4k \ln M, \qquad n = 2k, 2k+1, \ldots,$$

where M *is defined in (A1) and* $[a]$ *denotes the integer part of the real number* a.

Proof. Let $m_n = \max\left\{p \in \mathbb{N} : \frac{n}{p} \geq k\right\}$. Then from Lemma 2.3, Lemma 2.2 and Remark 2.2 we have

$$\rho(T_n^u \nu, T_n^u \tilde{\nu}) = \rho(T_k^u(T_{n-k}^u \nu), T_k^u(T_{n-k}^u \tilde{\nu})) \leq \delta \rho(T_{n-k}^u \nu, T_{n-k}^u \tilde{\nu}) \leq \ldots \leq$$
$$\leq \delta^{m_n - 1} \rho(T_{n-(m_n-1)k}^u \nu, T_{n-(m_n-1)k}^u \tilde{\nu}) \leq \delta^{m_n-1} 4k \ln M.$$

Noticing that $m_n = \left[\frac{n}{k}\right]$ we complete the proof.

■

Theorem 2.1. *Under (A1) for any* $u \in \mathcal{U}$ *there is a unique probability measure* $\bar{\nu}_u$ *such that* $T^u \bar{\nu}_u = \bar{\nu}_u$ *and moreover*

$$\sup_{\nu \in \mathcal{P}\left(\left(\mathbb{R}^d\right)^k \times E\right)} \|T_n^u \nu - \bar{\nu}_u\| \leq \frac{2}{\ln 2} \sup_{\nu \in \mathcal{P}\left(\left(\mathbb{R}^d\right)^k \times E\right)} \rho(T_n^u \nu, \bar{\nu}_u) \leq \delta^{\left[\frac{n}{k}\right]-1} \frac{8}{\ln 2} k \ln M, \tag{2.9}$$

with δ *defined in Lemma 2.3 and* M *defined in (A1).*

Proof. From Proposition 2.1 for all $\nu \in \mathcal{M}\left(\left(\mathbb{R}^d\right)^k \times E\right)$, $u \in \mathcal{U}$, $n \geq 2k$ and $m \geq 0$ we have

$$\rho(T_n^u \nu, T_{n+m}^u \nu) = \rho(T_n^u \nu, T_n^u(T_m^u \nu)) \leq \delta^{\left[\frac{n}{k}\right]-1} 4k \ln M. \tag{2.10}$$

It follows from Theorem 2.2 of [2] that the space $\left(\mathcal{P}\left(\left(\mathbb{R}^d\right)^k \times E\right), \rho(\cdot, \cdot)\right)$ is complete. Since by Remark 1.1 and Remark 2.1 $T^u \nu \in \mathcal{P}\left(\left(\mathbb{R}^d\right)^k \times E\right)$ for any $\nu \in \mathcal{P}\left(\left(\mathbb{R}^d\right)^k \times E\right)$ there exists of a limit measure $\bar{\nu}_u \in \mathcal{P}\left(\left(\mathbb{R}^d\right)^k \times E\right)$ such that

$$\rho(T_n^u \nu, \bar{\nu}_u) \to 0 \qquad \text{as } n \to \infty. \tag{2.11}$$

From Proposition 2.1 and Theorem 3.1 of [2] we have that $\bar{\nu}_u$ is a unique fixed point for the operator T^u. From Lemma 2.4, the fact that $\bar{\nu}_u$ is the unique fixed point for T^u, using (2.9) we finally obtain that

$$\sup_{\nu' \in \mathcal{P}\left(\left(\mathbb{R}^d\right)^k \times E\right)} \|T_n^u \nu' - \bar{\nu}\| \leq \frac{2}{\ln 2} \sup_{\nu' \in \mathcal{P}\left(\left(\mathbb{R}^d\right)^k \times E\right)} \rho(T_n^u \nu', \bar{\nu}) \leq$$

$$\leq \frac{2}{\ln 2} \sup_{\nu' \in \mathcal{P}\left(\left(\mathbb{R}^d\right)^k \times E\right)} \rho(T_n^u \nu', T_n^u \bar{\nu}_u) \leq \delta^{\left[\frac{n}{k}\right]-1} \frac{8k}{\ln 2} \ln M.$$

■

We can now construct an approximate controlled filter with the memory k

$$\tilde{\pi}_n^{u,k}(A) := \frac{\tilde{\sigma}_n^{u,k}(A)}{\tilde{\sigma}_n^{u,k}(E)}, \qquad n = 2k, 2k+1, \ldots,\ u \in \mathcal{U},\ A \in \mathcal{B}(E) \qquad \mathbb{P}^0 - a.e. \quad (2.12)$$

where

$$\tilde{\sigma}_n^{u,k}(A) := \int\limits_{\left(\mathbb{R}^d\right)^k \times E} \int\limits_{E^{k-1}} \int\limits_{A} \frac{r(x_n, y_n)}{g(y_n)} \mathbb{P}^{u(y_{n-k}, \ldots, y_{n-1})}(x_{n-1}, dx_n) \frac{r(x_{n-1}, y_{n-1})}{g(y_{n-1})} \times$$

$$\times \mathbb{P}^{u(y_{n-k-1}, \ldots, y_{n-2})}(x_{n-2}, dx_{n-1}) \frac{r(x_{n-2}, y_{n-2})}{g(y_{n-2})} \times \ldots \times \frac{r(x_{n-k+1}, y_{n-k+1})}{g(y_{n-k+1})} \times$$

$$\times \mathbb{P}^{u(y_{n-2k+1}, \ldots, y_{n-k})}(x_{n-k}, dx_{n-k+1}) \bar{\nu}_u(dy_{n-2k+1}, \ldots, dy_{n-k}, dx_{n-k}), \quad \mathbb{P}^0 - a.e. \quad (2.13)$$

for $n = 2k, 2k+1, \ldots$, $u \in \mathcal{U}$, $A \in \mathcal{B}(E)$, letting $\tilde{\pi}_n^{u,k}(A) = 0$ whenever $\tilde{\sigma}_n^{u,k}(E) = 0$.
Let

$$\tilde{T}^{u,n}\nu(A) = \int\limits_{\left(\mathbb{R}^d\right)^k \times E} \int\limits_{E^{k-1}} \int\limits_{A} \frac{r(x_n, y_n)}{g(y_n)} \mathbb{P}^{u(y_{n-k}, \ldots, y_{n-1})}(x_{n-1}, dx_n) \times$$

$$\times \frac{r(x_{n-1}, y_{n-1})}{g(y_{n-1})} \mathbb{P}^{u(y_{n-k-1}, \ldots, y_{n-2})}(x_{n-2}, dx_{n-1}) \frac{r(x_{n-2}, y_{n-2})}{g(y_{n-2})} \times \ldots \times$$

$$\times \frac{r(x_{n-k+1}, y_{n-k+1})}{g(y_{n-k+1})} \mathbb{P}^{u(y_{n-2k+1}, \ldots, y_{n-k})}(x_{n-k}, dx_{n-k+1}) \nu(dy_{n-2k+1}, \ldots, dy_{n-k}, dx_{n-k}), \quad (2.14)$$

for $n = 2k, 2k+1, \ldots$ and for some $\nu \in \mathcal{P}\left(\left(\mathbb{R}^d\right)^k \times E\right)$, $u \in \mathcal{U}$, $A \in \mathcal{B}(E)$.

Lemma 2.5. *Under (A1)*

$$\tilde{\Delta} := \sup_{n \geq 2k} \sup_{u \in \mathcal{U}} \sup_{\nu, \tilde{\nu} \in \mathcal{M}\left(\left(\mathbb{R}^d\right)^k \times E\right)} \rho(\tilde{T}^{u,n}\nu, \tilde{T}^{u,n}\tilde{\nu}) \leq 4k \ln M,$$

where M is defined in (A1).

Proof. The proof is analogous to the that of Lemma 2.2. and therefore is omitted.

■

From Lemma 2.5 we have that $\tilde{\delta} := \tanh \dfrac{\tilde{\Delta}}{4} < 1$.

Theorem 2.2. *Under (A1) for all $u \in U$, $n = 3k, 3k+1, \ldots$*

$$\|\pi_n^{u,k} - \tilde{\pi}_n^{u,k}\| \leq \delta^n \tilde{K}, \qquad \mathbb{P} - a.e. \quad (2.15)$$

where $\tilde{K}$ is a fixed constant.
Proof. From Lemma 2.5 and Lemma 2.1 we have

$$\rho(\tilde{T}^{u,n}\nu, \tilde{T}^{u,n}\tilde{\nu}) \le \tilde{\delta}\rho(\nu,\tilde{\nu}), \quad n = 3k, 3k+1, \dots,\ u \in U,\ \nu, \tilde{\nu} \in \mathcal{M}\left(\left(\mathbb{R}^d\right)^k \times E\right).$$

Notice that by (1.7) and (2.12)

$$\sigma_n^{u,k}(A) = \tilde{T}^{u,n} F_{n-k}(A)$$

and

$$\tilde{\sigma}_n^{u,k}(A) = \tilde{T}^{u,n}\bar{\nu}_u(A).$$

Therefore

$$\rho(\sigma_n^{u,k}, \tilde{\sigma}_n^{u,k}) \le \tilde{\delta}\rho(F_{n-k}, \bar{\nu}_u). \tag{2.16}$$

From Lemma 2.4, (2.16), Remark 2.1 and Theorem 2.1 we have

$$\|\pi_n^{u,k} - \tilde{\pi}_n^{u,k}\| \le \frac{2}{\ln 2}\rho(\pi_n^{u,k}, \tilde{\pi}_n^{u,k}) = \frac{2}{\ln 2}\rho(\sigma_n^{u,k}, \tilde{\sigma}_n^{u,k}) \le \frac{2}{\ln 2}\tilde{\delta}\rho(F_{n-k}, \bar{\nu}_u) \le$$

$$\le \delta^n \frac{8}{\ln 2} k\tilde{\delta}\ln M, \qquad n = 3k, 3k+1, \dots,\ u \in \mathcal{U}, \qquad \mathbb{P}^0 - a.e.$$

and since the righthand side of the last inequality is deterministic also $\mathbb{P} - a.e.$, which completes the proof with $\tilde{K} := \frac{8}{\ln 2} k\tilde{\delta}\ln M$.

■

Remark 2.3. Theorem 2.1 is important from practical point of view. Direct calculation of $\pi_n^{u,k}$ is rather difficult. But from Theorem 2.1 we see that we need only an information about the last k observations of the process (x_n) and the limit measure $\bar{\nu}_u$ to approximate our controlled filter $\pi_n^{u,k}$ using approximate controlled filter $\tilde{\pi}_n^{u,k}$ with large enough n. Notice moreover that by (2.9) $\bar{\nu}_u$ can be approximated by the iterations of the operator T^u.

Given $u \in \mathcal{U}$ we have the following form of the cost functional (1.8)
Theorem 2.3. *Under (A1)*

$$J_\mu(u) = \int\limits_{\left(\mathbb{R}^d\right)^k \times E} \int\limits_E \int\limits_{\mathbb{R}^d} c(x, u(y_2, \dots, y_{k+1})) r(x, y_{k+1}) dy_{k+1} \mathbb{P}^{u(y_1,\dots,y_k)}(z, dx) \times$$

$$\times \bar{\nu}_u(dy_1, \dots, dy_k, dz). \tag{2.17}$$

Proof. One can show that

$$\mathbb{E}[c(x_n, u(y_{n-k+1}, \dots, y_n))] = \int\limits_{\left(\mathbb{R}^d\right)^k \times E} \int\limits_E \int\limits_{\mathbb{R}^d} c(x_n, u(y_{n-k+1}, \dots, y_n)) r(x_n, y_n) dy_n \times$$

$$\times\mathbb{P}^{u(y_{n-k},\dots,y_{n-1})}(x_{n-1},dx_n)F_{n-1}(dy_{n-k},\dots,dy_{n-1},dx_{n-1}),\quad n=k+1,k+2,\dots,\ u\in\mathcal{U},$$

with (F_n) defined by (1.8).
Since F_{n-1}, $\bar{\nu}_u$ are probability measures, from (1.1), Remark 2.1 and Theorem 2.1 we obtain

$$\left\|\mathbb{E}[c(x_n,u(y_{n-k+1},\dots,y_n))]-\int\limits_{(\mathbb{R}^d)^k\times E}\int\limits_{E}\int\limits_{\mathbb{R}^d}c(x,u(y_2,\dots,y_{k+1}))r(x,y_{k+1})dy_{k+1}\times\right.$$

$$\left.\times\mathbb{P}^{u(y_1,\dots,y_k)}(z,dx)\bar{\nu}_u(dy_1,\dots,dy_k,dz)\right\|=$$

$$=\left\|\int\limits_{(\mathbb{R}^d)^k\times E}\int\limits_{E}\int\limits_{\mathbb{R}^d}c(x_n,u(y_{n-k+1},\dots,y_n))r(x_n,y_n)dy_n\mathbb{P}^{u(y_{n-k},\dots,y_{n-1})}(x_{n-1},dx_n)\times\right.$$

$$\times F_{n-1}(dy_{n-k},\dots,dy_{n-1},dx_{n-1})-\int\limits_{(\mathbb{R}^d)^k\times E}\int\limits_{E}\int\limits_{\mathbb{R}^d}c(x,u(y_2,\dots,y_{k+1}))r(x,y_{k+1})dy_{k+1}\times$$

$$\left.\times\mathbb{P}^{u(y_1,\dots,y_k)}(z,dx)\bar{\nu}_u(dy_1,\dots,dy_k,dz)\right\|\le\int\limits_{(\mathbb{R}^d)^k\times E}\int\limits_{E}\int\limits_{\mathbb{R}^d}c(x,u(y_2,\dots,y_{k+1}))\times$$

$$\times r(x,y_{k+1})dy_{k+1}\mathbb{P}^{u(y_1,\dots,y_k)}(z,dx)\|F_{n-1}(dy_1,\dots,dy_k,dz)-\bar{\nu}_u(dy_1,\dots,dy_k,dz)\|\le$$

$$\le\|c\|\cdot\|F_{n-1}-\bar{\nu}_u\|=\|c\|\cdot\|T^uF_{n-2}-\bar{\nu}_u\|\le\delta^{\left[\frac{n-1}{k}\right]-2}\frac{8}{\ln 2}\|c\|k\ln M\to 0,$$

which completes the proof.

■

3 Continuity properties

In this section we study continuity of the limit measure ν_u, cost functional $J(u)$ and approximate unnormalized filters with respect to control function u.

Let λ^j denote the Lebesgue measure on $(\mathbb{R}^d)^j$ for some $j=1,2,\dots$. We shall assume through this section that $k\ge 3$ and

(A2) if $u_n:(\mathbb{R}^d)^k\to U$ converges to $u:(\mathbb{R}^d)^k\to U$, λ^k-a.e. then for $x\in E$

$$\|\mathbb{P}^{u_n(y_1,\dots,y_k)}(x,\cdot)-\mathbb{P}^{u(y_1,\dots,y_k)}(x,\cdot)\|\to 0\qquad\text{for}\quad\lambda^k-a.e.(y_1,\dots,y_k)\qquad\text{as}\quad n\to\infty \tag{3.1}$$

and

$$\int_{\mathbb{R}^d} \sup_{x\in E} r(x,y)dy \le \bar{M} < \infty.$$

Proposition 3.1. *Under (A1) and (A2) if* $u_n : (\mathbb{R}^d)^k \to U$ *converges to* $u : (\mathbb{R}^d)^k \to U$, λ^k*-a.e. then* $\|\bar{\nu}_{u_n} - \bar{\nu}_u\| \to 0$ *as* $n \to \infty$.
Proof. Take $\nu \in \mathcal{P}((\mathbb{R}^d)^k \times E)$ such that $\nu(\cdot, E)$ is absolutely continuous with respect to λ^k measure. Then

$$\|\bar{\nu}_{u_n} - \bar{\nu}_u\| \le \|\bar{\nu}_{u_n} - T_m^{u_n}\nu\| + \|T_m^{u_n}\nu - T_m^u\nu\| + \|T_m^u\nu - \bar{\nu}_u\| = I_1 + I_2 + I_3.$$

Notice that I_1 and I_3 tend to 0 for all $n \ge 0$ as $m \to \infty$ from Theorem 2.1. We shall prove that $I_2 \to 0$ as $n \to \infty$ for all $m \ge 1$ by induction. For $m = 1$ from the definition of T^u, (1.1), (A2) and the absolute continuity ν with respect to λ^k we obtain

$$\|T^{u_n}\nu - T^u\nu\| \le \int_{(\mathbb{R}^d)^k\times E} \int_{\mathbb{R}^d} \sup_{x\in E} r(x, y_{k+1})dy_{k+1}\times$$

$$\times\|\mathbb{P}^{u_n(y_1,\dots,y_k)}(z,\cdot) - \mathbb{P}^{u(y_1,\dots,y_k)}(z,\cdot)\|\nu(dy_1,\dots,dy_k,dz) \le$$

$$\le \bar{M} \int_{(\mathbb{R}^d)^k\times E} \|\mathbb{P}^{u_n(y_1,\dots,y_k)}(z,\cdot) - \mathbb{P}^{u(y_1,\dots,y_k)}(z,\cdot)\|\nu(dy_1,\dots,dy_k,dz) \to 0$$

as $n \to \infty$, λ^k-a.e., using Lebesgue's dominated convergence theorem.
Assume that $I_2 \to 0$ for $m = l$ as $n \to \infty$. Then, for $m = l + 1$ we have

$$I_2 = \|T_{l+1}^{u_n}\nu - T_{l+1}^u\nu\| \le \int_{(\mathbb{R}^d)^k\times E} \int_{\mathbb{R}^d} \sup_{x\in E} r(x, y_{k+1})dy_{k+1}\times$$

$$\times\|\mathbb{P}^{u_n(y_1,\dots,y_k)}(z,\cdot) - \mathbb{P}^{u(y_1,\dots,y_k)}(z,\cdot)\|T_l^u\nu(dy_1,\dots,dy_k,dz) + \|T_l^{u_n}\nu - T_l^u\nu\| \to 0$$

as $n \to \infty$, , by (3.1) and the induction assumption. The proof of Proposition is therefore complete.

∎

Corollary 3.1. *Under (A1) and (A2) if* $u_n : (\mathbb{R}^d)^k \to U$ *converges to* $u : (\mathbb{R}^d)^k \to U$, λ^k*-a.e. and* $c(x, .)$ *is continuous for* $x \in E$ *we have* $J_\mu(u_n) \to J_\mu(u)$, *with* $J_\mu(u)$ *defined in (1.8).*
Proof. It follows from Theorem 2.3, the boundness of $\|c\|$, the continuity of $c(x,\cdot)$, $x \in E$, (A2) and Proposition 3.1 that

$$\|J_\mu(u) - J_\mu(u_n)\| \le \int_{(\mathbb{R}^d)^k\times E} \int_E \int_{\mathbb{R}^d} |c(x, u(y_2,\dots,y_{k+1}))-$$

$$-c(x,u_n(y_2,\dots,y_{k+1}))|r(x,y_{k+1})dy_{k+1}\mathbb{P}^{u(y_1,\dots,y_k)}(z,dx)\bar{\nu}_u(dy_1,\dots,dy_k,dz)+$$

$$+\|c\| \int\limits_{(\mathbb{R}^d)^k\times E} \int\limits_E \int\limits_{\mathbb{R}^d} r(x,y_{k+1})dy_{k+1}\times$$

$$\times|\mathbb{P}^{u(y_1,\dots,y_k)}(z,dx)-\mathbb{P}^{u_n(y_1,\dots,y_k)}(z,dx)|\bar{\nu}_u(dy_1,\dots,dy_k,dz)+$$

$$+\|c\| \int\limits_E \int\limits_{\mathbb{R}^d} r(x,y_{k+1})dy_{k+1} \sup_{z\in E} \sup_{y_1,\dots,y_k\in\mathbb{R}^d} \mathbb{P}^{u_n(y_1,\dots,y_k)}(z,dx)\|\bar{\nu}_u-\bar{\nu}_{u_n}\| \to 0,$$

as $n\to\infty$, which completes the proof.

■

Finally, using Proposition 3.1 and Lemma 2.4 one can show the following

Corollary 3.2. *Assume that* $u_n:(\mathbb{R}^d)^k\to U$ *converges to* $u:(\mathbb{R}^d)^k\to U$, λ^k-*a.e. and* $\sup_{x\in E}\frac{r(x,y)}{g(y)}$ *is uniformly bounded. Then under (A1) and (A2) for all* $m=2k,2k+1,\dots$

$$\|\tilde{\pi}_m^{u,k}-\tilde{\pi}_m^{u_n,k}\|\to 0 \quad as \quad n\to\infty.$$

Proof. By the definition (2.12) of the approximate unnormalized controlled finite memory filter $\tilde{\sigma}_m^{u,k}$, (A2) and Proposition 3.1 we obtain that

$$\|\tilde{\sigma}_m^{u,k}-\tilde{\sigma}_m^{u_n,k}\|\le$$

$$\le \int\limits_{(\mathbb{R}^d)^k\times E} \int\limits_{E^{k-1}} \sup_{x_m\in E}\frac{r(x_m,y_m)}{g(y_m)}\|\mathbb{P}^{u(y_{m-k},\dots,y_{m-1})}(x_{m-1},\cdot)-\mathbb{P}^{u_n(y_{m-k},\dots,y_{m-1})}(x_{m-1},\cdot)\|\times$$

$$\times\frac{r(x_{m-1},y_{m-1})}{g(y_{m-1})}\mathbb{P}^{u(y_{m-k-1},\dots,y_{m-2})}(x_{m-2},dx_{m-1})\frac{r(x_{m-2},y_{m-2})}{g(y_{m-2})}\times\dots\times$$

$$\times\frac{r(x_{m-k+1},y_{m-k+1})}{g(y_{m-k+1})}\mathbb{P}^{u(y_{m-2k+1},\dots,y_{m-k})}(x_{m-k},dx_{m-k+1})\times$$

$$\times\bar{\nu}_u(dy_{m-2k+1},\dots,dy_{m-k},dx_{m-k})+ \int\limits_{(\mathbb{R}^d)^k\times E} \int\limits_{E^{k-1}} \frac{r(x_m,y_m)}{g(y_m)}\mathbb{P}^{u_n(y_{m-k},\dots,y_{m-1})}(x_{m-1},dx_m)\times$$

$$\times\sup_{x_{m-1}\in E}\frac{r(x_{m-1},y_{m-1})}{g(y_{m-1})}\|\mathbb{P}^{u(y_{m-k-1},\dots,y_{m-2})}(x_{m-2},\cdot)-\mathbb{P}^{u_n(y_{m-k-1},\dots,y_{m-2})}(x_{m-2},\cdot)\|\times$$

$$\times\frac{r(x_{m-2},y_{m-2})}{g(y_{m-2})}\times\dots\times\frac{r(x_{m-k+1},y_{m-k+1})}{g(y_{m-k+1})}\mathbb{P}^{u(y_{m-2k+1},\dots,y_{m-k})}(x_{m-k},dx_{m-k+1})\times$$

$$\times\bar{\nu}_u(dy_{m-2k+1},\dots,dy_{m-k},dx_{m-k})+\dots+$$

$$+\int\limits_{\left(\mathbb{R}^d\right)^k\times E}\int\limits_{E^{k-1}}\frac{r(x_m,y_m)}{g(y_m)}\mathbb{P}^{u_n(y_{m-k},\ldots,y_{m-1})}(x_{m-1},dx_m)\frac{r(x_{m-1},y_{m-1})}{g(y_{m-1})}\times$$

$$\times\mathbb{P}^{u_n(y_{m-k-1},\ldots,y_{m-2})}(x_{m-2},dx_{m-1})\frac{r(x_{m-2},y_{m-2})}{g(y_{m-2})}\times\ldots\times\sup_{x_{m-k+1}\in E}\frac{r(x_{m-k+1},y_{m-k+1})}{g(y_{m-k+1})}\times$$

$$\times\|\mathbb{P}^{u(y_{m-2k+1},\ldots,y_{m-k})}(x_{m-k},\cdot)-\mathbb{P}^{u_n(y_{m-2k+1},\ldots,y_{m-k})}(x_{m-k},\cdot)\|\times$$

$$\times\bar{\nu}_u(dy_{m-2k+1},\ldots,dy_{m-k},dx_{m-k})+$$

$$+\int\limits_{E^k}\sup_{y_{m-k+1},\ldots,y_{m-k}\in\mathbb{R}^d}\frac{r(x_m,y_m)}{g(y_m)}\mathbb{P}^{u_n(y_{m-k},\ldots,y_{m-1})}(x_{m-1},dx_m)\frac{r(x_{m-1},y_{m-1})}{g(y_{m-1})}\times$$

$$\times\mathbb{P}^{u_n(y_{m-k-1},\ldots,y_{m-2})}(x_{m-2},dx_{m-1})\frac{r(x_{m-2},y_{m-2})}{g(y_{m-2})}\times\ldots\times\frac{r(x_{m-k+1},y_{m-k+1})}{g(y_{m-k+1})}\times$$

$$\times\sup_{x_{m-k}\in E}\mathbb{P}^{u_n(y_{m-2k+1},\ldots,y_{m-k})}(x_{m-k},dx_{m-k+1})\|\bar{\nu}_u-\bar{\nu}_{u_n}\|\to 0\quad\text{as}\quad n\to\infty,\quad\lambda^k-a.e.$$

Notice that if $\tilde{\sigma}_m^{u,k}(E)>0$, then for a sufficiently large n_j we have $\tilde{\sigma}_m^{u_{n_j},k}(E)>0$. Consequently

$$\|\tilde{\pi}_m^{u,k}-\tilde{\pi}_m^{u_{n_j},k}\|=\left\|\frac{\tilde{\sigma}_m^{u,k}}{\tilde{\sigma}_m^{u,k}(E)}-\frac{\tilde{\sigma}_m^{u_{n_j},k}}{\tilde{\sigma}_m^{u_{n_j},k}(E)}\right\|\to 0\quad as\quad j\to\infty,\qquad\lambda^k-a.e.$$

If $\tilde{\sigma}_m^{u_n,k}(E)\to 0=\tilde{\sigma}_m^{u,k}(E)$ as $n\to\infty$ then by the definition $\tilde{\pi}_m^{u_n,k}(E)\to 0=\tilde{\pi}_m^{u,k}(E)$. The proof is complete.

■

Remark 3.1. By Corollary 3.2, under assumptions of Corrolary 3.2 we immediately have that

$$\|\pi_m^{u,k}-\pi_m^{u_n,k}\|\to 0\qquad\text{as}\quad n\to\infty.$$

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