Moment stability for linear systems with a random parametric excitation

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Abstract

Moment stability for linear systems with a non-white parametric noise is considered. A method of reduction of the study of this stability to the study of stability for large-scale matrices is proposed. Mean square stability diagrams for random harmonic oscillator are presented.

Keywords: Stochastic linear systems; Moment stability; Stability diagrams.

1 Introduction.

Stability and stabilization is a fundamental objective in the design of controllers. This work deals with a stability for the following linear ordinary differential equation in R^N

$$\frac{dx}{dt} = Ax + \xi(t)Cx, \qquad 0 < t < \infty, \tag{1}$$

where A, C are $N \times N$ deterministic matrices and $\xi(t)$ is a stochastic process. We investigate p-stability, that is, asymptotic stability of the pth moments of the solution of equation (1) for p to be a positive integer.

There are necessary and sufficient conditions of such stability, if $\xi(t)$ is the white noise [5-7]. If the process $\xi(t)$ is non-white and has continuous trajectories, the study of stability turn out to be complicated. In this case there are necessary and sufficient conditions of moment stability only if $\xi(t)$ is a Gaussian process and the Lie algebra generated by the matrices A, C is solvable [1,10-12].

In this paper we consider the process $\xi(t)$ of the form

$$\xi(t) = sin(\alpha w(t)), \tag{2}$$

where α is a nonrandom parameter and $w(t)$ is Wiener process. We propose a method that reduces the study of moment stability for the equation (1) with the noise of the form (2) to the stability of nonrandom large-scale matrices. Using this method and numeric computation we get diagrams of 2-stability (mean square stability) for the random harmonic oscillator.

2 Equations for moments.

Let us start first from the mean value $Ex(t)$ of the solution of equation (1) with excitation (2) and nonrandom initial value $x(0)$. The solution of (1) is a functional with respect to the process $w(s)$, $0 \le s \le t$, and therefore we shall write $x(t) = x(t; w(s))$.
Let

$$v_0(t) := Ex(t; w(s)), \quad v_k(t) := \frac{1}{(2i)^k} \exp\{\frac{-k^2\alpha^2 t}{2}\}[Ex(t; w(s) + ik\alpha s)$$

$$+(-1)^k Ex(t; w(s) - ik\alpha s)], \quad k = 1, 2, 3, ...$$

To evaluate the mean value $v_0(t) = Ex(t)$ of the solution of equation (3) with nonrandom initial value $x(0)$ the following infinite chain of linear differential equations was obtained in the paper [2]:

$$\frac{dv_0}{dt} = Av_0 + Cv_1,$$

$$\frac{dv_1}{dt} = -\frac{\alpha^2}{2}v_1 + Av_1 + Cv_2 + \frac{1}{2}Cv_0, \tag{3}$$

$$\frac{dv_k}{dt} = -\frac{k^2\alpha^2}{2}v_k + Av_k + Cv_{k+1} + \frac{1}{4}Cv_{k-1}, \quad k = 2, 3, \dots,$$

$$v_0(0) = x(0), \quad v_k(0) = \frac{1}{(2i)^k}[1 + (-1)^k]x(0), \quad k = 1, 2, 3, \dots .$$

Consequently, the 1-stability of equation (1), (2) is equivalent to the asymptotic stability of the chain (4). Note that (see [3]) one can also obtain a similar chain when $\xi(t)$ is an Ornstein-Uhlenbeck process.
One can easily obtain a similar chain for higher order moments in the following way. Let us introduce under [4,5] the binominal coefficient $b(N, p) = \binom{N+p-1}{p}$ and the vector $x^{[p]}(t) \in R^{b(N,p)}$ of all p-th forms of the components of the vector $x(t)$:

$$x^{[p]}(t) = col(x_1(t)^p, \mu_1^p x_1(t)^{p-1} x_2(t), \dots, x_N(t)^p),$$

where parameters μ_i^j are chosen so as to obtain $\|x^{[p]}(t)\| = \|x(t)\|^p$ for the Euclidean norm of vectors.
For vector $x^{[p]}(t)$ we have under (1) the equation

$$\frac{dx^{[p]}}{dt} = A_{[p]}x^{[p]} + \xi(t)C_{[p]}x^{[p]},$$

where the matrices $A_{[p]}$, $C_{[p]}$ are determined from the matrices A, C [4,5].
Using the previous approach one can obtain for $Ex^{[p]}(t)$ the chain (3) with $A_{[p]}$, $C_{[p]}$ instead of A, C. So without loss of generality one can investigate only the 1-stability.

3 Stability for the chain.

Let Q_n be the following $nN \times nN$ block matrix

$$Q_n := \begin{pmatrix} A & C & 0 & 0 & 0 & \dots & \dots & \dots & 0 \\ \frac{1}{2}C & A & C & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \frac{1}{4}C & A & C & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \frac{1}{4}C & A \end{pmatrix}$$

and

$$P_n := diag(0, -\frac{\alpha^2}{2}E, \dots, -\frac{-(n-1)^2\alpha^2}{2}E), \quad S_n := diag(0, \dots, 0, C),$$

$$u_n(t) := col(v_0(t), \dots, v_{n-1}(t)),$$

with E - $N \times N$ identity matrix. We can rewrite (3) in the following form:

$$\frac{du_n}{dt} = P_n u_n + Q_n u_n + S_n v_n, \tag{4}$$

$$\frac{dv_k}{dt} = -\frac{k^2\alpha^2}{2}v_k + Av_k + Cv_{k+1} + \frac{1}{4}Cv_{k-1}, \quad k = n, n+1, \dots . \tag{5}$$

Denote by λ_j, $j = 1, \dots, nN$, eigenvalues of matrix $P_n + Q_n$. We show that the stability of this matrix, namely the signs of real parts of its eigenvalues, for sufficiently large n is crucial in the study of 1-stability for equation (1), (2).
Assume that there exists a $\gamma > 0$ such that for sufficiently large n

$$\max_j Re\ \lambda_j < -\gamma. \tag{6}$$

Denote by $||\cdot||$ the Euclidean norm for vectors and induced norm for matrices. It is known [8] that

$$||Q_n|| \le ||Q_n||_1 ||Q_n||_\infty,$$

where $|| \ \ ||_1$ and $|| \ \ ||_\infty$ is the maximum column sum norm and maximum row sum norm respectively. It follows from this and the form of matrix Q_n that there exists a $c_1 > 0$ such that for all n

$$||Q_n|| < c_1. \tag{7}$$

Using the triangular factorization [8], there exists a unitary matrix U_n such that

$$U_n(P_n+Q_n)U_n' = D_n + B_n,$$

where

$$D_n = diag(\lambda_1,\dots,\lambda_{nN}), \quad B_n = (a_{ij}),\ a_{ij}=0,\ \ i\geq j,\ \ i,j=1,\dots,nN.$$

Since (7) there exists a $c_2>0$ such that for all n

$$\|B_n\| < c_2. \tag{8}$$

Let V_n be a diagonal matrix:

$$V_n = diag(1,\delta^{-1},\delta^{-2+1/2},\dots,\delta^{-2+2^{2-nN}}), \quad \delta\in(0,1).$$

It follows from (4) that the vector $z_n = V_nU_nu_n$ satisfies the equation

$$\frac{dz_n}{dt} = D_nz_n + V_nB_nV_n^{-1}z_n + V_nU_nS_nv_n, \tag{9}$$

where

$$V_nB_nV_n^{-1} = (a_{ij}\delta^{2^{2-i}-2^{2-j}}).$$

Using (8), we have that for each $\varepsilon>0$ there exist $\delta\in(0,1)$, $n_0>0$ such that for $n>n_0$

$$\|V_nB_nV_n^{-1}\| < \varepsilon. \tag{10}$$

From (9) it follows that

$$\frac{d\|z_n\|^2}{dt} = (D_nz_n,\bar z_n) + (\bar D_n\bar z_n, z_n) + (V_n\bar B_nV_n^{-1}\bar z_n, z_n)$$

$$+(V_nB_nV_n^{-1}z_n,\bar z_n) + (V_n\bar U_n\bar S_n\bar v_n, z_n) + (V_nU_nS_nv_n,\bar z_n).$$

Taking into account that $\|V_n\|\leq\delta^{-2}$ and $\|S_n\| = \|C\|$, we obtain

$$|(V_n\bar U_nS_n\bar v_n, z_n) + (V_nU_nS_nv_n,\bar z_n)| \leq \varepsilon\|z_n\|^2 + \varepsilon^{-1}\delta^{-4}\|C\|^2\|v_n\|^2.$$

Hence,

$$\frac{d\|z_n\|^2}{dt} \leq -2\gamma\|z_n\|^2 + 3\varepsilon\|z_n\|^2 + \varepsilon^{-1}\delta^{-4}\|C\|^2\|v_n\|^2. \tag{11}$$

Using (5), we get

$$\frac{d\|v_k\|^2}{dt} = -k^2\alpha^2\|v_k\|^2 + (A\bar v_k, v_k) + (Av_k,\bar v_k) + (C\bar v_{k+1}, v_k)$$

$$+(Cv_{k+1},\bar v_k) + \frac{1}{4}(C\bar v_{k-1}, v_k) + \frac{1}{4}(Cv_{k-1},\bar v_k), \quad k=n,n+1,\dots.$$

$$\frac{d\|v_k\|^2}{dt} \le -k^2\alpha^2\|v_k\|^2 + 2\|A\|\|v_k\|^2 + \|C\|\|v_k\|^2 +$$
$$+\|C\|\|v_{k+1}\|^2 + \frac{1}{2}\|C\|\|v_k\|^2 + \frac{1}{2}\|C\|\|v_{k-1}\|^2.$$

From the definition of vectors $v_k(t)$ it follows that the series

$$\sum_{k=n}^{\infty} k^2\|v_k\|^2, \quad \sum_{k=n}^{\infty} \frac{d\|v_k\|^2}{dt}$$

are converged.
Therefore,

$$\sum_{k=n}^{\infty} \frac{d\|v_k\|^2}{dt} \le \frac{1}{2}\|C\|\|v_{n-1}\|^2 + \sum_{k=n}^{\infty}(-k^2\alpha^2 + 2\|A\| + 2,5\|C\|)\|v_k\|^2$$
$$\le \frac{1}{2}\|C\|\|v_{n-1}\|^2 + (-n^2\alpha^2 + 2\|A\| + 2,5\|C\|)\sum_{k=n}^{\infty}\|v_k\|^2.$$

If $q_n := n^2\alpha^2 - 2\|A\| - 2,5\|C\|$, then from the previous inequality we have

$$\frac{d}{dt}(e^{q_n t}\sum_{k=n}^{\infty}\|v_k(t)\|^2) \le \frac{1}{2}\|C\|\ e^{q_n t}\|v_{n-1}(t)\|^2,$$

and so

$$\sum_{k=n}^{\infty}\|v_k(t)\|^2 \le \frac{1}{2}\|C\|\int_0^t e^{-q_n(t-s)}\|v_{n-1}(s)\|^2 ds + e^{-q_n t}\sum_{k=n}^{\infty}\|v_k(0)\|^2.$$

Since $\|v_{n-1}\| \le \|u_n\| \le \|V_n^{-1}\|\|z_n\| \le \|z_n\|$ and

$$\sum_{k=n}^{\infty}\|v_k\|^2 \le \sum_{k=n}^{\infty}\frac{\|x(0)\|^2}{4^{k-1}} \le \frac{\|x(0)\|^2}{3\cdot 4^{n-2}},$$

we obtain

$$\|v_n(t)\|^2 \le \frac{1}{2}\|C\|\int_0^t e^{-q_n(t-s)}\|z_n(s)\|^2 ds + \frac{\|x(0\|^2}{3\cdot 4^{n-2}}e^{-q_n t}. \qquad (12)$$

Let $\eta := 2\gamma - 3\varepsilon$. Using Gronwall inequality to (11), we have

$$\|z_n(t)\|^2 \le \varepsilon^{-1}\delta^{-4}\|C\|^2\int_0^t \exp\{-\eta(t-s)\}\|v_n(s)\|^2 ds + e^{-\eta t}\|z_n(0)\|^2.$$

Substituting (12) for $\|v_n\|$, we obtain

$$\|z_n(t)\|^2 < \frac{1}{2}\varepsilon^{-1}\delta^{-4}\|C\|^3\int_0^t\int_0^s \exp\{-\eta(t-s) - q_n(s-s_1)\}\|z_n(s_1)\|^2 ds_1 ds$$

$$+\varepsilon^{-1}\delta^{-4}\|C\|^2\frac{\|x(0)\|^2}{3\cdot 4^{n-2}}\int_0^t \exp\{-\eta(t-s)-q_ns\}ds+e^{-\eta t}\|z_n(0)\|^2.$$

Let ε, $n_0>0$ be such that $\eta>0$ and $q_n>\eta$ for $n>n_0$. Therefore for $n>n_0$ we have

$$\|z_n(t)\|^2<\frac{\|C\|^3}{2(q_n-\eta)\varepsilon\delta^4}\int_0^t e^{-\eta(t-s)}\|z_n(s)\|^2ds$$

$$+[\|z_n(0)\|^2+\frac{\varepsilon^{-1}\delta^{-4}\|C\|^2\|x(0)\|^2}{3\cdot 4^{n-2}(q_n-\eta)}]e^{-\eta t}. \tag{13}$$

Let now $n_1>n_0$ such that for $n>n_1$

$$p_n:=\eta-\frac{\|C\|^3}{2(q_n-\eta)\varepsilon\delta^4}>0.$$

Taking into account that

$$\|z_n(0)\|^2=\|V_nU_nu_n(0)\|^2\le\|V_n\|^2\|u_n(0)\|^2$$

$$\le\delta^{-4}\sum_{k=0}^{n-2}4^{-k}\|x(0)\|^2<\frac{4}{3}\delta^{-4}\|x(0)\|^2$$

and applying Gronwall inequality to (13), we obtain

$$\|z_n(t)\|^2\le\frac{4}{3}\delta^{-4}[1+\frac{\varepsilon^{-1}\|C\|}{4^{n-1}(q_n-\eta)}]e^{-p_nt}\|x(0\|^2. \tag{14}$$

Thus, we have the mean square stability of equation (1), (2) whenever the condition (6) holds.
Now, we suppose that the matrix Q_n has eigenvalues with positive real parts for sufficiently large n. Without loss of generality let $Re\lambda_j>0,\ j=1,...,k,\ \ Re\lambda_j\le 0,\ j=k+1,...,nN$. Moreover, we suppose that there exists a $\sigma>0$ such that for sufficiently large n

$$Re\lambda_j>\sigma,\ j=1,...,k,\quad Re\lambda_j\le 0,\ j=k+1,...,nN. \tag{15}$$

Let $z_n^{(j)}$, $j=1,...,nN$, be components of vector z_n and

$$F(z_n,v_n,v_{n+1},...):=\frac{1}{2}[\sum_{j=1}^{k}\|z_n^{(j)}\|^2-\sum_{j=k+1}^{nN}\|z_n^{(j)}\|^2-\sum_{k=n}^{\infty}\|v_k\|^2].$$

By (5), we have

$$\frac{d\|v_n\|^2}{dt}=(\frac{dv_n}{dt},\bar v_n)+(v_n,\frac{d\bar v_n}{dt})=(-\frac{n^2\alpha^2}{2}v_n+Av_n+Cv_{n+1}+\frac{1}{4}Cv_{n-1},\bar v_n)$$

$$+(v_n,-\frac{n^2\alpha^2}{2}\bar v_n+A\bar v_n+C\bar v_{n+1}+\frac{1}{4}C\bar v_{n-1})=-n^2\alpha^2\|v_n\|^2+(Av_n,\bar v_n)$$

$$+(A\bar{v}_n, v_n) + (Cv_{n+1}, \bar{v}_n) + (C\bar{v}_{n+1}, v_n) + \frac{1}{4}[(Cv_{n-1}, \bar{v}_n) + (C\bar{v}_{n-1}, v_n)]$$

$$\leq (-n^2\alpha^2 + 2\|A\| + \frac{n}{4}\|C\|)\|v_n\|^2 + \|C\|\|v_{n+1}\|^2 + \frac{1}{4n}\|C\|\|z_{n-1}\|^2$$

$$\frac{d\|v_k\|^2}{dt} \leq (-k^2\alpha^2 + 2\|A\| + \frac{3}{2}\|C\|)\|v_k\|^2 + \|C\|\|v_{k+1}\|^2 + \frac{1}{2}\|C\|\|v_{k-1}\|^2,$$

$k = n+1, n+2, \ldots$.
Therefore by (9) we obtain

$$\frac{dF}{dt} \geq \sum_{j=1}^{k} Re\lambda_j \|z_n^{(j)}\|^2 - \sum_{j=k+1}^{nd} Re\lambda_j \|z_n^{(j)}\|^2 + \sum_{j=n}^{\infty} q_j \|v_j\|^2 + \rho_n \|z_n\|^2,$$

where $q_j = j^2\alpha^2 - 2\|A\| - 3\|C\|$ and by (10) for any $\varepsilon_1 > 0$ there exist $\delta \in (0,1)$ and n_2 such that for $n > n_2$ we have $|\rho_n| < \varepsilon_1$.

Consequently for sufficiently large n, we have

$$\frac{dF}{dt} \geq \frac{\sigma}{2}(\sum_{j=1}^{nN} \|z_n^{(j)}\|^2) \geq \sigma F$$

from which

$$F(z_n(t), v_n(t), v_{n+1}(t), \ldots) \geq F(z_n(0), v_n(0), v_{n+1}(0), \ldots)e^{\sigma t}.$$

Thus equation (1), (2) is not 1-stable when the condition (15) holds.

4 Random harmonic oscillator.

Consider the equation of harmonic oscillator

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + [1 + \beta\xi(t)]y = 0, \tag{16}$$

where $a > 0$, β are nonrandom parameters and $\xi(t)$ is a random process.
There are many works dealing with stability for this equation (see [9] for a rewiew).
Necessary and sufficient conditions of mean square stability of (15) are obtained in the case of white noise process $\xi(t)$ in [7].
We use the results of previous section to obtain exact regions of mean square stability for equation (16) with noise (2).
One can easily obtain that the vector

$$x := col\left(\left(\frac{dy}{dt}\right)^2, y\frac{dy}{dt}, y^2\right)$$

satisfies the equation (3) in R^3 with

$$A = \begin{pmatrix} -2a & -2 & 0 \\ 1 & -a & -1 \\ 0 & 2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -2\beta & 0 \\ 0 & 0 & -\beta \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues λ_j, $j = 1, ..., 3n$, of matrix $P_n + Q_n$ with maximum negative real parts were computed numerically for various n and $\alpha = 0,1;\ 1;\ 4,\ \ a = 0 \div 3$, $\beta = 0 \div 3$ (Fig. 1); $a = 0,01$, $\alpha = 0 \div 12$, $\beta = 0 \div 3$ (Fig. 2) with spacing 0, 01. The results for $n = 40$ and $n = 80$ up to third decimal place are the same. The stability diagrams are presented in Figs. 1-2. The regions of mean square stability are situated below the curves. These regions correspond the eigenvalues of matrix $P_n + Q_n$ for $n = 80$ such that

$$\max_j Re\ \lambda_j < -0.0001.$$

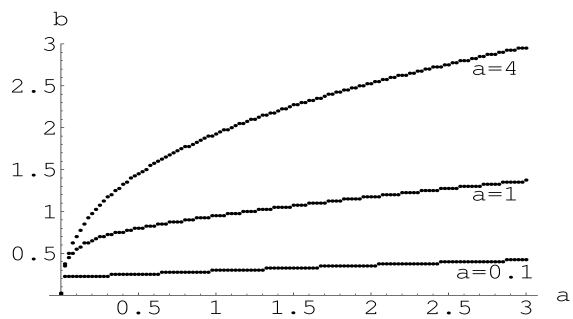

FIG. 1. Stability diagrams for the equation (16) and for the values $\alpha = 0,1;\ 1;\ 4$.

It follows from the Fig. 1 that the regions of mean square stability are increasing with α. The influence of a on growth the region of mean square stability is small when α is small.

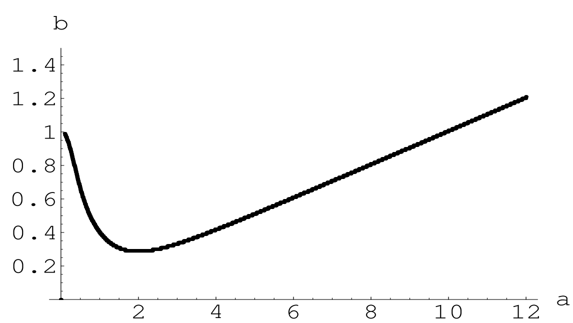

FIG. 2. Stability diagrams for the equation (16) and for the value $a = 0,01$.

It follows from Fig. 2 that we have the parametric resonance at $\alpha = 2$ if damping a is small. It is well known that this resonance exists also for nonrandom parametric excitation $\xi(t) = \beta sin(\alpha t)$.

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