Moment stability for linear systems with a random parametric excitation

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Abstract

Moment stability for linear systems with a non-white parametric noise is considered. A method of reduction of the study of this stability to the study of stability for large-scale matrices is proposed. Mean square stability diagrams for random harmonic oscillator are presented.

Keywords: Stochastic linear systems; Moment stability; Stability diagrams.

1 Introduction.

Stability and stabilization is a fundamental objective in the design of controllers. This work deals with a stability for the following linear ordinary differential equation in $\mathbb{R}^N$

$$\frac{dx}{dt} = Ax + \xi(t)Cx, \quad 0 < t < \infty, \quad (1)$$

where $A, C$ are $N \times N$ deterministic matrices and $\xi(t)$ is a stochastic process. We investigate $p$-stability, that is, asymptotic stability of the $p$th moments of the solution of equation (1) for $p$ to be a positive integer.

There are necessary and sufficient conditions of such stability, if $\xi(t)$ is the white noise [5-7]. If the process $\xi(t)$ is non-white and has continuous trajectories, the study of stability turn out to be complicated. In this case there are necessary and sufficient conditions of moment stability only if $\xi(t)$ is a Gaussian process and the Lie algebra generated by the matrices $A, C$ is solvable [1,10-12].

In this paper we consider the process $\xi(t)$ of the form

$$\xi(t) = \sin(\alpha w(t)), \quad (2)$$

where $\alpha$ is a nonrandom parameter and $w(t)$ is Wiener process. We propose a method that reduces the study of moment stability for the equation (1) with the
noise of the form (2) to the stability of nonrandom large-scale matrices. Using this
method and numeric computation we get diagrams of 2-stability (mean square
stability) for the random harmonic oscillator.

2 Equations for moments.

Let us start first from the mean value \( Ex(t) \) of the solution of equation (1)
with excitation (2) and nonrandom initial value \( x(0) \). The solution of (1) is a
functional with respect to the process \( w(s), \ 0 \leq s \leq t \), and therefore we shall
write \( x(t) = x(t; w(s)) \).

Let

\[
\begin{align*}
    v_0(t) &:= Ex(t; w(s)), \\
    v_k(t) &:= \frac{1}{(2i)^k} \exp \left\{ \frac{-k^2 \alpha^2 t}{2} \right\} [Ex(t; w(s) + ik\alpha)] \\
    &+ (-1)^k Ex(t; w(s) - ik\alpha)], \ k = 1, 2, 3, \ldots
\end{align*}
\]

To evaluate the mean value \( v_0(t) = Ex(t) \) of the solution of equation (3) with
nonrandom initial value \( x(0) \) the following infinite chain of linear differential
equations was obtained in the paper [2]:

\[
\begin{align*}
    \frac{dv_0}{dt} &= Av_0 + Cv_1, \\
    \frac{dv_1}{dt} &= -\frac{\alpha^2}{2} v_1 + Av_1 + Cv_2 + \frac{1}{2} Cv_0, \\
    \frac{dv_k}{dt} &= -\frac{k^2 \alpha^2}{2} v_k + Av_k + Cv_{k+1} + \frac{1}{4} Cv_{k-1}, \ k = 2, 3, \ldots, \\
    v_0(0) &= x(0), \ v_k(0) = \frac{1}{(2i)^k} [1 + (-1)^k] x(0), \ k = 1, 2, 3, \ldots.
\end{align*}
\]

Consequently, the 1-stability of equation (1), (2) is equivalent to the asymptotic
stability of the chain (4). Note that (see [3]) one can also obtain a similar chain
when \( \xi(t) \) is an Ornstein-Uhlenbeck process.

One can easily obtain a similar chain for higher order moments in the following
way. Let us introduce under [4,5] the binomial coefficient \( b(N, p) = \binom{N+p-1}{p} \)
and the vector \( x^{[p]}(t) \in R^{b(N,p)} \) of all \( p \)-th forms of the components of the vector
\( x(t) \):

\[
x^{[p]}(t) = col(x_1(t)^p, \mu_1^p x_1(t)^{p-1} x_2(t), \ldots, x_N(t)^p),
\]

where parameters \( \mu_i^j \) are chosen so as to obtain \( \|x^{[p]}(t)\| = \|x(t)\|^p \) for the Eu-
clidean norm of vectors.

For vector \( x^{[p]}(t) \) we have under (1) the equation

\[
\frac{dx^{[p]}}{dt} = A^{[p]} x^{[p]} + \xi(t) C^{[p]} x^{[p]},
\]

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where the matrices $A[p], C[p]$ are determined from the matrices $A, C [4,5].$

Using the previous approach one can obtain for $E x^{(p)}(t)$ the chain (3) with $A[p], C[p]$ instead of $A, C$. So without loss of generality one can investigate only the 1-stability.

3 Stability for the chain.

Let $Q_n$ be the following $nN \times nN$ block matrix

$$Q_n := \begin{pmatrix} A & C & 0 & 0 & 0 & \ldots & \ldots & 0 \\
\frac{1}{2}C & A & C & 0 & 0 & \ldots & \ldots & 0 \\
0 & \frac{1}{4}C & A & C & 0 & \ldots & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \ldots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & \ldots & \frac{1}{7}C & A \end{pmatrix}$$

and

$$P_n := \text{diag}(0, -\frac{\alpha^2}{2}E, \ldots, -\frac{(n-1)^2\alpha^2}{2}E), \quad S_n := \text{diag}(0, \ldots, 0, C),$$

$$u_n(t) := \text{col}(v_0(t), \ldots, v_{n-1}(t)),$$

with $E - N \times N$ identity matrix. We can rewrite (3) in the following form:

$$\frac{du_n}{dt} = P_n u_n + Q_n u_n + S_n v_n,$$  \(4\)

$$\frac{dv_k}{dt} = -\frac{k^2\alpha^2}{2}v_k + Av_k + Cv_{k+1} + \frac{1}{4}Cv_{k-1}, \quad k = n, n + 1, \ldots$$  \(5\)

Denote by $\lambda_j, \ j = 1, \ldots, nN$, eigenvalues of matrix $P_n + Q_n$. We show that the stability of this matrix, namely the signs of real parts of its eigenvalues, for sufficiently large $n$ is crucial in the study of 1-stability for equation (1), (2).

Assume that there exists a $\gamma > 0$ such that for sufficiently large $n$

$$\max_j \text{Re} \ \lambda_j < -\gamma,$$  \(6\)

Denote by $\| \cdot \|$ the Euclidean norm for vectors and induced norm for matrices. It is known [8] that

$$\|Q_n\| \leq \|Q_n\|_1\|Q_n\|_\infty,$$

where $\| \|$ and $\| \|$ is the maximum column sum norm and maximum row sum norm respectively. It follows from this and the form of matrix $Q_n$ that there exists a $c_1 > 0$ such that for all $n$

$$\|Q_n\| < c_1.$$  \(7\)
Using the triangular factorization [8], there exists a unitary matrix $U_n$ such that

$$U_n(P_n + Q_n)U_n' = D_n + B_n,$$

where

$$D_n = \text{diag}(\lambda_1, \ldots, \lambda_{nN}), \quad B_n = (a_{ij}), \quad a_{ij} = 0, \quad i \geq j, \quad i, j = 1, \ldots, nN.$$ \hspace{1cm} (7)

Since (7) there exists a $c_2 > 0$ such that for all $n$

$$\|B_n\| < c_2. \hspace{1cm} (8)$$

Let $V_n$ be a diagonal matrix:

$$V_n = \text{diag}(\delta^{-1}, \delta^{-2+1/2}, \ldots, \delta^{-2+2^{2-nN}}), \quad \delta \in (0, 1).$$

It follows from (4) that the vector $z_n = V_nU_nu_n$ satisfies the equation

$$\frac{dz_n}{dt} = D_nz_n + V_nB_nV_n^{-1}z_n + V_nU_nS_nv_n, \hspace{1cm} (9)$$

where

$$V_nB_nV_n^{-1} = (a_{ij} \delta^{2^{i-j} - 2^j}).$$

Using (8), we have that for each $\varepsilon > 0$ there exist $\delta \in (0, 1), n_0 > 0$ such that for $n > n_0$

$$\|V_nB_nV_n^{-1}\| < \varepsilon. \hspace{1cm} (10)$$

From (9) it follows that

$$\frac{d\|z_n\|^2}{dt} = (D_nz_n, \bar{z}_n) + (\bar{D}_n\bar{z}_n, z_n) + (V_n\bar{D}_nV_n^{-1}z_n, \bar{z}_n) + (V_nB_nV_n^{-1}z_n, \bar{z}_n) + (V_nU_nS_nv_n, \bar{z}_n).$$

Taking into account that $\|V_n\| \leq \delta^{-2}$ and $\|S_n\| = \|C\|$, we obtain

$$|(V_n\bar{D}_nS_n\bar{z}_n, z_n) + (V_nU_nS_nv_n, \bar{z}_n)| \leq \varepsilon\|z_n\|^2 + \varepsilon^{-1}\delta^{-4}\|C\|^2\|v_n\|^2.$$

Hence,

$$\frac{d\|z_n\|^2}{dt} \leq -2\gamma\|z_n\|^2 + 3\varepsilon\|z_n\|^2 + \varepsilon^{-1}\delta^{-4}\|C\|^2\|v_n\|^2. \hspace{1cm} (11)$$

Using (5), we get

$$\frac{d\|v_k\|^2}{dt} = -k^2\alpha^2\|v_k\|^2 + (A\bar{v}_k, v_k) + (Av_k, \bar{v}_k) + (C\bar{v}_{k+1}, v_k) + (Cv_{k+1}, \bar{v}_k) + \frac{1}{4}(C\bar{v}_{k-1}, v_k) + \frac{1}{4}(Cv_{k-1}, \bar{v}_k), \quad k = n, n + 1, \ldots.$$
\[
\frac{d\|v_k\|^2}{dt} \leq -k^2 \alpha^2 \|v_k\|^2 + 2\|A\|\|v_k\|^2 + \|C\|\|v_k\|^2 + \\
\|C\|\|v_{k+1}\|^2 + \frac{1}{2}\|C\|\|v_k\|^2 + \frac{1}{2}\|C\|\|v_{k-1}\|^2.
\]

From the definition of vectors \(v_k(t)\) it follows that the series
\[
\sum_{k=n}^{\infty} k^2\|v_k\|^2, \quad \sum_{k=n}^{\infty} \frac{d\|v_k\|^2}{dt}
\]
are converged.
Therefore,
\[
\sum_{k=n}^{\infty} \frac{d\|v_k\|^2}{dt} \leq \frac{1}{2}\|C\|\|v_{n-1}\|^2 + \sum_{k=n}^{\infty} (-k^2 \alpha^2 + 2\|A\| + 2, 5\|C\|)\|v_k\|^2
\]
\[
\leq \frac{1}{2}\|C\|\|v_{n-1}\|^2 + (-n^2 \alpha^2 + 2\|A\| + 2, 5\|C\|) \sum_{k=n}^{\infty} \|v_k\|^2.
\]
If \(q_n := n^2 \alpha^2 - 2\|A\| + 2, 5\|C\|\), then from the previous inequality we have
\[
\frac{d}{dt}(e^{qn\tau} \sum_{k=n}^{\infty} \|v_k(t)\|^2) \leq \frac{1}{2}\|C\|\|v_{n-1}(t)\|^2,
\]
and so
\[
\sum_{k=n}^{\infty} \|v_k(t)\|^2 \leq \frac{1}{2}\|C\| \int_{0}^{t} e^{-q\tau} \|v_{n-1}(s)\|^2 ds + e^{-qn\tau} \sum_{k=n}^{\infty} \|v_k(0)\|^2.
\]
Since \(\|v_{n-1}\| \leq \|u_n\| \leq \|V_n\| \leq \|z_n\|\) and
\[
\sum_{k=n}^{\infty} \|v_k\|^2 \leq \sum_{k=n}^{\infty} \frac{\|x(0)\|^2}{4^{k-1}} \leq \frac{\|x(0)\|^2}{3 \cdot 4^{n-2}},
\]
we obtain
\[
\|v_n(t)\|^2 \leq \frac{1}{2}\|C\| \int_{0}^{t} e^{-q\tau} \|v_{n-1}(s)\|^2 ds + \frac{\|x(0)\|^2}{3 \cdot 4^{n-2}} e^{-qn\tau}. \quad (12)
\]
Let \(\eta := 2\gamma - 3\varepsilon\). Using Gronwall inequality to (11), we have
\[
\|z_n(t)\|^2 \leq \varepsilon^{-1}\delta^{-4}\|C\|^2 \int_{0}^{t} \exp\{-\eta(t-s)\} \|v_n(s)\|^2 ds + e^{-\theta\tau}\|z_n(0)\|^2.
\]
Substituting (12) for \(\|v_n\|\), we obtain
\[
\|z_n(t)\|^2 \leq \frac{1}{2}\varepsilon^{-1}\delta^{-4}\|C\|^3 \int_{0}^{t} \int_{0}^{s} \exp\{-\eta(t-s) - q_n(s-s_1)\} \|z_n(s_1)\|^2 ds_1 ds
\]
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\[ + \epsilon^{-1} \delta^{-4} \| C \| \int_0^t \exp \{- \eta (t - s) - q_n s \} ds + \epsilon^{-\eta t} \| z_n(0) \|^2. \]

Let \( \epsilon, n_0 > 0 \) be such that \( \eta > 0 \) and \( q_n > \eta \) for \( n > n_0 \). Therefore for \( n > n_0 \) we have

\[
\| z_n(t) \|^2 < \frac{\| C \|^3}{2(q_n - \eta) \epsilon \delta^4} \int_0^t e^{-\eta (t-s)} \| z_n(s) \|^2 ds
\]

\[
+ [\| z_n(0) \|^2 + \frac{\epsilon^{-1} \delta^{-4} \| C \|^2 \| x(0) \|^2}{3 \cdot 4^{n-2} (q_n - \eta)}] e^{-\eta t}. \tag{13}
\]

Let now \( n_1 > n_0 \) such that for \( n > n_1 \)

\[ p_n := \eta - \frac{\| C \|^3}{2(q_n - \eta) \epsilon \delta^4} > 0. \]

Taking into account that

\[
\| z_n(0) \|^2 = \| V_n U_n u_n(0) \|^2 \leq \| V_n \|^2 \| u_n(0) \|^2
\]

\[
\leq \delta^{-4} \sum_{k=0}^{n-2} 4^{-k} \| x(0) \|^2 < \frac{4}{3} \delta^{-4} \| x(0) \|^2
\]

and applying Gronwall inequality to (13), we obtain

\[
\| z_n(t) \|^2 \leq \frac{4}{3} \delta^{-4} \left[ 1 + \frac{\epsilon^{-1} \| C \|}{4^{n-1} (q_n - \eta)} \right] e^{-\eta \epsilon p_n t} \| x(0) \|^2. \tag{14}
\]

Thus, we have the mean square stability of equation (1), (2) whenever the condition (6) holds.

Now, we suppose that the matrix \( Q_n \) has eigenvalues with positive real parts for sufficiently large \( n \). Without loss of generality let \( Re \lambda_j > 0, j = 1, \ldots, k, \ Re \lambda_j \leq 0, j = k + 1, \ldots, nN \). Moreover, we suppose that there exists a \( \sigma > 0 \) such that for sufficiently large \( n \)

\[
Re \lambda_j > \sigma, j = 1, \ldots, k, \ Re \lambda_j \leq 0, j = k + 1, \ldots, nN. \tag{15}
\]

Let \( z_n^{(j)}, j = 1, \ldots, nN \), be components of vector \( z_n \) and

\[
F(z_n, v_n, v_{n+1}, \ldots) := \frac{1}{2} \left[ \sum_{j=1}^k \| z_n^{(j)} \|^2 - \sum_{j=k+1}^{nN} \| z_n^{(j)} \|^2 - \sum_{k=n}^{\infty} \| v_k \|^2 \right].
\]

By (5), we have

\[
\frac{d\| v_n \|^2}{dt} = \left( \frac{dv_n}{dt}, \bar{v}_n \right) + (v_n, \frac{d\bar{v}_n}{dt}) = \left( -\frac{n^2 \alpha^2}{2} v_n + Av_n + C v_{n+1} + \frac{1}{4} C v_{n-1}, \bar{v}_n \right)
\]

\[
(\bar{v}_n, -\frac{n^2 \alpha^2}{2} \bar{v}_n + A \bar{v}_n + C \bar{v}_{n+1} + \frac{1}{4} C \bar{v}_{n-1}) = -n^2 \alpha^2 \| v_n \|^2 + (Av_n, \bar{v}_n).
\]
\begin{align*}
&+ (A \bar{v}_n, v_n) + (C^* v_{n+1}, \bar{v}_n) + (C^* v_{n+1}, v_n) + \frac{1}{4} [(C v_{n-1}, \bar{v}_n) + (C^* v_{n-1}, v_n)] \\
&\leq (-n^2 \alpha^2 + 2\|A\| + \frac{n}{4}\|C\|) \|v_n\|^2 + \|C\| \|v_{n+1}\|^2 + \frac{1}{4n} \|C\| \|z_{n-1}\|^2 \]
\end{align*}

\frac{d\|v_k\|^2}{dt} \leq (-k^2 \alpha^2 + 2\|A\| + \frac{3}{2}\|C\|) \|v_k\|^2 + \|C\| \|v_{k+1}\|^2 + \frac{1}{2}\|C\| \|v_{k-1}\|^2,

k = n + 1, n + 2, \ldots.

Therefore by (9) we obtain

\begin{align*}
\frac{dF}{dt} &\geq \sum_{j=1}^{n} Re \lambda_j \|z_n^{(j)}\|^2 - \sum_{j=k+1}^{\infty} Re \lambda_j \|z_n^{(j)}\|^2 + \sum_{j=n}^{\infty} q_j \|v_j\|^2 + \rho_n \|z_n\|^2;
\end{align*}

where \( q_j = j^2 \alpha^2 - 2\|A\| - 3\|C\| \) and by (10) for any \( \varepsilon_1 > 0 \) there exist \( \delta \in (0, 1) \) and \( n_2 \) such that for \( n > n_2 \) we have \( |\rho_n| < \varepsilon_1 \).

Consequently for sufficiently large \( n \), we have

\begin{align*}
\frac{dF}{dt} &\geq \frac{\sigma}{2} \sum_{j=1}^{nN} \|z_n^{(j)}\|^2 \geq \sigma F;
\end{align*}

from which

\begin{align*}
F(z_n(t), v_n(t), v_{n+1}(t), \ldots) \geq F(z_n(0), v_n(0), v_{n+1}(0), \ldots) e^{\sigma t}.
\end{align*}

Thus equation (1), (2) is not 1-stable when the condition (15) holds.

4 Random harmonic oscillator.

Consider the equation of harmonic oscillator

\begin{equation}
\frac{d^2 y}{dt^2} + a \frac{dy}{dt} + [1 + \beta \xi(t)] y = 0,
\end{equation}

where \( a > 0 \), \( \beta \) are nonrandom parameters and \( \xi(t) \) is a random process.

There are many works dealing with stability for this equation (see [9] for a review). Necessary and sufficient conditions of mean square stability of (15) are obtained in the case of white noise process \( \xi(t) \) in [7].

We use the results of previous section to obtain exact regions of mean square stability for equation (16) with noise (2).

One can easily obtain that the vector

\begin{equation}
x := col \left( \left( \frac{dy}{dt} \right)^2, y \frac{dy}{dt}, y^2 \right)
\end{equation}

satisfies the equation (3) in \( \mathbb{R}^3 \) with

\begin{equation}
A = \begin{pmatrix}
-2a & -2 & 0 \\
1 & -a & -1 \\
0 & 2 & 0
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & -2\beta & 0 \\
0 & 0 & -\beta \\
0 & 0 & 0
\end{pmatrix}.
\end{equation}
The eigenvalues $\lambda_j$, $j = 1, \ldots, 3n$, of matrix $P_n + Q_n$ with maximum negative real parts were computed numerically for various $n$ and $\alpha = 0, 1; 1; 4$, $a = 0 \div 3$, $\beta = 0 \div 3$ (Fig. 1); $a = 0, 01$, $\alpha = 0 \div 12$, $\beta = 0 \div 3$ (Fig. 2) with spacing 0, 01. The results for $n = 40$ and $n = 80$ up to third decimal place are the same. The stability diagrams are presented in Figs. 1-2. The regions of mean square stability are situated below the curves. These regions correspond the eigenvalues of matrix $P_n + Q_n$ for $n = 80$ such that

$$\max_j \Re \lambda_j < -0.0001.$$ 

![Graph](image1.png)

FIG. 1. Stability diagrams for the equation (16) and for the values $\alpha = 0, 1; 1; 4$.

It follows from the Fig. 1 that the regions of mean square stability are increasing with $\alpha$. The influence of $a$ on growth the region of mean square stability is small when $\alpha$ is small.

![Graph](image2.png)

FIG. 2. Stability diagrams for the equation (16) and for the value $a = 0, 01$.

It follows from Fig. 2 that we have the parametric resonance at $\alpha = 2$ if damping $a$ is small. It is well known that this resonance exists also for nonrandom parametric excitation $\xi(t) = \beta \sin(\alpha t)$. 

8
References.