Risk Sensitive Portfolio Optimization with Completely and Partially Observed Factors

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Abstract— In the paper optimal portfolio maximizing the long run risk sensitized growth rate of the capital process in the case when the dynamics of the asset prices depend on some economical factors, which are completely or partially observed, using a discounted cost approach is shown.

Index Terms— optimal portfolio, risk sensitive cost, factors, partial observation.

I. Introduction

Assume we are given a market consisting of m securities and k_1+k_2 factors. The price of the i-th security $S_i(t)$ at time t depends on the values of economic factors $x(t)=(z_1(t),z_2(t))$ as e.g. dividend yields, price - earning ratios, rate of inflation (see e.g. [1] and references therein for examples of such factors), in discrete time moments for simplicity denoted by $n=0,1,\ldots$. Given a probability space (Ω,F,P) assume that

$$S_i(n+1)=S_i(n)w_i(n), \tag{1}$$

where $w(n)=(w_1(n),\ldots,w_m(n))$ is a random vector with positive coordinates.
The first k_1 factors $(z_1(n))$ are completely observed and belong to a metric space E_1, while the remaining k_2 factors $(z_2(n))$ are observed only partially and are elements of a metric space E_2. The observation process $y(n)$ of $z_2(n)$ takes values in R^k and has conditional law dependent on the value of $z_2(n)$, i.e. for $n=1,2\ldots$ and $A\in\mathcal{B}(R^k)$

$$P\left\{y(n)\in A|Y^{n-1},W^{n-1},X^n\right\}:=R(z_2(n),A) \tag{2}$$

with $Y^{n-1}=\sigma\left\{y(t),t=1,\ \ldots,n-1\right\}$, $Y^0=\{\emptyset,\Omega\}$, $W^{n-1}=\sigma\left\{w(t),t=0,\ \ldots,n-1\right\}$, and $X^n=\sigma\left\{x(t),t=0,\ \ldots,n\right\}$. Assume that transition kernels $R(z,\cdot)$ are equivalent for $z\in E_2$ and for a given equivalent probability measure η there is a suitably measurable function r such that for $A\in\mathcal{B}(R^k)$ we have

$$R(z,A)=\int_A r(z,y)\eta(dy). \tag{3}$$

Notice that the above observation structure is quite general and includes typical observation models: function of $z_2(n)$ plus an additive noise (see for more details in [12]). Another model with an observation of $z_2(n)$ coming only through the asset prices $S_i(n)$ will be discussed briefly in section 6.
Assume furthermore that for $n=1,\ldots,$ and $A\in\mathcal{B}(R^m)$ conditional law

$$P\left\{w(n)\in A|W^{n-1},X^n,Y^n\right\}:=Q(x(n),A) \tag{4}$$

depends on $x(n)$ only and transition kernels $Q(x,\cdot)$, i.e. the measures $Q(x,\cdot)$ for $x\in E:=E_1\times E_2$ are equivalent. Denote by ν a probability measure equivalent to $Q(x,\cdot)$ and by $q(x,r)$ a density of $Q(x,\cdot)$ with respect to ν, so that for $A\in\mathcal{B}(R^m)$ we have

$$Q(x,A)=\int_A q(x,r)\nu(dr). \tag{5}$$

Notice that formula (4) describes a dependence of the asset prices on the factors x. In a number of papers concerning to the continuous time model (see e.g. [1], [5], [9]) linear dependence of the dynamics of factors on the dynamics of asset prices was considered. The first k_1 factors $(z_1(n))$ form a sequence of random variables satisfying

$$P\left\{z_1(n+1)\in A_1|X^n,W^n,Y^n\right\}:=P_1(x(n),w(n),A_1), \tag{6}$$

for $A_1\in\mathcal{B}(E_1)$ and a transition kernel P_1. Assume that measures $P_1(x,w,\cdot)$ for $x\in E$, $w\in R^d$ are equivalent. Fix $\bar{x}=(\bar{z}_1,\bar{z}_2)\in E$, $\bar{w}\in R^d$ and let $p_1(\cdot)=P_1(\bar{x},\bar{w},\cdot)$. Then there are positive densities $\bar{p}_1$ and p_1 such that for $A_1\in\mathcal{B}(E_1)$

$$P_1(x,w,A_1)=\int_{A_1}\bar{p}_1(x,w,z)p_1(dz), \tag{7}$$

and

$$P_1(z_1,\bar{z}_2,\bar{w},A_1)=\int_{A_1}p_1(z_1,z)p_1(dz), \tag{8}$$

The partially observed factors $(z_2(n))$ form a sequence of random variables satisfying

$$\begin{aligned}&P\left\{z_2(n+1)\in A_2|X^n,W^n,Y^n,z_1(n+1)\right\}\\&:=P_2(x(n),w(n),z_1(n+1),A_2),\end{aligned} \tag{9}$$

for $A_2\in\mathcal{B}(E_2)$ and a transition kernel P_2. Similarly as above for P_1 assume that measures $P_2(x,w,z,\cdot)$ for $x\in E$, $w\in R^d$, $z\in E_1$ are equivalent and fix $\bar{x}\in E$, $\bar{w}\in R^d$ and $\bar{z}\in E_1$. Letting $p_2(\cdot)=P_2(\bar{x},\bar{w},\bar{z},\cdot)$ there are positive densities $\bar{p}_2$ and p_2 such that for $A_2\in\mathcal{B}(E_2)$ we have

$$P_2(x,w,z,A_2)=\int_{A_2}\bar{p}_2(x,w,z,z')p_2(dz'). \tag{10}$$

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and

$$P_2(\bar{z}_1, z_2, \bar{w}, \bar{z}, A_2) = \int_{A_2} p_2(z_2, z')p_2(dz'). \qquad (11)$$

Notice that at generic time n the dynamics of $z_1(n)$ depends on $x(n-1)$, $w(n-1)$, the dynamics of $z_2(n)$ on $z_1(n)$, $x(n-1)$, $w(n-1)$, the dynamics of $w(n)$ on $x(n)$ and $w(n-1)$, while the dynamics of $y(n)$ depends on $z_2(n)$.

Assume that starting with an initial capital v we invest in the securities. Denote by $V(t)$ our wealth process at time t. Let $h_i(t)$ be the portion of our capital (wealth) invested in the i-th security at time t. At each time $n = 0, 1, \ldots,$ we can change our portfolio choosing vector $h(n) = (h_1(n), \ldots, h_m(n))^T$, which as chosen basing on the all available observations up to time n, is a $\sigma\left\{z_1(l), l \le n, Y^n, W^{n-1}\right\}$ measurable element of a fixed compact subset U of R^m. The portion of the capital which is not invested i.e. $1 - \sum_{i=1}^m h_i(n)$ remains on the banking account with rate of interest for simplicity equal to 0. We restrict the class of admissible controls to such controls $h(n)$ for which P a.e. $\frac{V(n+1)}{V(n)} > 0$ (which in particular is satisfied when we do not allow shortselling and shortborrowing, i.e. when $h_i(n) \ge 0$ and $1 - \sum_{i=1}^m h_i(n) \ge 0$). We easily see that $V(0) = v$ and

$$V(n+1) = \sum_{i=1}^m \frac{h_i(n)V(n)}{S_i(n)} S_i(n+1) + (1 - \sum_{i=1}^m h_i(n))V(n).$$

Consequently

$$\begin{aligned} \frac{V(n+1)}{V(n)} &= \sum_{i=1}^m h_i(n)w_i(n) + (1 - \sum_{i=1}^m h_i(n)) \\ &:= e^{G(w(n),h(n))}, \end{aligned} \qquad (12)$$

defining implicitly function G. Our purpose is to maximize for fixed $\Theta \in (0,1)$ the long run risk sensitized growth rate of our capital process given by the following cost functional

$$\begin{aligned} &\liminf_{t\to\infty} \frac{-2}{\Theta} t^{-1} \ln E\left\{V(t)^{\frac{-\Theta}{2}}\right\} \\ &= \liminf_{t\to\infty} \frac{-2}{\Theta} t^{-1} \ln E\left\{\exp\left\{\frac{-\Theta}{2} \ln V(t)\right\}\right\}, \end{aligned} \qquad (13)$$

over all admissible processes $(h(n))$ taking values in U, which at time n are adapted to the available information consisting of the observations of the prices $(S_i(t))$, factors $z_1(t)$ and $y(t)$ with $t \le n$, $i = 1, \ldots, m$. Such cost functional measures (see e.g. [1] and [13] for more details) $liminf_{t\to\infty} t^{-1}\left(E\left\{\ln V(t)\right\} - \frac{\Theta}{4} Var(\ln V(t)) + O(\Theta^2)\right)$. Maximizing (13) we therefore maximize the expected value of the portfolio growth $(\ln V(t))$ minimizing its variance (with weight $\frac{\Theta}{4}$) over long time horizon. The problem is equivalent to minimization of

$$\limsup_{n\to\infty} n^{-1} \ln E\left\{\exp\left\{\sum_{j=0}^{n-1} \frac{-\Theta}{2} G(w(j), h(j))\right\}\right\}. \qquad (14)$$

To guarantee integrability as well as to have nice properties of the term under expected value in (14) assume that there exist deterministic continuous functions $\underline{\phi}$, $\overline{\phi} : [0,1] \mapsto (0, \infty)$, with $\underline{\phi}(0) = \overline{\phi}(0) = 1$, such that for $\beta \in (0,1)$, $\gamma \in [0,1]$, $0 < \prod_{i=1}^\infty \underline{\phi}(\gamma\beta^{i-1})$, $\prod_{i=1}^\infty \overline{\phi}(\gamma\beta^{i-1}) < \infty$ and we have

$$\begin{aligned} &\underline{\phi}(\gamma) \le \inf_{x\in E} \inf_h \int_{R^m} e^{-\frac{\gamma\Theta}{2} G(w,h)} Q(x, dw) \qquad (15) \\ &\sup_{x\in E} \sup_h \int_{R^m} e^{-\frac{\gamma\Theta}{2} G(w,h)} Q(x, dw) \le \overline{\phi}(\gamma) \end{aligned}$$

with $\inf_h$ or $\sup_h$ over all admissible portfolio strategies h.

In the paper we shall assume also that the densities $r(\cdot,\cdot)$, $q(\cdot,\cdot)$, $p_1(\cdot,\cdot)$, $\bar{p}_1(\cdot,\cdot,\cdot)$, $p_2(\cdot,\cdot)$, $\bar{p}_2(\cdot,\cdot,\cdot,\cdot)$ are continuous bounded and the mapping

$$E \times U \times (0,1] \ni (x, h, \gamma) \mapsto \int_{R^m} e^{-\frac{\gamma\Theta}{2} G(w,h)} Q(x, dw) \qquad (16)$$

is continuous. Comments on the assumptions (15) and (16) are given in Section 5.

The study of risk sensitive portfolio optimization with prices of securities depending on factors has been originated in the paper [1] and continued then in a sequence of papers (see e.g. [5] and references therein). Discrete time dynamics of factors was considered in [3] and [13]. The case of unobserved factors (i.e. observed only through the asset prices) was studied in the papers [7], [9] and [8] under linear dependence on factors and linear Gaussian dynamics of factors. Partially observed control problems with risk sensitive cost functional (without financial applications) were investigated in [4] and [6]. In this paper risk sensitive portfolio optimization with both completely and partially observed factors is considered. The unobserved factors can have noisy observation, or as is pointed out in section 6 using our techniques we can solve the problem with some factors observed only through the asset prices. The fact that simultaneously completely and partially observed factors are allowed, together with multiplicative cost functional create an additional difficulty. To describe the evolution of our model we change a probability measure and use stochastic processes with values in E_1 and in the space of finite measures on E_2. In this paper we prove the existence and the form of optimal investment strategies for the cost functional (14). The methods are based on the adaptations and generalizations of some techniques from [4]. Although we minimize the cost functional (14) with $\Theta \in (0,1)$, the problem is also interesting for Θ negative (see [10]) as a dual problem to maximization of the probability that average rate of growth of portfolio is greater than a given value. To study the cost function (14) consider first (similarly as in [4]) a discounted control problem with the cost functional

$$J^{\beta,\Theta}_{z_1,\mu,s}((h(n)) = E\left\{\exp\left\{\sum_{j=0}^\infty \beta^j \frac{-\Theta}{2} G(w(j), h(j))\right\}\right\}. \qquad (17)$$

with discount factor $\beta \in (0,1)$, $z_1(0) = z_1$, $S(0) = s$ and initial law μ of the partially observed factors $(z_2(n))$. To solve the control problem with cost functional (17) we have to introduce a certain recurrence procedure (see (20) and (21) below) for which we need independence, under new measure P^o, of the processes $(y(n))$, $(w(n))$, $(z_1(n))$ and $(z_2(n))$. The new probability measure P^o on Ω is constructed in such way that its restrictions P^o_n and restrictions P_n of P respectively to the σ-field $F_n = \sigma\left\{X^n, Y^n, W^{n-1}\right\}$ satisfy

$$P_n(d\omega) = \Lambda_n(\omega) P^o_n(d\omega) \tag{18}$$

where

$$\begin{aligned}\Lambda_n(\omega) &= \prod_{i=1}^{n} r(z_2(i), y(i)) q(x(i-1), w(i-1)) \\ &\quad \frac{\bar{p}_1(x(i-1), w(i-1), z_1(i))}{p_1(z_1(i-1), z_1(i))} \\ &\quad \frac{\bar{p}_2(x(i-1), w(i-1), z_1(i), z_2(i))}{p_2(z_2(i-1), z_2(i))}.\end{aligned} \tag{19}$$

Following the consideration of Lemma 1.8 of [12] one can show

Lemma 1: Under P^o: the sequence $(y(n))$ is i.i.d. with density 1 with respect to η, the sequence $(w(n))$ is i.i.d. with density 1 with respect to ν, the process $(z_1(n))$ is Markov with transition density $p_1(\cdot,\cdot)$ with respect to the measure $p_1(\cdot)$, and $(z_2(n))$ forms a Markov process with transition density $p_2(\cdot,\cdot)$ with respect to the measure $p_2(\cdot)$. Furthermore the processes $(y(n))$, $(w(n))$, $(z_1(n))$ and $(z_2(n))$ are independent under measure P^o.

II. Discounted risk sensitive Bellman equation

Let σ^γ_n with $\gamma \in [0,1]$ be a stochastic process taking values in $M^+(E_2)$, the space of nontrivial finite measures on $\mathcal{B}(E_2)$, defined as follows

$$\begin{aligned}\sigma^\gamma_0(\varphi) &= \mu(\varphi) \\ \sigma^\gamma_n(\varphi) &= E^o\left\{\exp\left\{\sum_{j=0}^{n-1} \beta^j \frac{-\gamma\Theta}{2} G(w(j), h(j))\right\}\right. \\ &\quad \varphi(z_2(n))\Lambda_n \mid \mathcal{Y}^n\}\end{aligned} \tag{20}$$

for $\varphi \in b\mathcal{B}(E_2)$ the space of bounded Borel functions on E_2, where μ is the initial law of $(z_2(n))$ and $\mathcal{Y}^n = \sigma\left\{z_1(l), l \le n, Y^n, W^{n-1}\right\}$. Notice that the σ - field $\mathcal{Y}^n$ corresponds to the information available up to time n and

$$\begin{aligned}&E\left\{\sigma^\gamma_n(\phi)\right\} = \\ &E\left\{\exp\left\{\sum_{j=0}^{n-1} \beta^j \frac{-\gamma\Theta}{2} G(w(j), h(j))\right\} \varphi(z_2(n))\right\},\end{aligned}$$

so that the measure valued process σ^γ_n will be useful to evaluate the discounted cost functional (17).

By direct calculation (compare to section 2 of [4]), taking into account Lemma 1 in particular the independence of the process $(z_2(n))$ of $(y(n))$, $(w(n))$ and $(z_1(n))$ we obtain that for $n \ge 1$, $\varphi \in b\mathcal{B}(E_2)$

$$\begin{aligned}&\sigma^\gamma_n(\varphi) = \sigma^\gamma_{n-1}(\Phi^{\beta^{n-1}\gamma}(y(n), z_1(n), z_1(n-1), \\ &w(n), h(n-1))\varphi) = (\Phi^{\beta^{n-1}\gamma})^*(y(n), z_1(n), \\ &z_1(n-1), w(n), h(n-1))\sigma^\gamma_{n-1}(\varphi)\end{aligned} \tag{21}$$

P^o a.e. with

$$\begin{aligned}&\Phi^\delta(y, z_1(1), z_1(0), w, h)\varphi(z_2) = \exp\left\{\frac{-\delta\Theta}{2} G(w,h)\right\} \\ &q(z_1(0), z_2, w)\frac{\bar{p}_1(z_1(0), z_2, w, z_1(1))}{p_1(z_1(0), z_1(1))} \\ &\int_{E_2} \varphi(z) r(z,y) \bar{p}_2(z_1(0), z_2, w, z_1(1), z) p_2(dz)\end{aligned} \tag{22}$$

for $\delta \ge 0$, where $(\Phi^\delta)^*$ is an adjoint operator defined on $M^+(E_2)$.

The recursive formula (21) allows us to Markovianize (under the measure P^o) the discounted control problem. Namely, let $\mathcal{P}(E_2)$ be the space of probability measures on E_2 endowed with the weak convergence topology. Given a bounded Borel measurable function $f: \mathcal{P}(E_2) \to R^+$ we can extend it to a bounded Borel measurable function on $M^+(E_2)$ using an operator $S: b\mathcal{B}(\mathcal{P}(E_2)) \to b\mathcal{B}(M^+(E_2))$ defined as follows

$$Sf(\nu) = \nu(E_2) f(\frac{\nu}{\nu(E_2)}) \tag{23}$$

for $\nu \in M^+(E_2)$. The operator S introduced first in [13] allows us to consider value functions depending on probability measures on E_2 instead of larger space of finite measures on E_2.

Notice that by continuity assumptions imposed in Introduction for continuous bounded function $\phi: E_2 \mapsto R$ the mapping

$$(\delta, y, z_1(1), z_1(0), w, h, z_2) \mapsto \Phi^\delta(y, z_1(1), z_1(0), w, h)\phi(z_2)$$

is continuous and consequently for a continuous bounded function $E_1 \times \mathcal{P}(E_2) \times [0,1] \ni (z_1, \mu, \gamma) \mapsto f(z_1, \mu, \gamma) \in R$ the mapping

$$\begin{aligned}&(z_1, \mu, \gamma, h) \mapsto \int_{R^m}\int_{R^k}\int_{E_1} Sf(z', (\Phi^\gamma)^*(y, z', \\ &z_1, w, h)\mu, \gamma\beta) p_1(z_1, z') p_1(dz')\eta(dy)\nu(dw)\end{aligned} \tag{24}$$

is also continuous. Moreover let

$$\begin{aligned}&T_\beta f(z_1, \mu, \gamma) = \inf_{h \in U}\int_{R^m}\int_{R^k}\int_{E_1} Sf(z', (\Phi^\gamma)^*(y, z', \\ &z_1, w, h)\mu, \gamma\beta) p_1(z_1, z') p_1(dz')\eta(dy)\nu(dw)\end{aligned} \tag{25}$$

We have the following theorem

Theorem 1: There is a unique solution $w_\beta(z_1, \mu, \gamma)$ to the following equation

$$\begin{aligned}&w_\beta(z_1, \mu, \gamma) = \inf_{h \in U}\int_{R^m}\int_{R^k}\int_{E_1} Sw_\beta(z', (\Phi^\gamma)^*(y, z', \\ &z_1, w, h)\mu, \gamma\beta) p_1(z_1, z') p_1(dz')\eta(dy)\nu(dw)\end{aligned} \tag{26}$$

such that $\lim_{n\to\infty} T^n_\beta \underline{\phi}(z_1, \mu, \gamma) = w_\beta(z_1, \mu, \gamma) = \lim_{n\to\infty} T^n_\beta \overline{\phi}(z_1, \mu, \gamma)$ for $z_1 \in E_1$, $\mu \in \mathcal{P}(E_2)$ and $\gamma \in$

$[0,1]$. For fixed $z_1 \in E_1$ and $\gamma \in [0,1]$ the function $\mathcal{P}(E_2) \ni \mu \mapsto w_\beta(z_1,\mu,\gamma)$ is concave and $w_\beta(z_1,\mu,\gamma)$ is a continuous function of its coordinates.
Furthermore

$$w_\beta(z_1,\mu,\gamma) = \inf_{(h(n))} J^{\beta,\gamma\Theta}_{z_1,\mu,s}((h(n)) \tag{27}$$

and optimal strategies are time dependent functions of the current values of $z_1(n)$ and $\frac{\sigma_n^\gamma}{\sigma_n^\gamma(E_2)}$ only.

Proof: Notice first that if $\psi(\gamma)$, $\gamma \in [0,1]$ is a function of γ only then the iterations of the operator T_β are of the form

$$T^n_\beta \psi(z_1,\mu,\gamma) = \psi(\gamma\beta^n) \inf_{(h(i))} E\left\{\exp\left\{\sum_{j=0}^{n-1} \beta^j \frac{-\gamma\Theta}{2} G(w(j),h(j))\right\}\right\} \tag{28}$$

Consequently when ψ is right continuous at 0 with $\psi(0) = 1$ by (15) there exists a uniform limit when $n \to \infty$. Furthermore by continuity of the mapping (24) if ψ is continuous, $T^n_\beta\psi(z_1,\mu,\gamma)$ is also continuous function of its coordinates and the same property inherits the limit function w_β. One can show inductively that the mapping $\mu \mapsto T^n_\beta\psi(z_1,\mu,\gamma)$ for fixed $z_1 \in E_1$ and $\gamma \in [0,1]$ is concave. Therefore $w_\beta(z_1,\cdot,\gamma)$ is also concave. The limit if $\psi(0) = 1$ and ψ is continuous does not depend on the form of function ψ. In particular, $w_\beta(z_1,\mu,\gamma) = \lim_{n\to\infty} T^n_\beta \overline{\phi}(z_1,\mu,\gamma) = \lim_{n\to\infty} T^n_\beta \underline{\phi}(z_1,\mu,\gamma)$. The formula (27) follows from the inequality

$$w_\beta(z_1(n), \frac{\sigma_n^\gamma}{\sigma_n^\gamma(E_2)}, \gamma\beta^n) \leq E^o\left\{Sw_\beta(z_1(n+1)(\Phi^\gamma)^* (y(n+1), z_1(n+1), z_1(n), w(n), h(n)), \gamma\beta)|\mathcal{Y}^n\right\} \tag{29}$$

with equality for $h(n) = \hat{u}(z_1(n), \frac{\sigma_n^\gamma}{\sigma_n^\gamma(E_2)}, \gamma\beta^n)$, where $\hat{u}$ is a measurable selector corresponding to the infimum on right-hand side of (26).

■

III. Properties of the discounted value function

To study the cost functional of the form (14) we let discount factor β go to 1 in the discounted Bellman equation (26). For this purpose we shall need certain nice properties of the discounted value functions w_β. Starting from this section we assume that (we use notation below that $\frac{0}{0} = 0$)

$$\sup_{x,x'\in E} \sup_{w\in R^m} \frac{q(x,w)}{q(x',w)} := \lambda_q < \infty, \tag{30}$$

$$\sup_{z_1,z_1'\in E_1} \sup_{z'\in E_1} \frac{p_1(z_1,z')}{p_1(z_1',z')} := \lambda_{p_1} < \infty, \tag{31}$$

$$\sup_{x,x'\in E} \sup_{w\in R^m} \sup_{z_1(1)\in E_1} \frac{\bar{p}_1(x,w,z_1(1))}{\bar{p}_1(x',w,z_1(1))} := \lambda_{\bar{p}_1} < \infty, \tag{32}$$

and

$$\sup_{x,x'\in E} \sup_{w\in R^m} \sup_{z_1(1)\in E_1} \sup_{z\in E_2} \frac{\bar{p}_2(x,w,z_1(1),z)}{\bar{p}_2(x',w,z_1(1),z)} := \lambda_{\bar{p}_2} < \infty. \tag{33}$$

The above assumptions say that suitable transition probabilities should be absolutely continuous with densities bounded uniformly from above and from below. Such conditions do not seem to be restrictive if we consider the factor processes taking values in compact sets with continuous transition densities. The assumption (30) is more restrictive since it excludes the case of Gaussian w with expected value or variance depending on the factor x. It is satisfied when we consider $w(n)$ taking bounded values or unbounded but with tails of the same order (with finite ratio $\frac{q(x,w)}{q(x',w)}$ letting coordinates of w to minus or plus ∞). We have

Proposition 1: Under (30)-(33) for $z_1, z_1' \in E_1$, $\mu,\mu' \in \mathcal{P}(E_2)$ and $\beta \in (0,1)$, $\gamma \in [0,1]$

$$\lambda^{-1} \leq \frac{w_\beta(z_1,\mu,\gamma)}{w_\beta(z_1',\mu',\gamma)} \leq \lambda. \tag{34}$$

with $\lambda = \lambda_q\lambda_{p_1}^2\lambda_{\bar{p}_1}\lambda_{\bar{p}_2}$.

Proof: Notice first that for $y \in R^k$, $z_1(0), z_1'(0), z_1(1) \in E_1$, $w \in R^m$, $z_2, z_2' \in E_2$, $h \in U$, $\delta \in [0,1]$ and $\varphi \in b\mathcal{B}(E_2)$ by (30)-(33) we have

$$\begin{aligned}&\Phi^\delta(y, z_1(1), z_1(0), w, h)\varphi(z_2)\\ &\leq \lambda_1 \Phi^\delta(y, z_1(1), z_1'(0), w, h)\varphi(z_2')\end{aligned} \tag{35}$$

with $\lambda_1 = \lambda_q\lambda_{p_1}\lambda_{\bar{p}_1}\lambda_{\bar{p}_2}$. Therefore also for $\mu,\mu' \in \mathcal{P}(E_2)$

$$\begin{aligned}&(\Phi^\gamma)^*(y, z_1(1), z_1(0), w, h)\mu\\ &\leq \lambda_1(\Phi^\gamma)^*(y, z_1(1), z_1'(0), w, h)\mu'.\end{aligned} \tag{36}$$

Let

$$\begin{aligned}&m^\gamma(y, z_1(1), z_1(0), z_1'(0), w, h) =\\ &(\Phi^\gamma)^*(y, z_1(1), z_1'(0), w, h)\mu'\\ &-\lambda_1^{-1}(\Phi^\gamma)^*(y, z_1(1), z_1(0), w, h)\mu.\end{aligned} \tag{37}$$

Assume that $m^\gamma(y, z_1(1), z_1(0), z_1'(0), w, h, h')(E_2) > 0$. Notice that when $\lambda_1 = 1$ by (30)-(33) we easily see that $w_\beta(z_1,\mu,\gamma) = w_\beta(z_1',\mu',\gamma)$. Therefore assume that $\lambda_1 > 1$. By Lemma 2 of [4] and Theorem 1 the mapping $M^+(E_2) \ni \zeta \mapsto Sw_\beta(z_1,\zeta,\gamma)$ is concave. Then

$$\begin{aligned}&Sw_\beta(z_1(1), (\Phi^\gamma)^*(y, z_1(1), z_1'(0), w, h)\mu', \gamma\beta) \geq\\ &\lambda_1^{-1} Sw_\beta(z_1(1), (\Phi^\gamma)^*(y, z_1(1), z_1(0), w, h)\mu, \gamma\beta)\\ &+(1-\lambda_1^{-1})Sw_\beta(z_1(1),\\ &(1-\lambda_1^{-1})^{-1}m^\gamma(y, z_1(1), z_1(0), z_1'(0), w, h), \gamma\beta)\end{aligned}$$

and since w_β is positive we have

$$\begin{aligned}&Sw_\beta(z_1(1), (\Phi^\gamma)^*(y, z_1(1), z_1(0), w, h)\mu, \gamma\beta) \leq \lambda_1\\ &Sw_\beta(z_1(1), (\Phi^\gamma)^*(y, z_1(1), z_1'(0), w, h)\mu', \gamma\beta).\end{aligned} \tag{38}$$

If $m^\gamma(y, z_1(1), z_1(0), z_1'(0), w, h, h')(E_2) = 0$ then we have equality in (36) and by the definition of the operator S

$$\begin{aligned}&Sw_\beta(z_1(1), (\Phi^\gamma)^*(y, z_1(1), z_1(0), w, h)\mu, \gamma\beta) = \lambda_1\\ &Sw_\beta(z_1(1), (\Phi^\gamma)^*(y, z_1(1), z_1'(0), w, h)\mu', \gamma\beta).\end{aligned} \tag{39}$$

Consequently by (38) and (39)

$$\int_{R^m}\int_{R^k}\int_{E_1} Sw_\beta(z',(\Phi^\gamma)^*(y,z',z_1(0),w,h)\mu,\gamma\beta)$$
$$p_1(z_1(0),z')p_1(dz')\eta(dy)\nu(dw) \le \lambda_{p_1}\lambda_1 \int_{R^m}\int_{R^k}\int_{E_1}$$
$$Sw_\beta(z',(\Phi^\gamma)^*(y,z',z_1'(0),w,h)\mu',\gamma\beta)$$
$$p_1(z_1'(0),z')p_1(dz')\eta(dy)\nu(dw)$$

and for $h \in U$ such that

$$w_\beta(z_1'(0),\mu',\gamma) = \inf_{h\in U}\int_{R^m}\int_{R^k}\int_{E_1} Sw_\beta(z',(\Phi^\gamma)^*(y$$
$$,z',z_1'(0),w,h)\mu',\gamma\beta)p_1(z_1'(0),z')p_1(dz')\eta(dy)\nu(dw)$$

we have

$$w_\beta(z_1(0),\mu,\gamma) \le \lambda_{p_1}\lambda_1$$
$$\int_{R^m}\int_{R^k}\int_{E_1} Sw_\beta(z',(\Phi^\gamma)^*(y,z',z_1'(0),w,h)\mu',\gamma\beta)$$
$$p_1(z_1'(0),z')p_1(dz')\eta(dy)\nu(dw) = \lambda_{p_1} w_\beta(z_1'(0),\mu',\gamma)$$

and therefore we obtain the righthand side of (34). Interchanging z_1 by z_1', and μ by μ' in our consideration we obtain the left-hand side of (34), which completes the proof.

■

Proposition 1 shows the boundedness of the ratio $\frac{w_\beta(z_1,\mu,\gamma)}{w_\beta(z_1',\mu',\gamma)}$. We would like also to have its uniform continuity with respect to γ. Let

$$\lambda(z_1(0),z_1'(0),\mu,\mu') :=$$
$$\sup_{y\in R^k}\ \sup_{z_1(1)\in E_1}\ \sup_{w\in R^m}\ \sup_{h\in U}\ \sup_{\delta\in[0,1]}\ \sup_{A\in\mathcal{B}(E_2)}$$
$$\frac{(\Phi^\delta)^*(y,z_1(1),z_1(0),w,h)\mu(A)}{(\Phi^\delta)^*(y,z_1(1),z_1'(0),w,h)\mu'(A)} \tag{40}$$

and

$$\lambda_{p_1}(z_1,z_1') := \sup_{z'\in E_1}\frac{p_1(z_1,z')}{p_1(z_1',z')}. \tag{41}$$

Under (30)-(33) we have that $\lambda(z_1(0),z_1'(0),\mu,\mu') < \infty$ and $\lambda_{p_1}(z_1,z_1') < \infty$. In addition

Proposition 2: Under (30)-(33) for z_1, $z_1' \in E_1$ and μ, $\mu' \in \mathcal{P}(E_2)$ we have

$$(\lambda(z_1,z_1',\mu,\mu')\lambda_{p_1}(z_1,z_1'))^{-1} \le \frac{w_\beta(z_1,\mu,\gamma)}{w_\beta(z_1',\mu',\gamma)} \le$$
$$\lambda(z_1,z_1',\mu,\mu')\lambda_{p_1}(z_1,z_1'). \tag{42}$$

Proof: When $\lambda(z_1,z_1',\mu,\mu') = 1$ by (26) we easily obtain (42). Assume therefore that $\lambda(z_1,z_1',\mu,\mu') > 1$. By analogy to (37) define

$$m^\gamma(y,z_1(1),z_1,z_1',w,h) =$$
$$(\Phi^\gamma)^*(y,z_1(1),z_1',w,h)\mu' -$$
$$\lambda(z_1,z_1',\mu,\mu')^{-1}(\Phi^\gamma)^*(y,z_1(1),z_1,w,h)\mu. \tag{43}$$

When $m^\gamma(y,z_1(1),z_1,z_1',w,h)(E_2) > 0$ analogously as in proof of (38) taking into account positivity of w_β we obtain

$$Sw_\beta(z_1(1),(\Phi^\gamma)^*(y,z_1(1),z_1',w,h)\mu',\gamma\beta) \ge$$
$$(\lambda(z_1,z_1',\mu,\mu'))^{-1}$$
$$Sw_\beta(z_1(1),(\Phi^\gamma)^*(y,z_1(1),z_1,w,h)\mu,\gamma\beta) \tag{44}$$

If $m^\gamma(y,z_1(1),z_1,z_1',w,h)(E_2) = 0$, then

$$Sw_\beta(z_1(1),(\Phi^\gamma)^*(y,z_1(1),z_1',w,h)\mu',\gamma\beta) =$$
$$(\lambda(z_1,z_1',\mu,\mu'))^{-1}$$
$$Sw_\beta(z_1(1),(\Phi^\gamma)^*(y,z_1(1),z_1,w,h)\mu,\gamma\beta)$$

Therefore in both cases we have that

$$\int_{R^m}\int_{R^k}\int_{E_1} Sw_\beta(z',(\Phi^\gamma)^*(y,z',z_1,w,h)\mu,\gamma\beta)$$
$$p_1(z_1,z')p_1(dz')\eta(dy)\nu(dw)$$
$$\le \lambda_{p_1}(z_1,z_1')\lambda(z_1,z_1',\mu,\mu')$$
$$\int_{R^m}\int_{R^k}\int_{E_1} Sw_\beta(z',(\Phi^\gamma)^*(y,z',z_1',w,h)\mu',\gamma\beta)$$
$$p_1(z_1',z')p_1(dz')\eta(dy)\nu(dw)$$

Choosing h optimal for $w_\beta(z_1',\mu',\gamma)$ we obtain the right-hand side of (42). Interchanging the role of z_1 with z_1' and μ with μ' we obtain the assertion of Proposition.

■

IV. Risk sensitive asymptotics

In what follows we shall use a suitable discount vanish approach to obtain Bellman equation corresponding to the cost functional (14). Fix $\bar{z}_1 \in E_1$, $\bar{\mu} \in \mathcal{P}(E_2)$ and define functions

$$v_\beta(z_1,\mu,\gamma) = \frac{w_\beta(z_1,\mu,\gamma)}{w_\beta(\bar{z}_1,\bar{\mu},\gamma)} \tag{45}$$

By Proposition 1 functions v_β are bounded from below and above uniformly in β. Moreover by (26) we have

$$v_\beta(z_1,\mu,\gamma)\kappa_\beta(\gamma) = \inf_{h\in U}\int_{R^m}\int_{R^k}\int_{E_1} Sv_\beta(z',(\Phi^\gamma)^*(y,$$
$$z',z_1,w,h)\mu,\gamma\beta)p_1(z_1,z')p_1(dz')\eta(dy)\nu(dw) \tag{46}$$

with

$$\kappa_\beta(\gamma) = \frac{w_\beta(\bar{z}_1,\bar{\mu},\gamma)}{w_\beta(\bar{z}_1,\bar{\mu},\gamma\beta)}. \tag{47}$$

Assume that the mappings

$$E_1\times E_1\times\mathcal{P}(E_2)\times\mathcal{P}(E_2) \ni (z_1,z_1',\mu,\mu') \mapsto \lambda(z_1,z_1',\mu,\mu') \tag{48}$$

and

$$E_1\times E_1 \ni (z_1,z_1') \mapsto \lambda_{p_1}(z_1,z_1') \tag{49}$$

are continuous. Then since by Proposition 2

$$(\lambda(z_1,z_1',\mu,\mu')\lambda_{p_1}(z_1,z_1'))^{-1} \le \frac{v_\beta(z_1,\mu,\gamma)}{v_\beta(z_1',\mu',\gamma)} \le$$
$$\lambda(z_1,z_1',\mu,\mu')\lambda_{p_1}(z_1,z_1') \tag{50}$$

the family of functions $v_\beta(z_1,\mu,\gamma)$ is bounded continuous uniformly in β and γ. Therefore by Ascoli theorem (see theorem 33 of chapter 9 in [11]) using a diagonal procedure for a fixed γ one can find a subsequence $\beta_n \to 1$, and sequences of numbers $\kappa_l(\gamma)$, and continuous functions $E_1 \times \mathcal{P}(E_2) \ni (z_1,\mu) \mapsto v^l(z_1,\mu,\gamma)$ such that $l = 0,1,\ldots$

$$v_{\beta_n}(z_1,\mu,\gamma\beta_n^l) \to v^l(z_1,\mu,\gamma) \tag{51}$$

uniformly in (z_1,μ) from compact subsets of $E_1\times\mathcal{P}(E_2)$ and

$$\kappa_{\beta_n}(\gamma\beta_n^l)\to\kappa^l(\gamma) \quad (52)$$

as $m\to\infty$. We have

Lemma 2: The sequences (v^l), (κ^l) are solutions to the following equations

$$v^l(z_1,\mu,\gamma)\kappa^l(\gamma)=\inf_{h\in U} \int_{R^m}\int_{R^k}\int_{E_1} Sv^{l+1}(z',(\Phi^\gamma)^*(y,z',z_1,w,h)\mu,\gamma) p_1(z_1,z')p_1(dz')\eta(dy)\nu(dw) \quad (53)$$

for $l=0,1,\ldots$. Moreover for $l=0,1,\ldots$, $v^l(\cdot,\cdot,\gamma)$ is a continuous function and

$$(\lambda(z_1,z_1',\mu,\mu')\lambda_{p_1}(z_1,z_1'))^{-1}\le\frac{v^l(z_1,\mu,\gamma)}{v^l(z_1',\mu',\gamma)}\le \lambda(z_1,z_1',\mu,\mu')\lambda_{p_1}(z_1,z_1'). \quad (54)$$

Proof: By (46) we have

$$v_{\beta_n}(z_1,\mu,\gamma\beta_n^l)\kappa_{\beta_n}(\gamma\beta_n^l)=\inf_{h\in U}\int_{R^m}\int_{R^k}\int_{E_1} Sv_{\beta_n}(z',(\Phi^{\gamma\beta_n^l})^*(y,z',z_1,w,h)\mu,\gamma\beta_n^{l+1}) p_1(z_1,z')p_1(dz')\eta(dy)\nu(dw). \quad (55)$$

Since the mapping

$$(\delta,y,z',z_1,w,h,\mu)\mapsto(\Phi^\delta)^*(y,z',z_1,w,h)\mu$$

is continuous we have that

$$Sv_{\beta_n}(z',(\Phi^{\gamma\beta_n^l})^*(y,z',z_1,w,h)\mu,\gamma\beta_n^{l+1})\to Sv^{l+1}(z',(\Phi^\gamma)^*(y,z',z_1,w,h)\mu,\gamma) \quad (56)$$

as $n\to\infty$, uniformly in (y,z',z_1,w,h,μ) from compact subsets. If μ is from a compact subset $\Gamma\subset\mathcal{P}(E_2)$ then for a given $\varepsilon>0$ there is a compact subset $K_2\subset E_2$ such that $\nu(K_2)\ge 1-\epsilon$ for $\nu\in\Gamma$. By (16) there is a compact set $W\subset R^m$ such that $\sup_h\int_{W^c}e^{-\frac{\delta\Theta}{2}G(w,h)}Q(x,dw)\le\varepsilon$ uniformly in $x=(z_1,z_2)$ from a compact subset $K_1\times K_2$ of $E_1\times E_2$ and δ from a compact neighbourhood of γ with W^c standing for complement $R^m\setminus W$ of set W. Let compact subsets $K_1'\subset E_1$, $K_2'\subset E_2$ and $Z\subset R^k$ be such that

$$\sup_{z_1\in K_1}\sup_{z_2\in K_2}\sup_{w\in W}P_1(z_1,z_2,w,K_1'^c)\le\varepsilon,$$

$$\sup_{z_1\in K_1}\sup_{z_2\in K_2}\sup_{w\in W}\sup_{z'\in K_1'}P_2(z_1,z_2,w,z',K_2'^c)\le\varepsilon$$

and $sup_{z_2\in K_2'}R(z_2,Z^c)\le\varepsilon$. Then we have

$$\sup_{h\in U}\int_W\int_Z\int_{K_1'}|Sv_{\beta_n}(z',(\Phi^{\gamma\beta_n^l})^*(y,z',z_1,w,h)\mu, \gamma\beta_n^{l+1})-Sv^{l+1}(z',(\Phi^\gamma)^*(y,z',z_1,w,h)\mu,\gamma)| p_1(z_1,z')p_1(dz')\eta(dy)\nu(dw)\to 0$$

as $n\to\infty$ uniformly in $z_1\in K_1$, $\mu\in\Gamma$ and integrals $\int_{W^c}\int_{R^k}\int_{E_1}$, $\int_{R^m}\int_{Z^c}\int_{E_1}$, $\int_{R^m}\int_{R^k}\int_{K_1'^c}$ of the difference $|Sv_{\beta_n}-Sv^{l+1}|$ can be approximated from above respectively by $\lambda\varepsilon(2+\overline{\phi}(\gamma\beta_n^l)+\overline{\phi}(\gamma))$, $\lambda\varepsilon(2+4\overline{\phi}(\gamma\beta_n^l)+4\overline{\phi}(\gamma))$, and $\lambda\varepsilon(\overline{\phi}(\gamma\beta_n^l)+\overline{\phi}(\gamma))$, which completes the proof of (53). The bounds (54) follow directly from (50).

■

Consequently by (34) and (54) the family $v^l(\cdot,\cdot,\gamma)$ is equicontinuous and bounded. Therefore there is a constant $\kappa(\gamma)$ and a function $v(\cdot,\gamma)$ such that for a suitably chosen subsequence $l_n\to\infty$ we have

$$\kappa^{l_n}(\gamma)\to\kappa(\gamma) \quad (57)$$

and uniformly in (z_1,μ) from compact subsets of $E_1\times\mathcal{P}(E_2)$

$$v^{l_n}(z_1,\mu,\gamma)\to v(z_1,\mu,\gamma). \quad (58)$$

By a similar consideration as in the proof of Lemma 2 letting $l_n\to\infty$ in (53) we obtain

$$v(z_1,\mu,\gamma)\kappa(\gamma)=\inf_{h\in U} \int_{R^m}\int_{R^k}\int_{E_1} Sv(z',(\Phi^\gamma)^*(y,z',z_1,w,h)\mu,\gamma) p_1(z_1,z')p_1(dz')\eta(dy)\nu(dw). \quad (59)$$

Let for $\varphi\in b\mathcal{B}(E_2)$

$$\begin{aligned}\zeta_0^\gamma(\varphi) &:= \mu(\varphi)\\ \zeta_n^\gamma(\varphi) &:= \zeta_{n-1}^\gamma((\kappa(\gamma))^{-1}\Phi^\gamma(y(n),z_1(n),z_1(n-1), w(n),h(n-1))\varphi)\\ &:= (\kappa(\gamma))^{-1}(\Phi^\gamma)^*(y(n),z_1(n),z_1(n-1), w(n),h(n-1))\zeta_{n-1}^\gamma(\varphi) \end{aligned} \quad (60)$$

for $n=1,2,\ldots$. We have

Theorem 2: For fixed $\gamma\in[0,1]$ there exist a constant $\kappa(\gamma)$ and a continuous function $E_1\times\mathcal{P}\ni(z_1,\mu)\mapsto v(z_1,\mu,\gamma)$ which are solutions to the equation (59). Moreover $v(z_1,\cdot)$ is concave and

$$(\lambda(z_1,z_1',\mu,\mu')\lambda_{p_1}(z_1,z_1'))^{-1}\le\frac{v(z_1,\mu,\gamma)}{v(z_1',\mu',\gamma)}\le \lambda(z_1,z_1',\mu,\mu')\lambda_{p_1}(z_1,z_1') \quad (61)$$

Furthermore

$$\ln\kappa(\gamma)=\inf_{(h(n))}\limsup_{n\to\infty} \frac{1}{n}\ln E_\mu\left\{\exp\left\{\sum_{j=0}^{n-1}\frac{-\Theta\gamma}{2}G(w(j),h(j))\right\}\right\} \quad (62)$$

and the strategy $\hat{h}(n)=h(z_1(n),\frac{\zeta_n^\gamma}{\zeta_n^\gamma(E_2)},\gamma)$, where h is a selector for which the infimum in the right hand side of (59) is attained, is optimal.

Proof: The existence of solutions to (59) follows as written above by the limit argument as in the proof of Lemma 2. By (58) and (54) we immediately have (61). Notice now that by (59) and (60) for $n=1,2,\ldots$ we have

$$v(z_1,\mu,\gamma)\le E^o\left\{\zeta_n^\gamma(E_2)v(z_1(n),\frac{\zeta_n^\gamma}{\zeta_n^\gamma(E_2)},\gamma)\right\}$$

with equality for the strategy $(\hat{h}(i))$ defined in the statement of theorem. By Proposition 1 and the method of the construction of function v we have that $\lambda^{-1}\le$

$\frac{v(z_1,\mu,\gamma)}{v(z_1',\mu',\gamma)} \leq \lambda$ for $z_1, z_1' \in E_1$, $\mu, \mu' \in \mathcal{P}(E_2)$, and $\gamma \in [0,1]$. Consequently

$$0 \leq \limsup_{n\to\infty} \frac{1}{n} \ln E^o \left\{\zeta_n^\gamma(E_2)\right\},$$

with equality for the strategy $(\hat{h}(i))$. Since

$$\limsup_{n\to\infty} \ln E^o \left\{\zeta_n^\gamma(E_2)\right\} =$$
$$\limsup_{n\to\infty} \frac{1}{n} \ln E_\mu \left\{ \exp\left\{ \sum_{j=0}^{n-1} \frac{-\Theta\gamma}{2} G(w(j), h(j)) \right\}\right\}$$
$$- \ln \kappa(\gamma),$$

we obtain (62) with optimal strategy $(\hat{h}(i))$.

■

Remark 1: By theorem 2 we see that a control maximizing risk sensitized growth rate is a function of the observed factors $z_1(n)$ and the projections on the set of probability measures of measure valued processes ζ_n^γ. The result corresponds to the perfect observation case studied in [13], where using different approach it was shown that a control maximizing growth rate was a function of the factors only.

V. Remarks on assumptions

Notice first that to solve discounted Bellman equation we need equivalence of transition kernels $R(z_2,\cdot)$, $Q(x,\cdot)$, $P_1(x,w,\cdot)$ and $P_2(x,w,z_1,\cdot)$ for different values of $z_2 \in E_2$, $x \in E$, $w \in R^m$, $z_1 \in E_1$ respectively, as well as the continuity of their transition densities. If

$$s_n = h(r_n, \xi_n), \tag{63}$$

for $n = 0, 1, \ldots$, where $r_n \in W$ a given metric space and (ξ_n) is a sequence of i.i.d. R^d valued random variables with commmon strictly positive density g, ξ_n is independent of r_j for $j \leq n$ and moreover $h(r,\cdot)$ is a C^1 diffeomorphism of R^d with inverse function $k(x,\cdot)$ and Jacobian of $k(x,y)$ denoted by $\Delta(x,y)$ then

$$P\left\{s_n \in A | s_0, \ldots, s_{n-1}, r_0, \ldots, r_n\right\} = K(r_n, A), \tag{64}$$

where

$$K(r_n, A) = \int_A g(k(r_n,\xi)) |\Delta(r_n,\xi)| d\xi \tag{65}$$

with $A \in \mathcal{B}(R^d)$ and $|\Delta(r_n,\xi)|$ standing for the determinant of Jacobian $\Delta(r_n,\xi)$. Consequently for all $r \in W$ the transition kernels are equivalent. The continuity and boundedness of transition densities $k(r,\cdot)$ of $K(r,\cdot)$ with respect to $K(\bar{r},\cdot)$ with a fixed $\bar{r} \in W$ corresponds to the continuity and boundedness of the function

$$\xi \mapsto \frac{g(k(r,\xi))|\Delta(r,\xi)|}{g(k(\bar{r},\xi))|\Delta(\bar{r},\xi)|}. \tag{66}$$

The boundedness assumptions (30)-(33) have their analog of the form

$$\sup_{r,r' \in W} \sup_{\xi \in R^d} \frac{g(k(r,\xi))|\Delta(r,\xi)|}{g(k(r',\xi))|\Delta(r',\xi)|} < \infty. \tag{67}$$

A general model (63) includes in particular the case of additive or multiplicative noise ξ_n. Furthermore, in particular applications s_n may be an element of a finite set D. This is the case when our observation is quantized. Then if $s_n = h(r_n, \xi_n)$ where ξ_n is a sequence of i.i.d., D valued random variables with positive probability function $p(b) = P\left\{\xi_n = b\right\}$ for $b \in D$ and $h(w,\cdot)$ is $1-1$ mapping of D, then transition kernels $K(r,\cdot)$ defined (64) are for $r \in W$ equivalent and assumption (67) is replaced by the finitness of

$$\sup_{r,r'} \frac{p(k(r,b))}{p(k(r',b))}$$

for all $b \in D$.

The form (12) of implicitly defined function G corresponds to the continuous time lognormal market model with drift and volatility parameters depending on discrete time factors and with discrete time portfolio changes (see (11) and (12) in [13]). In such case the functions $\underline{\phi}(\gamma)$ and $\overline{\phi}(\gamma)$ can be easily constructed. If $\sup_{x\in E} \sup_h \int_{R^m} G(w,h) Q(x,dw) := \|G\| < \infty$ then by Jensen's lemma we can choose $\underline{\phi}(\gamma) = e^{-\frac{\gamma\theta}{2}\|G\|}$ in the formula (15). The existence of the function $\overline{\phi}(\gamma)$ from (15) requires certain properties of the transition kernel $Q(x,\cdot)$ satisfied in particular by many probability laws when we assume additionally that $(w_i(n))$ form a sequence of i.i.d. random variables. In particular when E is compact and we don't allow shortselling and shortborrowing i.e. $U = \left\{(h_1,\ldots,h_m)^T, h_i \geq 0, \sum_{i=1}^m h_i \leq 1\right\}$ then under (16) the functions $\underline{\phi}(\gamma)$ and $\overline{\phi}(\gamma)$ may be defined as

$$\underline{\phi}(\gamma) = \inf_{x\in E} \inf_{h\in U} \int_{R^m} e^{-\frac{\gamma\Theta}{2} G(w,h)} Q(x,dw)$$

and

$$\overline{\phi}(\gamma) = \sup_{x\in E} \sup_{h\in U} \int_{R^m} e^{-\frac{\gamma\Theta}{2} G(w,h)} Q(x,dw).$$

In this case for the continuity of (16), assuming (30) and using the continuity of the transition density q, we need only integrability of $e^{-\frac{\gamma\Theta}{2}G(w,h)}$ for each $h \in U$ and $\gamma \in (0,1]$.
The continuity of the mappings (48) and (49) is clearly satisfied when the spaces E_1 and E_2 are compact.

Notice finally that, if we consider as in [3] finite valued factors in the form of ergodic Markov chains then: assumptions (31)-(33) correspond to one step irreducibility, and the mappings (48) as well as (49) are clearly continuous with discrete topology on E_1. In this case, if additionally there are no shortselling and shortborrowing, the only further assumptions we have to impose concern (as mentioned above) suitable integrability of $e^{-\frac{\gamma\Theta}{2}G(w,h)}$ and regularity of the density q i.e. continuity with respect to w and boundedness (30).

VI. Other related models

In the paper we assumed that transition probability P_2 defined in (9) depends besides of $x(n)$ and $w(n)$ on $z_1(n+1)$ whereras transition probability P_1 defined in (6) depended on $x(n)$ and $w(n)$ only. One can consider an alternative model in which P_1 depends on $x(n)$, $w(n)$, $z_2(n+1)$ while P_2 on $x(n)$ and $w(n)$ only. To be more precise let for $A_1 \in \mathcal{B}(E_1)$

$$P\left\{z_1(n+1) \in A_1 | X^n, W^n, Y^n, z_2(n+1)\right\} := P_1(x(n), w(n), z_2(n+1), A_1), \quad (68)$$

and measures $P_1(x,w,z,\cdot)$ for $x \in E$, $w \in R^d$, $z \in E_2$ be equivalent. For fixed $\bar{x} = (\bar{z}_1, \bar{z}_2) \in E$, $\bar{w} \in R^d$, $\bar{z} \in E_2$ let $p_1(\cdot) = P_1(\bar{x}, \bar{w}, \bar{z}, \cdot)$. Define positive densities $\bar{p}_1$ and p_1 such that

$$P_1(x,w,z,A_1) = \int_{A_1} \bar{p}_1(x,w,z,z')p_1(dz'), \quad (69)$$

and

$$P_1(z_1, \bar{z}_2, \bar{w}, \bar{z}, A_1) = \int_{A_1} p_1(z_1,z')p_1(dz'). \quad (70)$$

Similarly let

$$P\left\{z_2(n+1) \in A_2 | X^n, W^n, Y^n\right\} := P_2(x(n), w(n), A_2), \quad (71)$$

for $A_2 \in \mathcal{B}(E_2)$ and assume that measures $P_2(x,w,\cdot)$ for $x \in E$, $w \in R^d$ are equivalent. Fix $\bar{x} \in E$ and $\bar{w} \in R^d$. Letting $p_2(\cdot) = P_2(\bar{x}, \bar{w}, \bar{z}, \cdot)$ there are positive densities $\bar{p}_2$ and p_2 such that we have

$$P_2(x,w,A_2) = \int_{A_2} \bar{p}_2(x,w,z')p_2(dz'). \quad (72)$$

and

$$P_2(\bar{z}_1, z_2, \bar{w}, A_2) = \int_{A_2} \bar{p}_2(z_2,z')p_2(dz'). \quad (73)$$

We can continue as in the previous case changing the form of Λ_n in (18) (by adding $z_2(i)$ to $\bar{p}_1$ and deleting $z_1(i)$ in $\bar{p_2}$) and obtain Φ^δ in (22) of the form

$$\begin{aligned}
&\Phi^\delta(y, z_1(1), z_1(0), w, h)\varphi(z_2) \\
&= \exp\left\{\frac{-\delta\Theta}{2}G(w,h)\right\} q(z_1(0), z_2, w) \quad (74)\\
&\int_{E_2} \frac{\bar{p}_1(z_1(0), z_2, w, z, z_1(1))}{p_1(z_1(0), z_1(1))} \\
&\varphi(z)r(z,y)\bar{p}_2(z_1(0), z_2, w, z)p_2(dz) \quad (75)
\end{aligned}$$

Also, in assumptions (32) as well as (33) we have to correct the form of transition densities $\bar{p_1}$ and $\bar{p_2}$. With the above mentioned modifications for our new model one can obtain analogs of the results from sections 2, 3 and 4, i.e. Theorems 1 and Theorem 2.

Consider now another modification of our model, which is closely related to the models considered in [7], [8] and [9], although we allow in our model simultanealously both observed and unobserved factors. Assume that the factors $z_2(n)$ are observed only through the asset prices $S_i(n)$ i.e. there are no observation process $y(n)$. In such situation our observation available at time n is the σ - field $\mathcal{Y}^n$ of the form $\sigma\left\{z_1(l), l \leq n, W^{n-1}\right\}$. Consequently the density Λ_n defined in (19) does not contain the term $r(z_2(i), y(i))$ and Lemma 1 remains true without the statements on $(y(n))$. We can define the process σ_n^γ as in (20) and obtain (21) with Φ^δ of the form

$$\begin{aligned}
&\Phi^\delta(y, z_1(1), z_1(0), w, h)\varphi(z_2) \\
&= \exp\left\{\frac{-\delta\Theta}{2}G(w,h)\right\} q(z_1(0), z_2, w) \\
&\frac{\bar{p}_1(z_1(0), z_2, w, z_1(1))}{p_1(z_1(0), z_1(1))} \\
&\int_{E_2} \varphi(z)\bar{p}_2(z_1(0), z_2, w, z_1(1), z)p_2(dz), \quad (76)
\end{aligned}$$

which means that we neglect the term $r(z,y)$ in (22). Then one can obtain analogous results to those of sections 2,3 and 4 leading to analogs of Theorems 1 and 2.

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