Duality and Risk Sensitive Portfolio Optimization

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Abstract. In this paper maximization over long time horizon of the probability that the averaged growth of portfolio is greater than a given benchmark, is shown to be dual problem to the risk sensitive portfolio management. Differentiability of the risk sensitive value function with respect to the risk parameter is then studied.

1. Introduction

Assume we are given a market consisting of m securities and k factors. The prices of securities depend on factors, the set of which may include dividend yields, rate of inflation, short term interest rates etc. Denote by $V(n)$ the value of portfolio at time n. Given portfolio strategy $h(n) = (h_1(n), \dots, h_m(n))^T$, which is an R^m vector (z^T stands for the transpose of z) representing parts of capital invested in the securities i.e. $h_i(n)$ is a part of the capital $V(n)$ at time n invested in the $i-th$ security, we assume the portfolio dynamics of the form

$$\frac{V(n+1)}{V(n)} = \exp\{F(x(n), h(n), W(n))\}, \tag{1.1}$$

where $(x(n)) = ((x_1(n), \dots, x_k(n))^T)$ is a process of factors and $(W(n))$ is standing for a sequence of i.i.d. random variables with law η, and F is a suitable Borel measurable function. The portfolio strategy $h(n)$ is a random variable taking values in a compact subset U of R^m, adapted to the available information about the prices of securities and the values of factors up to time n. In particular, when

$$U = \left\{ h \in R^m : h_i \geq 0 \quad \text{for} \quad i = 1, 2, \dots, n, \quad \sum_{i=1}^m h_i(n) \leq 1 \right\}$$

we have the absence of short selling and short borrowing and $1 - \sum_{i=1}^m h_i(n)$ corresponds to the part of capital invested in the banking account at time n. We shall denote by $\mathcal{A}$ the set of all portfolio strategies. The factor process is Markov with transition operator $P(x, \cdot)$. In particular case we shall consider the factor process $(x(n))$ of the form

$$x(n+1) = G(x(n), W(n)) \tag{1.2}$$

with the same $W(n)$ as in the portfolio dynamics (1.1).

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Example 1.1. Assume that the discrete time dynamics of assets is given by the formula

$$\frac{dS_i(t)}{S_i(t)} = a_i(x(n))dt + \sum_{j=1}^{k+m} \sigma_{ij}(x(n))dw_j(t) \tag{1.3}$$

for $t \in [n, n+1)$ with a_i, σ_{ij} being for $i = 1, \dots, m$, $j = 1, \dots, k+m$ measurable functions, and $(w_j(t))$ being independent Brownian motions. Then denoting by $W(n)$ the trajectory of $w_j(t) - w_j(n)$, $n+1 \geq t \geq n$, $j = 1, \dots, k+m$ using Ito's lemma (see [**S2**]) we have the function F of the form

$$\begin{aligned} F(x(n), h(n), W(n)) &= \sum_{i=1}^{m} \int_n^{n+1} a_i(x(n))h_i(s)ds \\ -\frac{1}{2}\sum_{j=1}^{k+m} \int_n^{n+1} \left(\sum_{i=1}^{m} h_i(s)\sigma_{ij}(x(n))\right)^2 &+ \int_n^{n+1} \sum_{i=1}^{m} h_i(s) \sum_{j=1}^{k+m} \sigma_{ij}(x(n))dw_j(s). \end{aligned} \tag{1.4}$$

The dynamics of factors $(x(n))$ maybe of the form

$$x_r(n+1) = b_r(x(n)) + \sum_{j=1}^{k+m} \lambda_{rj}(w_j(n+1) - w_j(n)) \tag{1.5}$$

where b_r is a measurable function and λ_{rj} is a constant for $r = 1, \dots, k$ and $j = 1, \dots, k+m$.

Example 1.2. We assume in general that security prices have dynamics

$$\frac{S_i(n+1)}{S_i(n)} = \zeta_i(x(n), W(n)) \tag{1.6}$$

where $(x(n))$ is a factor process with transition operator P (in particular of the form (1.2)), ζ is a given vector function and $(W(n))$ is a sequence of i.i.d. random variables. Then

$$\begin{aligned} V(n+1) &= V(n)\left(\sum_{i=1}^{m} h_i(n)\zeta_i(x(n), W(n)) + \left(1 - \sum_{i=1}^{m} h_i(n)\right)\right) \\ &= V(n)\exp\{F(x(n), h(n), W(n))\} \end{aligned} \tag{1.7}$$

defining implicitly the function F.

When $h \in \mathcal{A}$ is of the form $h_i(n) = h_i(x(n))$ for a given Borel measurable vector function $h : R^k \mapsto U$ it will be called the Markov portfolio strategy. The class of such strategies we shall denote by $\mathcal{M}$. We consider the following problems
Primary problem: For $c \in R$ find $h \in \mathcal{A}$ maximizing

$$\begin{aligned} I^h(c) &:= \liminf_{N\to\infty} \frac{1}{N} \ln P_{vx}\left\{\frac{1}{N}\ln V(N) \geq c\right\} \\ &= \liminf_{N\to\infty} \frac{1}{N} \ln P_{vx}\left\{\frac{1}{N}\sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \geq c\right\} \end{aligned} \tag{1.8}$$

which means that we are looking for a strategy which allows to maximize probability of the portfolio growth rate above a given benchmark c. We shall use notation

$$I(c) := \sup_{h\in\mathcal{A}} I^h(c), \quad I_{\mathcal{M}}(c) := \sup_{h\in\mathcal{M}} I^h(c) \tag{1.9}$$

Following [**P1**] and [**P2**] we shall consider the following control problem
Auxiliary problem: For $\theta > 0$ find $h \in \mathcal{A}$ maximizing

$$\begin{aligned}\Lambda^h(\theta) &:= \liminf_{N\to\infty} \frac{1}{N} \ln E_{vx}\left\{V(N)^\theta\right\} \\ &= \liminf_{N\to\infty} \frac{1}{N} \ln E_{vx}\left\{\exp\left\{\sum_{i=0}^{N-1} \theta F(x(i), h(i), W(i))\right\}\right\}\end{aligned} \tag{1.10}$$

Notice that the auxiliary problem with risk factor $\theta < 0$ was introduced as so called risk sensitive control problem in [**BP**]. Let

$$\Lambda(\theta) := \sup_{h\in\mathcal{A}} \Lambda^h(\theta), \quad \Lambda_{\mathcal{M}}(\theta) := \sup_{h\in\mathcal{M}} \Lambda^h(\theta) \tag{1.11}$$

In the formulae (1.8)-(1.11) above with an abuse of notation we neglect the dependence on initial capital v and initial value of factors x. In the paper we shall characterize the solution to the primary problem and the cost functional $I^h(c)$ in terms of the auxiliary problem and cost functional Λ^h with $h \in \mathcal{M}$ (in fact as a dual to the auxiliary problem). We extend the papers [**P1**] and [**P2**]. The duality method introduced in the section 2 strongly depends on differentiability of the risk sensitive costs functional Λ^h or of its value function Λ.

2. Main duality result

We have

Lemma 2.1. *For $\theta \geq 0$ and $h \in \mathcal{A}$*

$$\begin{aligned}&\liminf_{N\to\infty} \frac{1}{N} \ln P_{vx}\left\{\frac{1}{N}\sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \geq c\right\} \leq \\ &\liminf_{N\to\infty} \frac{1}{N} \ln E_{vx}\left\{\exp\left\{\sum_{i=0}^{N-1} \theta F(x(i), h(i), W(i))\right\}\right\} - \theta c\end{aligned} \tag{2.1}$$

Proof. The case $\theta > 0$ follows directly from the Chebychev's inequality. In the case $\theta = 0$ the inequality is trivially satisfied. □

In what follows we shall assume that for a given $h \in \mathcal{M}$ and $\theta \in [0, K)$ the values of

$$\frac{1}{N} \ln E_{vx}\left\{\exp\left\{\sum_{i=0}^{N-1} \theta F(x(i), h(i), W(i))\right\}\right\} \tag{2.2}$$

are finite and there exists a limit as $N \to \infty$ which not depend on the initial capital v and the initial state of factors x.

Since in (2.1) both h and $\theta \in [0, K)$ may be chosen arbitrarily we immediately have (with * denoting convex duality)

Lemma 2.2.

$$I^h(c) \le -\sup_{\theta\in[0,K)} \left\{\theta c - \Lambda^h(\theta)\right\} := -\Lambda^{h*}(c) \tag{2.3}$$

and

$$I(c) \le -\sup_{\theta\in[0,K)} \{\theta c - \Lambda(\theta)\} := -\Lambda^{*}(c) \tag{2.4}$$

Remark 2.3. One can easily notice that the mappings $\theta \mapsto \Lambda^h(\theta)$ and $\theta \mapsto \Lambda(\theta)$ are convex.

Let for $h \in \mathcal{M}$

$$\Lambda_N^h(\theta, x) := \ln E_x \left\{ \exp\left\{ \theta \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \right\} \right\}. \tag{2.5}$$

Define a new equivalent measure $Q_N^{\theta,h}$ by the formula

$$dQ_N^{\theta,h} = \exp\left\{ \theta \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) - \Lambda_N^h(\theta, x) \right\} dP \tag{2.6}$$

The main result of this section is preceded by the following sequence of lemmas.

Lemma 2.4. *For $c_n - \varepsilon > c$, $h \in \mathcal{M}$ and $\theta \in [0,K)$ we have*

$$\begin{aligned} &\liminf_{N\to\infty} \frac{1}{N} \ln P_{vx} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \ge c \right\} \ge \Lambda^h(\theta) - \theta(c_n + \varepsilon) \\ &+ \liminf_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta,h} \left\{ c_n - \varepsilon < \frac{1}{N} \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) < c_n + \varepsilon \right\}. \end{aligned} \tag{2.7}$$

Proof. We clearly have

$$\begin{aligned} &\frac{1}{N} \ln P_{vx} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \ge c \right\} \ge \\ &\ge \frac{1}{N} \ln P_{vx} \left\{ c_n - \varepsilon < \frac{1}{N} \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) < c_n + \varepsilon \right\} = \\ &= \frac{1}{N} \ln \int \exp\left\{ -\theta \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) + \Lambda_N^h(\theta, x) \right\} \\ &1_{c_n - \varepsilon < \frac{1}{N}\sum_{i=0}^{N-1} F(x(i),h(i),W(i)) < c_n+\varepsilon} dQ_N^{\theta,h} \ge \\ &\frac{1}{N} \ln \int \exp\left\{ -\theta N(c_n + \varepsilon) + \Lambda_N^h(\theta, x) \right\} \\ &1_{c_n - \varepsilon < \frac{1}{N}\sum_{i=0}^{N-1} F(x(i),h(i),W(i)) < c_n+\varepsilon} dQ_N^{\theta,h}. \end{aligned}$$

□

Lemma 2.5. *For $\gamma > 0$ and $h \in \mathcal{M}$*

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta,h} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \ge c_n + \varepsilon \right\} \le \\ &\le -\gamma(c_n + \varepsilon) + \Lambda^h(\theta + \gamma) - \Lambda^h(\theta). \end{aligned} \tag{2.8}$$

PROOF. Notice that using the existence of the limit in (2.2) we have for $\gamma \geq -\theta$

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta,h} \left\{ \exp\left\{ \gamma \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \right\} \right\} = \\ &\lim_{N\to\infty} \frac{1}{N} \left(\Lambda_N^h(\theta+\gamma, x) - \Lambda_N^h(\theta, x) \right) = \Lambda^h(\theta+\gamma) - \Lambda^h(\theta). \end{aligned} \tag{2.9}$$

Consequently for $\gamma \geq 0$ by Chebychev inequality we obtain

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta,h} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \geq c_n + \varepsilon \right\} \leq \\ &\limsup_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta,h} \left\{ \exp\left\{ \gamma \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \right\} \geq \exp\left\{ \gamma N(c_n+\varepsilon) \right\} \right\} \leq \\ &-\gamma(c_n+\varepsilon) + \limsup_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta,h} \left\{ \exp\left\{ \gamma \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \right\} \right\} \end{aligned}$$

which by (2.9) completes the proof. □

LEMMA 2.6. *For $-\theta \leq \gamma < 0$ and $h \in \mathcal{M}$ we have*

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta,h} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \leq c_n - \varepsilon \right\} \leq \\ &-\gamma(c_n - \varepsilon) + \Lambda^h(\theta+\gamma) - \Lambda^h(\theta). \end{aligned} \tag{2.10}$$

PROOF. By Chebychev inequality

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta,h} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \leq c_n - \varepsilon \right\} = \\ &= \limsup_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta,h} \left\{ \exp\left\{ \gamma \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \right\} \geq \exp\left\{ \gamma N(c_n-\varepsilon) \right\} \right\} \leq \\ &\limsup_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta,h} \left\{ \exp\left\{ \gamma \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \right\} \right\} - \gamma(c_n - \varepsilon) \end{aligned}$$

and the inequality (2.10) follows from (2.9). □

Denote by $\Lambda'^{+}(\theta)$ and $\Lambda^{h\prime+}(\theta)$ the righthand derivatives of functions $\Lambda(\theta)$ and $\Lambda^h(\theta)$ at θ and similarly $\Lambda'^{-}(K)$ the lefthand derivative of Λ at K.

THEOREM 2.7. *Assume that $\Lambda(\theta) = \Lambda_{\mathcal{M}}(\theta)$ for $\theta \in [0, K)$.*
For $c \in \{\Lambda'^{+}(\theta) : \theta \in [0, K)\}$ or $c \leq \Lambda'^{+}(0)$ we have

$$I(c) = -\Lambda^*(c) \tag{2.11}$$

Let c_n be defined as follows:

1. *for $c \in \{\Lambda'^{+}(\theta) : \theta \in [0, K)\}$, c_n is such that $c < c_n \leq c + \frac{1}{n} < \Lambda'^{-}(K)$ and there exists $\theta(c_n) \in [0, K)$ such that $c_n = \Lambda'(\theta(c_n))$,*
2. *for $c \leq \Lambda'^{+}(0)$, c_n is such that $\Lambda'^{+}(0) < c_n \leq \Lambda'^{+}(0) + \frac{1}{n} < \Lambda'^{-}(K)$ and there exists $\theta(c_n) \in [0, K)$ such that $c_n = \Lambda'(\theta(c_n))$.*

Let $h^\theta \in \mathcal{M}$ be an optimal control corresponding to the value function $\Lambda(\theta)$. Then

$$I(c) = \lim_{n\to\infty} I^{h^{\theta(c_n)}}(c) \tag{2.12}$$

Furthermore if $c \in \left\{\Lambda^{h\prime+}(\theta) : \theta \in [0, K)\right\}$ or $c \le \Lambda^{h\prime+}(0)$, for $h \in \mathcal{M}$ we have

$$I^h(c) = -\Lambda^{h*}(c) \tag{2.13}$$

Proof. Notice first that by the choice of c_n we have

$$\frac{\Lambda^{h^{\theta(c_n)}}(\theta(c_n)+\gamma) - \Lambda^{h^{\theta(c_n)}}(\theta(c_n))}{\gamma} \le \frac{\Lambda_{\mathcal{M}}(\theta(c_n)+\gamma) - \Lambda_{\mathcal{M}}(\theta(c_n))}{\gamma} < c_n + \varepsilon \tag{2.14}$$

for a sufficiently small $\gamma > 0$. Consequently by Lemma 2.5 we have

$$\limsup_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta(c_n),h^{\theta(c_n)}} \left\{\frac{1}{N}\sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \ge c_n + \varepsilon\right\} \le 0. \tag{2.15}$$

For $-\theta(c_n) \le \gamma < 0$ we have

$$\frac{\Lambda^{h^{\theta(c_n)}}(\theta(c_n)) - \Lambda^{h^{\theta(c_n)}}(\theta(c_n)+\gamma)}{-\gamma} \ge \frac{\Lambda_{\mathcal{M}}(\theta(c_n)) - \Lambda_{\mathcal{M}}(\theta(c_n)+\gamma)}{-\gamma} > c_n - \varepsilon \tag{2.16}$$

assuming additionally that γ is sufficiently close to 0. Therefore by Lemma 2.6 we have

$$\limsup_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta(c_n),h^{\theta(c_n)}} \left\{\frac{1}{N}\sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \le c_n - \varepsilon\right\} \le 0. \tag{2.17}$$

Summarizing we obtain

$$\liminf_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta(c_n),h^{\theta(c_n)}} \left\{c_n - \varepsilon < \frac{1}{N}\sum_{i=0}^{N-1} F(x(i), h(i), W(i)) < c_n + \varepsilon\right\} = 0 \tag{2.18}$$

and from Lemma 2.4 we have

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta(c_n),h^{\theta(c_n)}} \left\{\frac{1}{N}\sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \ge c\right\} \\
&\ge \Lambda^{h^{\theta(c_n)}}(\theta(c_n)) - \theta(c_n)(c_n+\varepsilon)) = -\theta(c_n)c_n - \Lambda_{\mathcal{M}}(\theta(c_n)) - \theta(c_n)\varepsilon \\
&= \Lambda^*_{\mathcal{M}}(c_n) - \theta(c_n)\varepsilon.
\end{aligned} \tag{2.19}$$

Since we assumed that $\Lambda_{\mathcal{M}}(\theta) = \Lambda(\theta)$ and the lefthand side of (2.19) does not depend on ε we have

$$\liminf_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta(c_n),h^{\theta(c_n)}} \left\{\frac{1}{N}\sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \ge c\right\} \ge \Lambda^*(c_n).$$

Consequently by continuity of $c \mapsto \Lambda^*(c)$ we obtain

$$\lim_{n\to\infty} \liminf_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta(c_n),h^{\theta(c_n)}} \left\{\frac{1}{N}\sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \ge c\right\} \ge \Lambda^*(c),$$

which by (2.4) proves (2.11) and (2.12). In the case when $c \in \left\{\Lambda^{h\prime+}(\theta) : \theta \in [0, K)\right\}$ or $c \leq \Lambda^{h\prime+}(0)$, for $h \in \mathcal{M}$ define c_n and $\theta(c_n)$ as in the statement of Theorem with Λ'^+ and Λ'^- replaced by $\Lambda^{h\prime+}$ and $\Lambda^{h\prime-}$. Then as in (2.14) and (2.15) we have

$$\frac{\Lambda^h(\theta(c_n) + \gamma) - \Lambda^h(\theta(c_n))}{\gamma} < c_n + \varepsilon$$

for sufficiently small $\gamma > 0$ and

$$\frac{\Lambda^h(\theta(c_n)) - \Lambda^h(\theta(c_n) + \gamma)}{-\gamma} > c_n - \varepsilon$$

for $-\theta(c_n) \leq \gamma < 0$ sufficiently close to 0. Therefore by Lemmas 2.5 and 2.6 we have

$$\liminf_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta(c_n),h} \left\{ c_n - \varepsilon < \frac{1}{N} \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) < c_n + \varepsilon \right\} = 0$$

and from Lemma 2.4

$$\liminf_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta(c_n),h} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \geq c \right\} \geq \Lambda^{h*}(c_n) - \theta(c_n)\varepsilon$$

and letting ε to 0, then $n \to \infty$ we obtain that

$$\lim_{n\to\infty} \liminf_{N\to\infty} \frac{1}{N} \ln Q_N^{\theta(c_n),h} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} F(x(i), h(i), W(i)) \geq c \right\} \geq \Lambda^{h*}(c),$$

which by (2.3) completes the proof. □

REMARK 2.8. Notice first that as soon as we are able to find a solution to the Bellman equation corresponding to the risk sensitive cost functional (1.10) we have that $\Lambda(\theta) = \Lambda_{\mathcal{M}}(\theta)$. Moreover, the right hand derivatives $\Lambda'^+(\theta)$ and $\Lambda^{h\prime+}(\theta)$ are right continuous so that c_n in Theorem 2.7 is well defined. Furthermore, by (2.13) we have that

$$I(c) \leq I_{\mathcal{M}}(c) = -\inf_{h\in\mathcal{M}} \Lambda^{h*}(c).$$

3. Regularity of value functions

3.1. The case of factors independent of the noise. In this section we shall assume that $(x(n))$ is a Markov process with transition operator $P(x, dx')$ is independent of $(W(n))$. Furthermore the operator P is Feller i.e. it maps the set $C(R^k)$ of bounded continuous R^k - valued functions into itself. Consider first the following system of Bellman equations

$$w_N(v, x, \theta) = v^\theta \tag{3.1}$$

$$\begin{aligned} w_{N-1}(v, x, \theta) &= \sup_{h\in U} E_{vx} \left\{ w_N(ve^{F(x,h,W(0))}), x(1), \theta) \right\} \\ &= v^\theta \sup_{h\in U} \int e^{\theta F(x,h,y)} \eta(dy) := v^\theta e^{g(x,\theta)} \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
w_{N-2}(v,x,\theta) &= \sup_{h\in U} E_{vx}\left\{w_{N-1}(ve^{F(x,h,W(0))},x(1),\theta)\right\} \\
&= v^{\theta}\sup_{h\in U} E_{vx}\left\{e^{\theta F(x,h,W(0))}\int_E e^{g(x',\theta)}P(x,dx')\right\} \qquad (3.3)\\
&= v^{\theta}E_x\left\{e^{g(x,\theta)+g(x(1),\theta)}\right\}.
\end{aligned}
$$

By an easy induction we obtain that

$$
\begin{aligned}
w_0(v,x,\theta) &:= \sup_{h\in\mathcal{A}}\left\{\exp\left\{\sum_{i=0}^{N-1}\theta F(x(i),h(i),W(i))\right\}\right\} \qquad (3.4)\\
&= v^{\theta}E_x\left\{\exp\left\{\sum_{i=0}^{N-1}g(x(i),\theta)\right\}\right\}.
\end{aligned}
$$

Notice that an optimal control is Markov defined by a Borel measurable function $h: R^k \mapsto U$ for which the supremum in $\sup_h \int e^{\theta F(x,h,y)}\eta(dy) = e^{g(x,\theta)}$ is attained. We have therefore an optimal control which is Markov, time independent and independent of the current value of our portfolio. One can notice that

$$
\Lambda(\theta) = \liminf_{N\to\infty}\frac{1}{N}E_x\left\{exp\left\{\sum_{i=0}^{N-1}g(x(i),\theta)\right\}\right\}. \qquad (3.5)
$$

It will be important for future analysis to determine the case when the function

$$
E \ni x \mapsto E_x\left\{exp\left\{\sum_{i=0}^{N-1}(g(x(i),\theta)-\Lambda(\theta))\right\}\right\}
$$

is bounded. Solution to this problem leads to so called multiplicative Poisson equation. We are looking for a constant $\Lambda(\theta)$ and a bounded function $w(x,\theta)$ such that

$$
e^{w(x,\theta)} = e^{g(x,\theta)-\Lambda(\theta)}E_x\left\{e^{w(x(1),\theta)}\right\}. \qquad (3.6)
$$

We say that the process (x_n) is uniformly ergodic if there exists $0<\gamma<1$ such that for all $x,x'\in R^k$ and $A\in\mathcal{B}(R^k)$, where $\mathcal{B}(R^k)$ is the set of Borel measurable subsets of R^k, we have

$$
P(x,A) - P(x',A) \le \gamma.
$$

If $(x(n))$ is uniformly ergodic and θ is sufficiently small, by [**D2**] we have the existence of a constant $\Lambda(\theta)$ and a continuous bounded function $w(x,\theta)$ which are solutions to the multiplicative Poisson equation. In the case when there is a probability measure ν and density p such that for any $A\in\mathcal{B}(R^k)$, $P(x,A) = \int_A p(x,y)\nu(dy)$ we have

$$
\sup_{x,x',y\in R^k}\frac{p(x,y)}{p(x',y)} < \infty, \qquad (3.7)
$$

then by Theorem 1 of [**D1**] for any $\theta\in R$ there is a constant $\Lambda(\theta)$ and a continuous bounded function $w(x,\theta)$ which are solutions to the multiplicative Poisson equation. Notice that under (3.7) we clearly have uniform ergodicity of $(x(n))$. For a more general discussion concerning multiplicative Poisson equation see [**KM**]. Since w is bounded $\Lambda(\theta)$ is uniquely defined by (3.5) and $\liminf$ maybe replaced by $\lim$. The function w is defined uniquely up to an additive constant.

THEOREM 3.1. *Assume that $(x(n))$ is uniformly ergodic, the mapping $[0,K) \ni \theta \mapsto g(x,\theta)$ is differentiable for $x \in R^k$ and for $\theta \in [0,K)$ there is a constant $\Lambda(\theta)$ and a continuous bounded function $w(x,\theta)$ which are solutions to the multiplicative Poisson equation. Then the mapping*

$$[0,K) \ni \theta \mapsto \Lambda(\theta) \tag{3.8}$$

is differentiable.

PROOF. We follow first some ergodicity arguments of Theorem 4 of [**S1**]. Let for $x \in R^k$ and $A \in \mathcal{B}(R^k)$

$$T_\theta(x,A) := \frac{1}{e^{w(x,\theta)}} e^{g(x,\theta)-\Lambda(\theta)} E_x\left\{1_A(x(1))e^{w(x(1),\theta)}\right\}. \tag{3.9}$$

By the form of multiplicative Poisson equation T_θ is a transition operator. Notice that Markov process corresponding to T is uniformly, also in $\theta \in [0,K)$, ergodic, i.e. there is $\delta \in (0,1)$ such that for $x, x' \in R^k$, $A \in \mathcal{B}(R^k)$

$$T_\theta(x,A) - T_\theta(x',A) \le \delta. \tag{3.10}$$

In fact, assume that there is x_n, $x_n' \in R^k$ and $A_n \in \mathcal{B}(R^k)$ and $\theta_n \in [0,1)$ such that

$$\lim_{n\to\infty} T_{\theta_n}(x_n,A_n) - T_{\theta_n}(x_n',A_n) = 1.$$

Therefore $\lim_{n\to\infty} T_{\theta_n}(x_n',A_n) = 0$ and $T_{\theta_n}(x_n, E\setminus A_n) = 0$. By the form of the transition operator T_θ we have that $\lim_{n\to\infty} P(x_n',A_n) = 0$ and $\lim_{n\to\infty} P(x_n, E\setminus A_n) = 0$ which contradicts uniform ergodicity of $(x(n))$.

By (3.10) and Section 5 of Chapter 5 of [**D**] there is a probability measure μ and a constant K such that for $n = 1,2,\ldots$, $x \in R^k$ and $f \in b\mathcal{B}(R^k)$ - the space of bounded Borel measurable functions on R^k

$$|T_\theta^n(x,f) - \mu(f)| \le K\delta^n \|f\|. \tag{3.11}$$

with the same $\delta \in (0,1)$ as in (3.10) and $\|f\|$ being the supremum norm of f. Therefore

$$|E_x\left\{e^{\sum_{i=0}^{n-1}(g(x(i),\theta)-\Lambda(\theta))} f(x(n))e^{w(x(n),\theta)}\right\} - e^{w(x,\theta)}\mu(f)| \le K\|e^{w_\theta}\|\delta^n\|f\|, \tag{3.12}$$

where $\|e^{w_\theta}\|$ stands for supremum norm of $e^{w(x,\theta)}$ with respect to x. Consequently

$$|E_x\left\{e^{\sum_{i=0}^{n-1}(g(x(i),\theta)-\Lambda(\theta))} f(x(n))\right\} - e^{w(x,\theta)}\mu(fe^{-w_\theta})| \le K\|e^{w_\theta}\|\delta^n|\|fe^{-w_\theta}\|. \tag{3.13}$$

Let

$$R_N(x,\theta) := \frac{1}{N}\ln E_x\left\{\exp\left\{\sum_{i=0}^{N-1} g(x(i),\theta)\right\}\right\}.$$

Clearly

$$\frac{\partial}{\partial\theta}R_N(x,\theta) = \frac{1}{N}\frac{E_x\left\{\sum_{j=0}^{N-1} g_\theta'(x(j),\theta)\exp\left\{\sum_{i=0}^{N-1} g(x(i),\theta)\right\}\right\}}{E_x\left\{\exp\left\{\sum_{i=0}^{N-1} g(x(i),\theta)\right\}\right\}} \tag{3.14}$$

and using the Poisson equation (3.6) we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial\theta} R_N(x,\theta) \\
&= \frac{1}{N} \frac{\sum_{j=0}^{N-1} E_x\left\{ g'_\theta(x(j),\theta) e^{w_j^N(x(j),\theta)} \exp\left\{ \sum_{i=0}^{j-1} (g(x(i),\theta) - \Lambda(\theta)) \right\}\right\}}{E_x\left\{ \exp\left\{ \sum_{i=0}^{N-1} (g(x(i),\theta) - \Lambda(\theta)) \right\}\right\}}
\end{aligned} \tag{3.15}
$$

with

$$
e^{w_j^N(x,\theta)} := E_x\left\{ \exp\left\{ \sum_{i=0}^{N-j-1} (g(x(j),\theta) - \Lambda(\theta)) \right\}\right\}.
$$

By (3.13) the value of $e^{w_j^N(x,\theta)}$ converges to $e^{w(x,\theta)}\mu(e^{-w_\theta})$, uniformly in $x \in R^k$ and θ, as $N \to \infty$. Therefore

$$
\frac{\partial}{\partial\theta} R_N(\theta) \to \mu(g'_\theta) \tag{3.16}
$$

uniformly in $x \in R^k$ and θ as $N \to \infty$ and

$$
\frac{\partial}{\partial\theta} \Lambda(\theta) = \mu(g'_\theta) \tag{3.17}
$$

which completes the proof. □

Remark 3.2. In the case, when $(x(n))$ is independent of $(W(n))$, by Theorem 3.1 we have differentiability of $\theta \mapsto \Lambda(\theta)$, so that by Theorem 2.7 (2.11) holds for any $c < \Lambda'(K)$.

3.2. General case. In this section we shall consider a general form (1.2) of the factor process $(x(n))$. Maximization of the risk sensitive cost functional $\Lambda^h(\theta)$ leads to the following Bellman equation

$$
e^{w(x,\theta)+\Lambda(\theta)} = \sup_{h\in U} E\left\{ e^{\theta F(x,h,W(0)) + w(G(x,W(0)),\theta)} \right\} \tag{3.18}
$$

For a given Borel measurable $h : R^k \mapsto U$ we have the following Poisson equation

$$
e^{w^h(x,\theta)+\Lambda^h(\theta)} = E\left\{ e^{\theta F(x,h(x),W(0)) + w^h(G(x,W(0)),\theta)} \right\}. \tag{3.19}
$$

Let for $f \in b\mathcal{B}(R^k)$

$$
S_\theta f(x) := \sup_{h\in U} \ln E\left\{ e^{\theta F(x,h,W(0)) + f(G(x,W(0))} \right\} \tag{3.20}
$$

and for a Borel measurable $h : R^k \mapsto U$

$$
S_\theta^h f(x) := \ln E\left\{ e^{\theta F(x,h(x),W(0)) + f(G(x,W(0))} \right\}. \tag{3.21}
$$

Assume

$$
E\left\{ |F(x,h,W(0))| e^{\theta F(x,h,W(0))} \right\} < \infty \tag{3.22}
$$

for $x \in R^k$, $h \in U$ and $\theta \in [0,K)$, and

$$
\sup_{x\in R^k} \sup_{h\in U} \sup_{\theta\in[0,K)} \max\left\{ E\left\{ e^{\theta F(x,h,W(0))} \right\}, E\left\{ e^{-\theta F(x,h,W(0))} \right\} \right\} < \infty. \tag{3.23}
$$

Assume moreover that there is $M > 0$ such that for $\theta > 0$ sufficiently small

$$\sup_{x\in R^k} \sup_{h\in U} \ln E\left\{e^{F(x,h,,W(0)}\right\} \le \theta M \tag{3.24}$$

and for $h \in \mathcal{M}$

$$E\left\{|F(x,h(x),W(0))|e^{\theta F(x,h(x),W(0))}\right\} < \infty. \tag{3.25}$$

THEOREM 3.3. *Assume that the operator S transforms the space $C(R^k)$ into itself, (3.22), (3.23) and (3.24) are satisfied and $(x(n))$ is uniformly ergodic. Then for a sufficiently small θ there is a unique up to an additive constant function $w_\theta \in C(R^k)$ and a constant $\Lambda(\theta)$ which are solutions to the equation (3.18). The constant $\Lambda(\theta)$ is an optimal value of the cost functional $\Lambda^h(\theta)$ defined in (1.11). Furthermore, when $\hat{h} : R^k \mapsto U$ is a Borel measurable function for which the maximum on the right hand side of (3.18) is attained, then the Markov portfolio strategy $\hat{h}(n) = \hat{h}(x(n))$ is optimal. Assume now that the operator S_θ^h transforms the space $b\mathcal{B}(R^k)$ into itself, (3.23), (3.24) and (3.25) are satisfied and $(x(n))$ is uniformly ergodic. Then for a sufficiently small θ there is a unique up to an aditive constant function $w_\theta^h \in b\mathcal{B}(R^k)$ and a constant $\Lambda^h(\theta)$ which are solutions to the equation (3.19). The constant $\Lambda^h(\theta)$ is the value of the cost functional (1.10) corresponding to the Markov portfolio strategy generated by the function h.*

PROOF. We follow the proof of Theorem 1 in [**S2**] where the case $\theta < 0$ was considered. We have the following representations for the operators S_θ and S_θ^h:

$$S_\theta f(x) = \sup_{h\in U} \sup_{\nu\in\mathcal{P}} \left(\int \theta\left\{F(x,h,z) + f(G(x,z))\right\}\nu(dz) - I(\nu,\eta)\right) \tag{3.26}$$

and

$$S_\theta^h f(x) = \sup_{\nu\in\mathcal{P}} \left(\int \theta\left\{F(x,h(x),z) + f(G(x,z))\right\}\nu(dz) - I(\nu,\eta)\right) \tag{3.27}$$

where $\mathcal{P}$ is the set of probability measures on the space where the random variables $(W(n))$ take values and $I(\nu,\eta)$ stands for the relative entropy of ν with respect to η. Furthermore, the suprema in (3.26) and (3.27) are attained by the measures

$$\nu_{x,h,f}(dz) := \frac{e^{\theta F(x,h,z)+f(G(x,z))}\eta(dz)}{E_x\left\{e^{\theta F(x,h,W(0))+f(G(x,W(0)))}\right\}},$$

and

$$\nu_{x,f}^h(dz) := \frac{e^{\theta F(x,h(x),z)+f(G(x,z))}\eta(dz)}{E_x\left\{e^{\theta F(x,h(x),W(0))+f(G(x,W(0)))}\right\}},$$

respectively.

Let for $h \in U$, $A \in \mathcal{B}(R^k)$

$$\bar{\nu}_{x,h,f}(A) := \frac{E_x\left\{1_A(G(x,W(0))e^{\theta F(x,h,W(0))+f(G(x,W(0)))}\right\}}{E_x\left\{e^{\theta F(x,h,W(0))+f(G(x,W(0)))}\right\}} \tag{3.28}$$

and for $h \in \mathcal{M}$

$$\bar{\nu}_{x,f}^h(A) := \frac{E_x\left\{1_A(G(x,W(0))e^{\theta F(x,h(x),W(0))+f(G(x,W(0)))}\right\}}{E_x\left\{e^{\theta F(x,h(x),W(0))+f(G(x,W(0)))}\right\}}. \tag{3.29}$$

Denote by $\|f\|_{sp} := \sup_{x\in R^k} f(x) - \inf_{x'\in R^k} f(x')$ be so called span norm of $f \in b\mathcal{B}(R^k)$ and by $\|\nu\|_{var}$ the variation norm of $\nu \in \mathcal{P}(R^k)$. We have (compare to Lemma 1 of [**S2**])

LEMMA 3.4. *For a given $M > 0$ there is an $0 < L(M) < 1$ such that for $f_1, f_2 \in b\mathcal{B}(R^k)$, $x_1, x_2 \in R^k$, $h_1, h_2 \in U$ if $\|f_1\|_{sp} \le M$ and $\|f_2\|_{sp} \le M$ we have*

$$\|\bar{\nu}_{x_1,h_1,f_1} - \bar{\nu}_{x_2,h_2,f_2}\|_{var} \le 2L(M) \tag{3.30}$$

and

$$\|\bar{\nu}^h_{x_1,f_1} - \bar{\nu}^h_{x_2,f_2}\|_{var} \le 2L(M) \tag{3.31}$$

PROOF. Notice that by Schwarz inequality we have

$$\begin{aligned}\bar{\nu}_{x,h,f}(A) &\ge e^{-\|f\|_{sp}} \frac{E_x\left\{1_A(G(x,W(0))e^{\theta F(x,h,W(0))}\right\}}{E_x\left\{e^{\theta F(x,h,W(0))}\right\}}\\ &\ge e^{-\|f\|_{sp}} \frac{E_x\left\{1_A(G(x,W(0))\right\}}{E_x\left\{e^{\theta F(x,h,W(0))}\right\} E_x\left\{e^{-\theta F(x,h,W(0))}\right\}},\end{aligned}$$

so that whenever $\bar{\nu}_{x^1_n,h^1_n,f^1_n}(A_n) \to 1$ and $\bar{\nu}_{x^2_n,h^2_n,f^2_n}(A_n) \to 0$ with $\|f^1_n\|$, $\|f^2_n\| \le M$ by (3.23) we immediately have that

$$E_{x^1_n}\left\{1_{R^k\setminus A_n}(G(x^1_n,W(0))\right\} \to 0$$

and

$$E_{x^2_n}\left\{1_{A_n}(G(x^2_n,W(0))\right\} \to 0$$

as $n \to \infty$, which contradicts the uniform ergodicity of $(x(n))$. □

Notice (for details see Theorem 1 of [**S2**]) that for $x_1, x_2 \in R^k$, $h_1, h_2 \in U$ we have

$$\begin{aligned}&S_\theta f_1(x_2) - S_\theta f_2(x_2) - (S_\theta f_1(x_1) - S_\theta f_2(x_1))\\ &\le \frac{1}{2}\|f_1 - f_2\|_{sp}\|\bar{\nu}_{x_2,h_2,f_1} - \bar{\nu}_{x_1,h_1,f_2}\|_{var}\end{aligned}$$

and for $h \in \mathcal{M}$

$$S^h_\theta f_1(x_2) - S^h_\theta f_2(x_2) - (S^h_\theta f_1(x_1) - S^h_\theta f_2(x_1)) \le \frac{1}{2}\|f_1 - f_2\|_{sp}\|\bar{\nu}^h_{x_2,f_1} - \bar{\nu}^h_{x_1,f_2}\|_{var}.$$

By Lemma 3.1 for $\|f_1\|_{sp} \le M$ and $\|f_2\|_{sp} \le M$ we obtain

$$\|S_\theta f_1 - S_\theta f_2\|_{sp} \le L(M)\|f_1 - f_2\|_{sp} \tag{3.32}$$

and

$$\|S^h_\theta f_1 - S^h_\theta f_2\|_{sp} \le L(M)\|f_1 - f_2\|_{sp}. \tag{3.33}$$

Consequently there exist at most unique solutions to the equations (3.18) and (3.19). By (3.24) for θ sufficiently small a less than $1 - L(M)$ we have

$$\|S_\theta 0\|_{sp} \le \theta M < M$$

and

$$\|S^n_\theta 0\|_{sp} \le \theta M(1 + L(M) + L(M)^2 + \ldots + L(M)^{n-1}) \le M$$

whenever $\theta < 1 - L(M)$ and

$$\|S^{n+1}_\theta 0 - S^n_\theta 0\|_{sp} \le L(M)^n\|S_\theta 0\|_{sp}.$$

Therefore for a sufficiently small θ the iterations $S^n_\theta 0$ converge to the unique (up to an additive constant) solution w_θ of (3.18) and $\Lambda(\theta)$ is defined in a unique way.

Optimality of the strategy $(\hat{h}(n))$ follows directly from Proposition 1 in [**D1**]. The existence of unique solution to (3.19) for $h \in \mathcal{M}$ can be shown as above by showing the convergence of the iterations $S_\theta^{hn}0$. □

Remark 3.5. The assumptions (3.22)-(3.24) are satisfied in the case of example 1.1 when a_i and σ_{ij} are bounded functions. Under continuity and boundedness of the functions a_i, σ_{ij} and b_r with $i = 1, \dots, m$, $j = 1, \dots, k+m$, $r = 1, \dots, k$, we have that the operator S transforms the space $C(R^k)$ into itself. If the rank of the matrix (λ_{ij}) is k by the proof of Lemma 1 of [**S2**] we see that $(x(n))$ is uniformly ergodic. The proof of Theorem 3.3 consists of two steps: first we show local contractivity in the span norm of the operator S_θ or S_θ^h and then we use small θ argument (introduced in fact first in [**D2**]) to show that the operators S_θ and S_θ^h do not increase the span norm (if it is sufficiently large). One can however try to exploit ideas of the paper [**D1**] and show that if the measures

$$\hat{\nu}_{x,h}(A) := \frac{\int e^{\theta F(x,h,z)} 1_A(G(x,z))\nu(dz)}{\int e^{\theta F(x,h,z)}\nu(dz)}$$

for $A \in \mathcal{B}(R^k)$ are equivalent for $x \in R^k$ and $h \in U$ with densities bounded from above and bounded away from 0 (uniformly in $x \in R^k$, $h \in U$ and θ) then, as in the proof of Theorem 1 in [**D1**] we have the convergence (in span norm) of $S_\theta^n 0$ and $S_\theta^{hn}0$ the solutions to (3.18) and (3.19) respectively.

In the remaining part of this section for fixed $h \in \mathcal{M}$ we study differentiability of $\Lambda^h(\theta)$. Let

$$H^h(x,\theta) := \int F(x,h(x),y) e^{\theta F(x,h(x),y)} e^{w^h(G(x,y))} \eta(dy) \tag{3.34}$$

We have

Theorem 3.6. *Assume that for $h \in \mathcal{M}$ we have unique (up to an additive constant) bounded solution w^h and a constant Λ^h to (3.19), the function H^h is bounded and continuous in θ and (3.22), (3.23) are satisfied. Then the mapping $[0,K) \ni \theta \mapsto \Lambda^h(\theta)$ is differentiable.*

Proof. By analogy to the proof of Theorem 3.1 consider the following transition operator

(3.35)
$$T_\theta^h(x,A) := e^{-w^h(x,\theta)} E\left\{ e^{\theta F(x,h(x),W(0)) - \Lambda^h(\theta) + w^h(G(x,W(0)),\theta)} 1_A(G(x,W(0))\right\}.$$

As in the proof of Theorem 3.1 using (3.23) we show that Markov process with transition operator T_θ^h is uniformly, also in $\theta \in [0,K)$ ergodic i.e. there is $\delta \in (0,1)$ such that for $x, x' \in R^k$, $A \in \mathcal{B}(R^k)$ and $\theta \in [0,K)$

$$T_\theta^h(x,A) - T_\theta^h(x',A) \le \delta \tag{3.36}$$

Consequently there are $K > 0$ and a probability measure μ_h such that for $f \in b\mathcal{B}(R^k)$

$$|(T_\theta^h)^n(x,f) - \mu_h(f)| \le K\delta^n \|f\|$$

and

$$
\begin{aligned}
&|E_x\left\{e^{\sum_{i=0}^{n-1}(\theta F(x(i),h(x(i),W(n)))-\Lambda^h(\theta))}f(x(n))\right\} - e^{w^h(x,\theta)}\mu_h(fe^{-w^h_\theta})| \\
&\le K\|e^{w^h_\theta}\|\delta^n\|fe^{-w^h_\theta}\|.
\end{aligned} \tag{3.37}
$$

Let now

$$R_N^h(x,\theta) := \frac{1}{N}\ln E_x\left\{\exp\left\{\sum_{i=0}^{N-1}(\theta F(x(i),h(x(i))) - \Lambda^h(\theta))\right\}\right\}$$

Clearly

$$
\begin{aligned}
&\frac{\partial}{\partial\theta}R_N^h(x,\theta) \\
&= \frac{1}{N}\frac{E_x\left\{\sum_{j=0}^{N-1}F(x(j),h(x(j)),W(n))\exp\left\{\sum_{i=0}^{N-1}\theta F(x(i),h(x(i)),W(n))\right\}\right\}}{E_x\left\{\exp\left\{\sum_{i=0}^{N-1}\theta F(x(i),h(x(i)),W(n))\right\}\right\}}.
\end{aligned} \tag{3.38}
$$

and by equation (3.19) we obtain

$$
\begin{aligned}
\frac{\partial}{\partial\theta}R_N^h(x,\theta) = \frac{1}{N}\sum_{j=0}^{N-1}E_x&\left\{F(x(j),h(x(j)),W(n))e^{w^{hN}_{j+1}(x(j+1),\theta)}\right. \\
&\left.\frac{\exp\left\{\sum_{i=0}^{j}(\theta F(x(i),h(x(i)),W(n)) - \Lambda^h(\theta))\right\}}{E_x\left\{\exp\left\{\sum_{i=0}^{N-1}(\theta F(x(i),h(x(i)),W(n)) - \Lambda^h(\theta))\right\}\right\}}\right\}.
\end{aligned} \tag{3.39}
$$

with

$$e^{w^{hN}_j(x,\theta)} := E_x\left\{\exp\left\{\sum_{i=0}^{N-j-1}(\theta F(x(j),h(x(j)) - \Lambda^h(\theta))\right\}\right\}.$$

Since by (3.37) the value of $e^{w^{hN}_j(x,\theta)} = (S^h_\theta)^{N-j-1}0$ converges uniformly in $x \in R^k$ and θ to $e^{w^h(x,\theta)}\mu(e^{-w^h_\theta})$, letting $N \to \infty$ the value of $R_N^h(x,\theta)$ converges to the same value as

$$
\begin{aligned}
\frac{1}{N}\sum_{j=0}^{N-1}E_x&\left\{F(x(j),h(x(j)),W(n))e^{w^h(x(j+1),\theta)}\right. \\
&\left.\frac{\exp\left\{\sum_{i=0}^{j}(\theta F(x(i),h(x(i)),W(n)) - \Lambda^h(\theta))\right\}}{E_x\left\{\exp\left\{\sum_{i=0}^{N-1}(\theta F(x(i),h(x(i)),W(n)) - \Lambda^h(\theta))\right\}\right\}}\right\}
\end{aligned} \tag{3.40}
$$

and using the boundedness of $H^h(x,\theta)$ by (3.37) we obtain that $\frac{\partial}{\partial\theta}R_N^h(x,\theta)$ converges uniformly to $\mu_h(H^h_\theta e^{-w^h_\theta})$, which completes the proof. □

Remark 3.7. Notice that all assumptions of Theorem 3.6 are satisfied in the case of example 1.1 provided that the functions a_i, σ_{ij} for $i = 1\dots,m$, $j = 1,\dots,k+m$ and b_r, for $r = 1,\dots,k$ are bounded. Consequently by Theorem 2.7, (2.13) holds for $c < \Lambda^{h\prime}(K)$.

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