Discrete Time Markets with Transaction Costs

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Abstract

In the paper various aspects of arbitrage and pricing with concave and proportional transaction costs are studied.

1 Introduction

Assume that we are two accounts: bank and stock account. Transaction costs corresponding to transfers from the bank to the stock account and vice versa are described by the pair of functions τ_1 and τ_2, which transform R^+ into R^+, $\tau_1(0) = \tau_2(0) = 0$, τ_1 is concave and τ_2 is convex. Moreover, we assume that both functions are differentiable with $\dot{\tau}_1(0) = 1+\lambda$, $\dot{\tau}_2(0) = 1-\mu$ for positive λ and μ, $\dot{\tau}_1$ is decreasing greater than 1, while $\dot{\tau}_2$ is increasing less than 1. When we transfer ammount $\tau_1(x)$, $x \geq 0$ from the bank to the stock account (buy assets), the last account is increased by x. In the case, when we transfer ammount x from the stock to the bank account (sell assets), the bank account is increased by $\tau_2(x)$. By the form of the functions τ_1 and τ_2 we see that when they are not linear (i.e. $\tau_1(x) \not\equiv (1+\lambda)x$ and $\tau_2(x) \not\equiv (1-\mu)x)$, buying or selling small ammounts of assets we pay almost proportional transaction costs, while for large transactions the transaction costs are relatively (per unit of the transaction) smaller. Such situation is significantly different from the one with proportional transaction costs, since the transaction costs are not additive (buying or selling assets for $x_1 + x_2$ is less expensive than buying or selling assets first for x_1 and then for x_2).

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In this paper we mainly concentrate on strictly concave transaction costs (the situation when τ_1 and τ_2 are nonlinear), trying to treat the proportional transaction costs as a particular case of the concave transaction costs. For simplicity of the presentation we restrict ourselves to the case with one asset or one stock account. All results except of those concerning Cox Ross Rubinstein model can be easily extended to finite number of assets. Our market position is usually characterized by the vector $\binom{x}{y}$, where x corresponds to the ammount on the bank account, while y to the ammount on the stock account. We admit negative values for x or y, which correspond to debits. Moreover for simplicity we assume that rate of interests on the banking account is equal to 0. Given a probability space (Ω, F, P) endowed with an increasing family of σ - fields (F_t), we consider the prices S_t of the asset at time $t = 0, \dots, T$ as F_t measurable random variables which are strictly positive. We assume for simplicity that σ - field F_0 is trivial. In what follows we denote by $L^0(G, F_T)$ the set of all F_T measurable random variables taking values in a set G, which may be also random. Moreover when $\tau_1(x) \equiv (1+\lambda)x$ and $\tau_2(x) \equiv (1-\mu)x$ we say that we have proportional transaction costs, while in the case when $\tau_1(x) \not\equiv (1+\lambda)x$ and $\tau_2(x) \not\equiv (1-\mu)x$ we say about nonlinear concave transaction costs.

2 Arbitrage

Let

$$M = \Big\{ \tbinom{x}{y} : y \geq -(\tau_1)^{-1}(x), \quad for \quad x \geq 0 \\ and \quad y \geq (\tau_2)^{-1}(-x), \quad for \quad x \leq 0 \Big\}, \tag{1}$$

where $(\tau_1)^{-1}$, $(\tau_2)^{-1}$ denote the inverse functions to τ_1, τ_2 respectively. One can notice that M is the set of all nonnegative market positions (portfolios), i.e. positions from which we can enter $\binom{0}{0}$. The set $-M$ is of the form

$$-M = \Big\{ \tbinom{x}{y} : y \leq (\tau_1)^{-1}(-x), \quad for \quad x \leq 0 \\ and \quad y \leq -(\tau_2)^{-1}(x), \quad for \quad x \geq 0 \Big\} \tag{2}$$

and is set of all positions which can be achieved starting from $\binom{0}{0}$.
We say that we have a *weak arbitrage at time* T (WA) if starting from the position $\binom{0}{0}$ at time 0 we enter the set M a.s. and with a positive probability the set $M \setminus \binom{0}{0}$ at time T. If starting from the position $\binom{0}{0}$ we enter the set

M a.s. and $intM$ with a positive probability at time T, we say that we have a *strict arbitrage opportunity at time* T (SA). Similarly we can characterize the absense of arbitrage. Denote by A_T the set of all portfolios which can be achieved at time T starting at time 0 from $\binom{0}{0}$. If there are no weak arbitrage at time T we have a *strict absense of arbitrage* (SAA), which means that

$$A_T \cap L^0(M, F_T) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}. \tag{3}$$

If there are no strict arbitrage at time T, which is the case when

$$A_T \cap L^0(M, F_T) \subset L^0(\partial M, F_T) \tag{4}$$

we have a *weak absense of arbitrage* (WAA).

Remark 1 *By the very definitions we immediately have:*

$$(SAA) \Rightarrow (WAA) \ \ and \ \ (SA) \Rightarrow (WA).$$

Let $\binom{x_t}{y_t}$ be our market position at time t before possible transactions. It can be completely characterized by initial position $\binom{x_0}{y_0}$ and the pair of (F_t) adapted nonnegative processes l_t, m_t by the following formula

$$\begin{aligned} x_{t+1} &= x_t - \tau_1(l_t) + \tau_2(m_t) \\ y_{t+1} &= \tfrac{S_{t+1}}{S_t}\left(y_t + l_t - m_t\right). \end{aligned} \tag{5}$$

Define for $t = 0, 1, \ldots, T$ sequences h_t, (H_t) of random transformations of R and R^2 respectively

$$h_t(y) := \frac{y}{S_t}, \qquad H_t \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} x \\ h_t(y) \end{pmatrix}. \tag{6}$$

Clearly

$$\begin{aligned} H_t M = \Big\{ \tbinom{x}{y} : y \geq \tfrac{-(\tau_1)^{-1}(x)}{S_t}, \ \ for \ \ x \geq 0 \\ and \ \ y \geq \tfrac{-(\tau_2)^{-1}(-x)}{S_t}, \ \ for \ \ x \leq 0 \Big\}, \end{aligned} \tag{7}$$

Let $\hat{y}_t = h_t(y_t)$, $\hat{l}_t = h_t(l_t)$, $\hat{m}_t = h_t(m_t)$ and $\hat{A}_T = H_T A_T$. We have

$$\hat{y}_{t+1} = \hat{y}_t + \hat{l}_t - \hat{m}_t, \tag{8}$$

where $\hat{y}_t$ is the number of assets in our portfolio before possible transactions at time t. Moreover from (8) we obtain

$$\hat{A}_T = \sum_{s=0}^{T} N_s, \tag{9}$$

where $N_s = -L^0(H_s M, F_s)$. Furthermore, (3) and (4) are equivalent respectively to

$$\hat{A}_T \cap L^0(H_T M, F_T) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \tag{10}$$

and

$$\hat{A}_T \cap L^0(H_T M, F_T) \subset L^0(\partial H_T M, F_T). \tag{11}$$

From the above consideration taking into account Remark 1 and (10), (11) we immediately have

Lemma 1 *(WA) with proportional transaction costs ($\tau_1(x) \equiv (1+\lambda)x$ and $\tau_2(x) \equiv (1-\mu)x$) implies (SA) with concave transaction costs. (WA) with concave transaction costs implies arbitrage without transaction costs. Conversely, absense of arbitrage without transaction costs implies (SAA) with concave transaction costs, while (WAA) with concave transaction costs implies (SAA) with proportional transaction costs.*

The following two examples explain the last lemma

Examples. Assume $T = 1$, and $S_1 = S_0(1+\xi)$, where the rate of return ξ is such that $P\{\xi = 0\} = p > 0$, $P\{\xi = r\} = 1 - p > 0$ for $r > 0$. We clearly see that we have an arbitrage without transaction costs. In the case with concave transaction costs starting from $\binom{x}{(\tau_1)^{-1}(-x)}$, $x < 0$ we would like to reach (or cross) the position $\binom{x}{(\tau_2)^{-1}(-x)}$ which is impossible, so that even (WA) with concave trasaction costs does not hold. Consider now deterministic rate of return $\xi = r > 0$. When $\frac{1+r}{1+\lambda} > \frac{1}{1-\mu}$ we have (SA) with proportional transaction costs. If

$$\lim_{x\to\infty} \dot{\tau}_1(x) = 1 = \lim_{x\to\infty} \dot{\tau}_2(x)$$

and $r > \lambda$, we have (SA) with concave transaction costs (we choose x so small that $(1+r)(\tau_1)^{-1}(-x) > (\tau_2)^{-1}(-x)$). On the other hand, when $\lambda < r < \frac{\lambda+\mu}{1-\mu}$, we have (WAA) with proportional transaction costs.

We have the following

Proposition 1 *Under nonlinear concave transaction costs (WAA) is equivalent to (SAA).*

Proof. By Remark 1 we have $(SAA) \Rightarrow (WAA)$. Assume we have (WAA) which means in its equivalent form that (11) holds. By the form of the set M if $\binom{x}{y} \in M$, then $k\binom{x}{y} \in intM$ for $k > 1$. The above property is preserved under the transformation H_s, so that whenever $n_s \in N_s$ then $kn_s \in L^0(intH_sM \cup \binom{0}{0}, F_s)$ for $k > 1$. If (10) is not satisfied (i.e. (SAA) does not hold), then there is a $v \in A_T \cap L^0(H_TM, F_T)$ such that $v \not\equiv \binom{0}{0}$. Consequently $kv \in A_T \cap L^0(H_TM, F_T)$ and $kv \notin L^0(\partial H_TM, F_T)$ for $k > 1$, which contradicts (WAA).

□

Remark 2 *Notice that for proportional transaction costs (WAA) does not imply (SAA) (see theorem 1 below). Moreover arbitrage without transaction costs at time T implies the existence of a one step arbitrage at certain deterministic time $t \leq T$ (see [1] theorem 3.3). This is no longer true in the case of concave transaction costs.*

Below we formulate a sufficient condition for arbitrage with concave transaction costs

Lemma 2 *If $\lim_{x\to\infty} \dot{\tau}_1(x) = \lim_{x\to\infty} \dot{\tau}_2(x)$ and there is $\epsilon > 0$ and $0 \leq t \leq T$ such that $\frac{S_t}{S_{t+1}} \leq 1 - \epsilon$, P a.e., then we have a strong arbitrage at time T with concave transaction costs.*

Proof. Let $K > 0$ be such that

$$\frac{(\tau_1)^{-1}(K)}{(\tau_2)^{-1}(K)} > 1 - \epsilon.$$

Choosing the position $\binom{-K}{(\tau_1)^{-1}(K)}$ at time t we obtain

$$(\tau_1)^{-1}(K) > (\tau_2)^{-1}(K)\frac{S_t}{S_{t+1}},$$

from which the arbitrage follows.

□

Another charaterizations of arbitrage use an analog of "positive dual set" $(H_tM)^*$ to the set H_tM defined as follows

$$(H_tM)^* := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \tau_2(S_tx) \leq y \leq \tau_1(S_tx) \;\; and \;\; x \geq 0 \right\}. \tag{12}$$

The following theorem characterizes arbitrage under proportional transaction costs

Theorem 1 *In the case of proportional transaction costs (SAA) is equivalent to the existence of an equivalent measure Q and an F_t adapted process ρ_t such that $1-\mu < \rho_t < 1+\lambda$, for $t = 0,1\ldots,T$ and $(\rho_t S_t)$ is a Q martingale. If the market is finite (the prices S_t of the asset admit a finite number of values only), in the case of proportional transaction costs, (WAA) is equivalent to the existence of an equivalent measure Q and an F_t adapted process ρ_t such that $1-\mu \leq \rho_t \leq 1+\lambda$, P a.e. for $t = 0,1\ldots,T$ and $(\rho_t S_t)$ is a Q martingale.*

Proof. Under (SAA) by theorem 2.1 of [3] there exists a bounded martingale $(\binom{z_1(t)}{z_2(t)})$ with strictly positive components such that

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \in L^0(int(H_tM)^*, F_t).$$

Let $dQ = \frac{z_1(T)}{z_1(0)}dP$ and $\rho_t := \frac{z_2(t)}{z_1(t)S_t}$. By Lemma 2 of [9] $(\rho_t S_t)$ is a Q martingale. Moreover by the form of the set $(H_tM)^*$ we have that $1-\mu \leq \rho_t \leq 1+\lambda$, P a.e.. On the other hand the process $(\binom{E[\frac{dQ}{dP}|F_t]}{\rho_t S_t E[\frac{dQ}{dP}|F_t]})$ is a P martingale taking values in $int(H_tM)^*$. Consequently for each $t \leq T$ there is a martingale (Z_s^t) such that Z_s^t takes values in $(H_sM)^*$, while Z_t^t in $int(H_tM)^*$. Adapting the proof of Lemma 3.6 of [3] we obtain (10), which completes the proof of the first equivalence. The second equivalence follows from theorem 3.2 of [4]. Namely, under (WAA) there is a martingale with strictly positive components $(\binom{z_1(t)}{z_2(t)})$ (see lemma 3.1 of [4]) taking values respectively in the set $(H_tM)^*$. As above we construct the process (ρ_t) and the measure Q. Conversely, if there exists an equivalent measure Q and an F_t adapted process ρ_t such that $1-\mu \leq \rho_t \leq 1+\lambda$, P a.e. for $t = 0,1\ldots,T$ such that $(\rho_t S_t)$ is a Q martingale, the process $(\binom{1}{\rho_t S_t})$ is a P martingale taking values in $(H_tM)^*$ and by theorem 3.2 of [4] (WAA) follows.

□

Remark 3 *By the proof of theorem 1 one can notice that under proportional transaction costs, (SAA) is also equivalent to the existence of an equivalent measure Q such that $\frac{dQ}{dP}$ is bounded and an F_t adapted process ρ_t such that $1-\mu < \rho_t < 1+\lambda$, for $t = 0,1\ldots,T$ and $(\rho_t S_t)$ is a Q martingale.*

In an analogy to theorem 1 we show below a sufficient condition for absense of arbitrage under nonlinear concave transaction costs. We denote by $(\cdot,\cdot)$ and $|\cdot|$ respectively the scalar product and Eulidean norm in R^2.

Proposition 2 *Assume that the market is finite and for each $r > 0$ there* *is a supermartingale* $Z(r)$ *such that* $|Z_s(r)| \geq r$, P *a.e. for* $s = 0, 1 \ldots, T$ *and*

1) $(Z_s(r), \xi) \leq 0$ *for any* $\xi \in N_s$ *such that* $|\xi| \leq |Z_s(r)|$,

2) if $(Z_T(r), \xi) = 0$ *for* $\xi \in N_T$ *such that* $|\xi| \leq |Z_T(r)|$, *then* $\xi = 0$,

Then we have (SAA).

Proof. If (10) does not hold, there exist $\xi_s \in N_s$ and $0 \neq \xi'_T \in N_T$ such that $\sum_{s=0}^{T} \xi_s = -\xi'_T$. Therefore $\sum_{s=0}^{T-1} \xi_s = -\xi_T - \xi'_T \in N_T$. Choose

$$r \geq \max\left\{\max_s |\xi_s|, |\xi_T + \xi'_T|\right\}.$$

Then

$$(Z_T(r), \sum_{s=0}^{T-1} \xi_s) = -(Z_T(r), (\xi_T + \xi'_T)) \geq 0$$

and by the supermartingale property

$$E\left\{(Z_T(r), \sum_{s=0}^{T-1} \xi_s)\right\} \leq \sum_{s=0}^{T-1} E\left\{(Z_s(r), \xi_s)\right\} \leq 0.$$

Therefore $(Z_T(r), \xi_T + \xi'_T) = 0$ and by (2) $\xi_T + \xi'_T = 0$ from which it follows that $\xi_T = \xi'_T = 0$, a contradiction.

□

3 Set of hedging endowments

Let $C = \binom{C_1}{C_2}$ be a pair of integrable F_T measurable random variables called a *contingent claim*. If for an initial portfolio $\binom{x}{y}$ there is an investment strategy (l_t, m_t) such that $\binom{x_T}{y_T} \in \binom{C_1}{C_2} + M$, we say that starting from $\binom{x}{y}$ we *hedge* the contingent claim C. Our purpose now is to characterize the set of hedging endowments Γ described as follows

$$\Gamma = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : C \in v + A_T \right\}. \tag{13}$$

Let $\mathcal{P}$ be a class of pairs $((\rho_t), Q)$ consisting of (F_t) adapted processes (ρ_t) taking values in the interval $[1-\mu, 1+\lambda]$ and an equivalent probability measure Q such that $\frac{dQ}{dP}$ is bounded and $(\rho_t S_t)$ is a Q martingale.
Following [2] and [3] (see also [9]) we have

Theorem 2 *Under (SAA) for proportional transaction costs we have*

$$\Gamma = \bigcap_{((\rho_t),Q)\in\mathcal{P}} \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : v_1 + \rho_0 v_2 \geq E^Q\{C_1\} + E^Q\{\rho_T C_2\} \right\}. \tag{14}$$

Proof. By theorem 4.2 [3]

$$\Gamma = \bigcap_{(Z(t))\in\mathcal{Z}} \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : E\{z_1(T)C_1 + h_T(z_2(T))C_2\} \leq z_1(0)v_1 + h_0(z_2(0))v_2 \right\} \tag{15}$$

where $\mathcal{Z}$ is the set of bounded nonnegative martingales $(\binom{z_1(t)}{z_2(t)})$ such that

$$(1-\mu)z_1(t) \leq h_T(z_2(t)) \leq (1+\lambda)z_1(t), \quad P\ a.e.$$

for $t = 0, 1 \ldots, T$. Notice that we can restrict the set $\mathcal{Z}$ to strictly positive martingales. Let $(Z(t)) \in \mathcal{Z}$ be a strictly positive martingale. Define $dQ = \frac{z_1(T)}{z_1(0)}dP$ and $\rho_t := \frac{z_2(t)}{z_1(t)S_t}$. Clearly $((\rho_t), Q) \in \mathcal{P}$. Therefore the set defined by the right hand side of (14) is contained in the set Γ given in (15). On the other hand if $v \in \Gamma$ defined in (13) then

$$\begin{pmatrix} C_1 \\ \frac{C_2}{S_T} \end{pmatrix} = \begin{pmatrix} v_1 \\ \frac{v_2}{S_0} \end{pmatrix} + \sum_{s=0}^{T} \xi_s,$$

with $\xi_s \in N_s$. Let $((\rho_t), Q) \in \mathcal{P}$. Since

$$[1 \quad \rho_T S_T] \sum_{a=0}^{T} \xi_s = C_1 + \rho_T C_2 - v_1 - v_2 \frac{S_T}{S_0}\rho_T$$

is Q integrable, by Lemma 3.5 [3] we obtain

$$v_1 + v_2 E^Q\left\{\frac{\rho_T S_T}{S_0}\right\} \geq E^Q\{C_1\} + E^Q\{\rho_T C_2\}.$$

Therefore v is a element of the set defined in (14). Thus the sets defined in (13), (14) and (15) coincide, which completes the proof.

□

Remark 4 *There is a problem to characterize the set Γ in the case of non-linear concave transaction costs even when we have a finite financial market. We conjecture that*

$$\Gamma = \Big\{ v = \binom{v_1}{v_2} : E\left\{z_1(T)C_1 + h_T(z_2(T))C_2\right\} \leq$$
$$z_1(0)v_1 + h_0(z_2(0))v_2 \quad for \quad each \quad martingale$$
$$Z(t) = \binom{z_1(t)}{z_2(t)} \in L^0((H_tM)^*, F_t) \; such \;\; that$$
$$|Z(t)| \geq |\xi_t|, \quad where \quad \xi_t \in N_t \quad for \quad t = 0, 1, \dots, T$$
$$and \quad H_T C - H_0 v = \sum_{s=0}^{T} \xi_s, \Big\} . \tag{16}$$

It is not clear that for each $r > 0$ there is a martingale $Z(t)$ taking values in $(H_tM)^*$ such that $|Z(t)| \geq r$, which we impose for a sufficient condition of (SAA) in Proposition 2.

The remaining part of this section we devote to the study of the form of the set Γ for a so called *Cox Ross Rubinstein model* (CRR) with concave transaction costs, which is a finite market model with i.i.d. rates of return $\xi_t := \frac{S_{t+1} - S_t}{S_t}$ such that $0 < P\{\xi_t = a\} = 1 - P\{\xi_t = b\} < 1$, for $-1 < a < 0 < b$. We assume now that $C_1 = c_1(S_T)$ and $C_2 = S_T c_2(S_T)$ for Borel functions c_1 and c_2.

Define

$$\tau(z) := \begin{cases} \tau_1(z) \text{ if } \; z \geq 0 \\ -\tau_2(-z) \text{ if } \; z \leq 0 \end{cases} \tag{17}$$

which is the ammount of money we spend (or we get) to achieve the position z on the stock account. Denote by (η_t, θ_t) the market position consisting of the ammount of money η_t on the bank account and ammount of assets θ_t held in the portfolio at time t after possible transactions. If $\eta_T = c_1(S_T)$ and $\theta_T = c_2(S_T)$, we say we *replicate* the contingent claim C. In the case of CRR model we have replication when

$$\eta_{T-1} - c_1(S_{T-1}(1+e)) = \tau((c_2(S_{T-1}(1+e)) - \theta_{T-1})S_{T-1}(1+e)) \tag{18}$$

holds for $e = a, b$. Let

$$\Phi(c_1, c_2)(\theta, s) = c_1((1+a)s) - c_1((1+b)s) +$$
$$\tau((c_2((1+a)s) - \theta)s((1+a)) - \tau((c_2((1+b)s) - \theta)s(1+b)). \tag{19}$$

One can notice (see proof of theorem 5 of [9]) that under

$$\frac{1+b}{1+a} > \frac{1+\lambda}{1-\mu} \tag{20}$$

the mapping $\theta \mapsto \Phi(c_1, c_2)(\theta, s)$ is strictly increasing with the range equal to R, so that $[\Phi(c_1, c_2)(\cdot, s)]^{-1}(0)$ is uniquely defined. Let

$$\Psi_2(c_1, c_2)(s) := [\Phi(c_1, c_2)(\cdot, s)]^{-1}(0),$$

$$\Psi_1(c_1, c_2)(s) := c_1((1+a)s) + \tau((c_2(1+a)s) - \Psi_2(c_1, c_2)(s)s(1+a))$$

and $\Psi(c_1, c_2)(s) := (\Psi_1(c_1, c_2)(s), \Psi_2(c_1, c_2)(s))$. Combining theorem 5 of [9] and [7] we obtain

Theorem 3 *Under (20) there is a unique replicating strategy (η_t, θ_t) for CRR model and is in the form of iterations of the operator Ψ as follows*

$$(\eta_t, \theta_t) = \Psi^{T-t}(c_1, c_2)(s_t) \tag{21}$$

If additionally

$$c_2((1+a)s) \leq \Psi_2(c_1, c_2)(s) \leq c_2((1+b)s) \tag{22}$$

for $s \geq 0$, then

$$\Psi_2(c_1, c_2)((1+a)s) \leq \Psi_2^2(c_1, c_2)(s) \leq \Psi_2(c_1, c_2)((1+b)s). \tag{23}$$

If $b > \lambda$ and $a + \mu > 0$ and (22) is satisfied then

$$\Gamma = \begin{pmatrix} \eta_0 \\ \theta_0 S_0 \end{pmatrix} + M. \tag{24}$$

Remark 5 *Under (22), when $S_{T-1} = (1+a)S_{T-2}$ we have*

$$\theta_{T-1} = \Psi_2(c_1, c_2)((1+a)S_{T-2}) \leq \theta_{T-2}.$$

Moreover if $S_{T-1} = (1+b)S_{T-2}$ we have

$$\theta_{T-1} = \Psi_2(c_1, c_2)((1+b)S_{T-2}) \geq \theta_{T-2}.$$

By induction we easily obtain the following property of the replicating strategy: when the price of the asset increases we buy assets, while when it decreases we sell assets. The form (24) of the set Γ is true in a number of cases (even when (20) does not hold) for proportional transaction costs (see [8] or [9] for details).

4 Pricing

Given a characterization of the set of initial endowments Γ we are interested to know the *seller price* $p_s(C)$ at time $t = 0$ of the contingent claim C. It is the minimal ammount of money on our bank account from which we can enter the set Γ.

Theorem 4 *Under (SAA) for proportional transaction costs we have*

$$p_s(C) = \sup_{(\rho_t),Q)\in\mathcal{P}} E^Q \left\{C_1 + \rho_T C_2\right\}. \tag{25}$$

Proof. Notice first that by theorem 2, $p_s(C) \geq E^Q \{C_1 + \rho_T C_2\}$. To show the inverse inequality it remains to follow the proof of Theorem 4 in [9] which is based on the arguments from the proof of theorem 1 in [6].

□

In the remaining part of this section we shall consider a contingent claim C of the form $C_1 = c_1(S_T)$ and $C_2 = S_T c_2(S_T)$. Let Θ denote the class of (F_t) adapted processes (θ_t) such that there exists an adapted process (η_t) for which we have

$$\eta_{t+1} - \eta_t + \tau((\theta_{t+1} - \theta_t)S_{t+1}) = 0 \tag{26}$$

for $t = 0, \ldots, T-1$, and

$$c_1(S_T) - \eta_{T-1} + \tau((c_2(S_T) - \theta_{T-1})S_T) \leq 0. \tag{27}$$

Notice that (26) means that strategy (η_t, θ_t) is *selffinacing* i.e. there are not exogenous infusion or withdrawal of money, while (27) says that strategy (η_t, θ_t) hedges the contingent claim C.

In what follows we extend functions τ_1 and τ_2 letting $\tau_1(x) = -\tau_1(-x)$ and $\tau_2(x) = -\tau_2(-x)$ for $x \leq 0$.

Given $(\theta_t) \in \Theta$ define the class $\Lambda((\theta_t))$ consisting of the pairs $((\rho_t), Q)$, where Q is an equivalent measure such that $\frac{dQ}{dP}$ is bounded, and ρ_t is an (F_t) adapted function valued process for which

$$\min\{\tau_1(x), \tau_2(x)\} \leq \rho_t(x) \leq \max\{\tau_1(x), \tau_2(x)\} \tag{28}$$

for $x \in R$,

$$E^Q[\rho_t(\theta_t S_t) - \rho_t((\theta_t - \theta_{t-1})S_t)|F_{t-1}] \leq \rho_{t-1}(\theta_{t-1}S_{t-1}) \tag{29}$$

Q a.e. for $t = 1, \dots, T-1$ and

$$E^Q[\rho_T(c_2(S_T)S_T) - \rho_T((c_2(S_T) - \theta_{T-1})S_T)|F_{T-1}] \leq \rho_{T-1}(\theta_{T-1}S_{T-1}) \quad (30)$$

Q a.e..
We have
Proposition 3 *If there is $(\theta_t) \in \Theta$ such that $\Lambda((\theta_t))$ is nonempty, then*

$$p_s(C) \geq \inf_{(\theta_t)\in\Theta} \sup_{(\rho_t),Q)\in\Lambda((\theta_t))} E^Q\left\{c_1(S_T) + \rho_T(c_2(S_T)S_T)\right\}. \quad (31)$$

Proof. Let (η_t, θ_t) be a strategy which hedges the contingent claim C and $((\rho_t), Q) \in \Lambda((\theta_t))$. By (27) and (30) taking into account that $\tau \geq \rho_T$ we obtain

$$\begin{aligned} E^Q[c_1(S_T) + \rho_T(S_T c_2(S_T))|F_{T-1}] \leq \\ E^Q[\eta_{T-1} + \rho_T(S_T c_2(S_T)) - \tau((c_2(S_T) - \theta_{T-1})S_T)|F_{T-1}] \\ \leq \eta_{T-1} + E^Q[\rho_T(S_T c_2(S_T)) - \rho_T((c_2(S_T) - \theta_{T-1})S_T)|F_{T-1}] \\ \leq \eta_{T-1} + \rho_{T-1}(\theta_{T-1}S_{T-1}) \end{aligned} \quad (32)$$

Q a.e.. Moreover, the process $(\eta_t + \rho_t(\theta_t S_t))$ is for $t \leq T-1$ a Q supermartingale. In fact, by (26), inequality $\tau \geq \rho_t$ and (29) we obtain that

$$E^Q[\eta_t + \rho_t(\theta_t S_t)|F_{t-1}] \leq \eta_{t-1} + \rho_{t-1}(\theta_{t-1}S_{t-1}),$$

Q a.e. for $t \leq T-1$. Therefore by (32) and inequality $\tau \geq \rho_0$

$$\sup_{(\rho_t),Q)\in\Lambda((\theta_t))} E^Q\left\{c_1(S_T) + \rho_T(c_2(S_T)S_T)\right\} \leq \eta_0 + \tau(\theta_0 S_0)$$

from which (31) follows.

□

Notice that when ρ_t is a linear function with linear coefficient denoted with an ambiguity by ρ_t, such that $(\rho_t S_T)$ is a Q martingale, the inequalities (29) and (30) are in fact equalities and comparing (31) with (25) we obtain
Corollary 1 *In the case of proportional transaction costs under (SAA) we have an equality in the formula (31).*
To study the formula (31) for CRR model with nonlinear concave transaction costs we have to introduce a new notation. Let

$$\tau'(\theta) := \begin{cases} \tau_1 \text{ if } \theta \geq 0 \\ -\tau_2 \text{ if } \theta \leq 0 \end{cases} \quad (33)$$

In the case of CRR model we can consider $\Omega = \{a, b\}^T$ and $\omega = (\omega_1, \ldots, \omega_T)$. For an F_{t+1} adapted random variable ξ_{t+1} define F_t measurable random variables ξ_t^a and ξ_t^b as follows: we let for $\xi_t^a(\omega)$ and $\xi_t^b(\omega)$ the values of $\xi_{t+1}(\omega')$, where ω' differs from ω only at ω_{t+1} which is equal respectively to a or b. Under (20) by theorem 3 there is a unique replicating strategy (η_t, θ_t) given by the formula (21). Let

$$q_t^b := \frac{n_t^b}{d_t^b}, \tag{34}$$

where

$$n_t^b := \tau((\theta_t^a - \theta_t)(1+a)S_t) - \tau'(\theta_t^a - \theta_t)(\theta_t^a S_t(1+a)) + \tau'(\theta_t - \theta_{t-1})(\theta_t S_t)$$

and

$$d_t^b := \tau((\theta_t^a - \theta_t)(1+a)S_t) - \tau'(\theta_t^a - \theta_t)(\theta_t^a S_t(1+a)) +$$
$$\tau'(\theta_t^b - \theta_t)(\theta_t^b S_t(1+b)) - \tau(\theta_t^b - \theta)(1+b)S_t).$$

We have

Lemma 3 *Under (20), (22), when $a + \mu > 0$ we have*

$$0 < q_t^b < 1. \tag{35}$$

Proof. Technical based on the properties of the functions τ_1 and τ_2 consists in consideration of all possible positions of $\theta_t^a \leq \theta_t \leq \theta_t^b$ (inequality holds by (23)) and 0.

□

We can now generalize Proposition 3.5 of [5]

Proposition 4 *Under (20), (22), assuming furthermore that $b > \lambda$ and $a + \mu > 0$ for the replicating strategy (η_t, θ_t) in CRR model we have*

$$E^Q\left\{c_1(S_T) + \rho_T(c_2(S_T)S_T)\right\} = \eta_0 + \tau(\theta_0 S_0) \tag{36}$$

where $Q\{\xi_t = a\} = 1 - q_t^b$, $\rho_T(x) = \tau'(\theta_T - \theta_{T-1})(x)$ and $\rho_t(x) = \tau'(\theta_t - \theta_{t-1})(x)$, for $0 \leq t \leq T-1$ with $\theta_{-1} := 0$.

Consequently we have the equality in (31).

Proof. Notice that the process $(\eta_t + \rho_t(\theta_t S_t))$ is a Q martingale. Since $\rho_0(\theta_0 S_0) = \tau(\theta_0 S_0)$ we therefore have (36). Combining (36) with (31) and (24) we obtain the equality in (31).

□

Remark 6 *We have showed equality in (31) for proportional transaction costs and for CCR model with nonlinear concave transaction costs. One can expect the equality (31) to be true as a general rule.*

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