Infinite horizon risk sensitive control of discrete time Markov processes under minorization property

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Abstract

Risk sensitive control of Markov processes satisfying minorization property is studied using splitting techniques. Existence of solutions to multiplicative Poisson equation is shown. Approximation by uniformly ergodic controlled Markov processes is introduced, which allows to show the existence of solutions to the infinite horizon risk sensitive Bellman equation.

Key words: risk sensitive control, discrete time Markov processes, splitting, Poisson equation, Bellman equation
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1 Introduction

On a probability space $(\Omega, \mathcal{F}, P)$ consider a controlled Markov process $X = (x_n)$ taking values on a complete separable metric state space E endowed with the Borel σ-algebra $\mathcal{E}$. Assume that x_n has a controlled transition operator $P^{a_n}(x_n, \cdot)$, where a_n is the control at time n taking values on a compact metric space U and adapted to the σ-algebra $\sigma\{x_0, x_1, \ldots, x_n\}$.

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Let $c : E \times U \to R$ be a continuous and bounded function. Our aim is to minimize the following exponential ergodic performance criterion

$$J_x^\gamma((a_n)) = \lim_{t\to\infty} \sup \frac{1}{t} \ln E_x^{(a_n)} \left\{ \exp \left\{ \sum_{i=0}^{t-1} \gamma c(x_i, a_i) \right\} \right\} \tag{1}$$

with risk factor $\gamma > 0$. In what follows we shall distinguish the following special classes of admissible controls (a_n): *Markov controls* $\mathcal{U}_M = \{(a_n) : a_n = u_n(x_n)\}$, where $u_n : E \mapsto U$ is a sequence of Borel measurable functions, and *stationary controls* $\mathcal{U}_s = \{(a_n) : a_n = u(x_n)\}$, where $u : E \mapsto U$ is a Borel measurable function.

Consider the following assumptions

(A1) $\exists_{\beta>0}\ \exists_{C compact \in \mathcal{E}}\ \exists_{\nu\in\mathcal{P}(E)}$ *with* $\nu(C) = 1$ such that $\forall_{A\in\mathcal{E}}$

$$\inf_{x\in C} \inf_{a\in U} P^a(x, A) \geq \beta\nu(A)$$

(A2) C given in (A1) is ergodic, i.e. $\forall_{(a_n)\in\mathcal{U}_M}\ \forall_{x\in E}\ E_x^{(a_n)}\{\tau_C\} < \infty$, where $\tau_C = \inf\{i > 0 : x_i \in C\}$.

In the paper risk sensitive control problem with cost functional (1) and general state space is studied. The paper generalizes [7] and [8] where uniform ergodicity assumption was required. Instead of uniform ergodicity we require minorization property (A1) which allows us to use splitting techniques arguments. Risk sensitive discrete time control problems has been studied in a number of papers [1], [3], [4], [5], [6], [9], [10] for finite or countable state spaces. General state space model in discrete time was considered in [7] and [8] only. Financial applications of risk sensitive control problems were introduced in [2] and continued in a number of papers e.g. see [12] and [14] and references therein. The first part of the paper is devoted to the study of so called Poisson equation. Although such equation was considered in [11] and [3], a rather simple characterization of the solution seems to be new. The main result of the paper is the existence of solutions to the Bellman equation corresponding to the risk sensitive control problem with a general state space (under minorization property).

2 Splitting of Markov processes

Let $\hat{E} = \{C \times \{0\} \cup C \times \{1\} \cup E \setminus C \times \{0\}\}$ and $\hat{x}_n = (x_n^1, x_n^2) \in \hat{E}$. Given a Markov control $a_n = u_n(x_n^1)$, where $u_n : E \mapsto U$ is a sequence of Borel measurable functions, consider the following Markov process defined on $\hat{E}$

(i) when $(x_n^1, x_n^2) \in C\times\{0\}$, x_n^1 moves to y accordingly to $(1-\beta)^{-1}(P^{a_n}(x_n^1, dy)-\beta\nu(dy))$ and whenever $y \in C$, x_n^2 is changed into $x_{n+1}^2 = \beta_{n+1}$, where β_n is i.i.d. $P\{\beta_n = 0\} = 1-\beta$, $P\{\beta_n = 1\} = \beta$,

(ii) when $(x_n^1, x_n^2) \in C \times \{1\}$, x_n^1 moves to y accordingly to ν and $x_{n+1}^2 = \beta_{n+1}$,

(iii) when $(x_n^1, x_n^2) \in E\setminus C\times\{0\}$, x_n^1 moves to y accordingly to $P^{a_n}(x_n^1, dy)$ and whenever $y \in C$, x_n^2 is changed into $x_{n+1}^2 = \beta_{n+1}$.

In what follows we shall write that the control (a_n) of $(\hat{x}_n)$ is in the class $\mathcal{U}_M$, whenever there is a sequence of Borel measurable functions $u_n : E \mapsto U$ such that $a_n = u_n(x_n^1)$. Let $C_0 = C \times \{0\}$, $C_1 = C \times \{1\}$. By direct calculation we obtain

Lemma 1 *For $n = 1, 2\ldots$ we have P a.e.*

$$P\{\hat{x}_n \in C_0|\hat{x}_n \in C_0 \cup C_1, \hat{x}_{n-1}, \ldots, \hat{x}_0\} = 1-\beta$$

$$P\{\hat{x}_n \in C_1|\hat{x}_n \in C_0 \cup C_1, \hat{x}_{n-1}, \ldots, \hat{x}_0\} = \beta.$$

Furthermore we have

Lemma 2 *Under Markov control $(a_n) \in \mathcal{U}_M$ the process $(\hat{x}_n = (x_n^1, x_n^2))$ is Markov with transition operator $\hat{P}^{a_n}(\hat{x}_n, dy)$ defined by (i)-(iii). Furthermore the first coordinate (x_n^1) is also a Markov process with transition operator $P^{a_n}(x_n^1, dy)$.*

Proof. The first statement follows from the construction (i)-(iii) of the split Markov process $(\hat{x}_n)$. For the second notice that for $A \in \mathcal{E}$

$$\begin{aligned} & P\left\{x_{n+1}^1 \in A|x_n^1, x_{n-1}^1, \ldots, x_0^1\right\} = \\ & P\left\{x_{n+1}^1 \in A|x_n^1, x_n^2 = 0, x_{n-1}^1, \ldots, x_0^1\right\} P\left\{x_n^2 = 0|x_n^1, x_{n-1}^1, \ldots, x_0^1\right\} \\ & +P\left\{x_{n+1}^1 \in A|x_n^1, x_n^2 = 1, x_{n-1}^1, \ldots, x_0^1\right\} P\left\{x_n^2 = 1|x_n^1, x_{n-1}^1, \ldots, x_0^1\right\} \end{aligned} \tag{2}$$

In the case when $x_n^1 \in C$, (2) is equal to

$$\frac{P^{a_n}(x_n^1, A) - \beta\nu(A)}{1-\beta}(1-\beta) + \beta\nu(A) = P^{a_n}(x_n^1, A).$$

For $x_n^1 \notin C$, (2) is equal to $P^{a_n}(x_n^1, A)$, which completes the proof of Markov property of (x_n^1).

□

Corollary 1 *For any bounded Borel measurable function $f : E^m \mapsto R$, $m = 1, 2, \ldots,$ and control $(a_n) \in \mathcal{U}_M$ we have*

$$E_x^{(a_n)}\left\{f(x_1, x_2, \ldots, x_m)\right\} = \hat{E}_{\delta_x^*}^{(a_n)}\left\{f(x_1^1, x_2^1, \ldots, x_m^1)\right\} \tag{3}$$

where $\delta_x^* = \delta_{(x,0)}$ for $x \in E \setminus C$ and $\delta_x^* = (1-\beta)\delta_{(x,0)} + \beta\delta_{(x,1)}$ for $x \in C$ and $\hat{E}_\mu$ stands for conditional law of Markov process $(\hat{x}_n)$ with initial law $\mu \in \mathcal{P}(\hat{E})$.

Proof. It follows from (2) that for a bounded Borel measurable $g : E \mapsto R$

$$\begin{aligned} &\hat{E}\left\{g(x_{i+1}^1)|x_i^1, x_{i-1}^1, \ldots, x_0^1\right\} = \hat{E}\left\{\hat{E}\left\{g(x_{i+1}^1)|\hat{x}_i, \hat{x}_{i-1}, \ldots, \hat{x}_0\right\}|x_i^1, x_{i-1}^1, \ldots, x_0^1\right\} \\ &= \hat{E}\left\{\hat{E}_{\hat{x}_i}\left\{g(x_1^1)\right\}|x_i^1, x_{i-1}^1, \ldots, x_0^1\right\} = \hat{E}_{\delta_{x_i^1}^*}\left\{g(x_1^1)\right\}. \end{aligned} \tag{4}$$

On the other hand by Markovianity of (x_n^1) we have

$$\hat{E}\left\{g(x_{i+1}^1)|x_i^1, x_{i-1}^1, \ldots, x_0^1\right\} = \int_E g(y)P^{a_i}(x_i^1, dy). \tag{5}$$

Consequently applying (4) and (5) to function $f : E^m \mapsto R$ we obtain (3).

□

3 Multiplicative Poisson equation (MPE)

In this section we shall assume that

(A3) $\forall_{(a_n)\in\mathcal{U}_s}$ $\exists_d$ such that $\forall_{x\in\hat{E}}$

$$\hat{E}_x^{(a_n)}\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}}\left(\gamma c(x_i^1, a_i) - d\right)\right\}\right\} < \infty$$

and for $x \in C_1$

$$\hat{E}_x^{(a_n)}\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}}\left(\gamma c(x_i^1, a_i) - d\right)\right\}\right\} \geq 1$$

Lemma 3 *Under (A3) for $(a_n) \in \mathcal{U}_s$ there is a unique $\lambda^\gamma((a_n))$ such that for*

$$\hat{E}_x^{(a_n)}\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}}\left(\gamma c(x_i^1, a_i) - \lambda^\gamma((a_n))\right)\right\}\right\} = 1 \tag{6}$$

for $x \in C_1$.

Proof. Notice that for $x \in C_1$ the mapping

$$D : \lambda \mapsto \hat{E}_x^{(a_n)} \left\{ \exp \left\{ \sum_{i=1}^{\tau_{C_1}} \gamma c(x_i^1, a_i) - \lambda \right\} \right\}$$

is strictly decreasing whenever it is finite valued. Moreover by (A3) we have that $\infty > D(d) \geq 1$. Since $\lim_{d \to \infty} D(d) = 0$ by continuity of D there is a unique $\lambda^\gamma((a_n))$ for which (6) holds.

□

Remark 1 *Notice that letting $d = \inf_{x \in E, a \in U} \gamma c(x, a)$ we have a sufficient condition for (A3) in the form:*
(D1) $\quad \hat{E}_x^{(a_n)} \{\exp\{\gamma \|c\|_{sp} \tau_{C_1}\}\} < \infty \quad$ *for $x \in \hat{E}$, where $\|c\|_{sp} := \sup_{x \in E, a \in U} c(x, a) - \inf_{x \in E, a \in U} c(x, a)$. In section 6 we shall formulate a sufficient condition for (D1) in terms of the expected value of the functional with respect to the original Markov process (x_n).*

For Borel measurable $u : E \mapsto U$ let

$$e^{\hat{w}^u(x)} = \hat{E}_x^u \left\{ \exp \left\{ \sum_{i=0}^{\tau_{C_1}} \left(\gamma c(x_i^1, u(x_i^1)) - \lambda^\gamma(u) \right) \right\} \right\}, \tag{7}$$

where by $\lambda^\gamma(u)$ we denote the value of $\lambda^\gamma((u(x_n^1))$ and expected value $\hat{E}_x^u$ stands for $\hat{E}_x^{(u(x_n^1))}$.
By (A3) clearly $\lambda^\gamma(u) \geq d$ and therefore $\hat{w}^u$ is well defined. Furthermore we have

Lemma 4 *Function $\hat{w}^u$ defined in (7) is a unique up to an additive constant solution to the multiplicative Poisson equation (MPE) for the split Markov process $(\hat{x}_n)$:*

$$e^{\hat{w}^u(x)} = e^{\gamma c(x^1, u(x^1)) - \lambda^\gamma(u)} \int_{\hat{E}} e^{\hat{w}^u(y)} \hat{P}^{u(x^1)}(x, dy) \tag{8}$$

Furthermore, if $\hat{w}$ and λ satisfy the equation

$$e^{\hat{w}(x)} = e^{\gamma c(x^1, u(x^1)) - \lambda} \int_{\hat{E}} e^{\hat{w}(y)} \hat{P}^{u(x^1)}(x, dy) \tag{9}$$

then $\lambda = \lambda^\gamma(u)$ defined in Lemma 4 and $\hat{w}$ differs from $\hat{w}^u$ by an additive constant.

Proof. In fact, we have using (6)

$$\hat{E}_x^u \{\exp\{w(\hat{x}_1)\}\} = \hat{E}_x^u \left\{\chi_{\hat{x}_1\in C_1} \hat{E}_{x_1}^u \left\{\exp\left\{\sum_{i=0}^{\tau_{C_1}} \gamma c(x_i^1, u(x_i^1)) - \lambda^\gamma(u)\right\}\right\}\right\}$$

$$+\hat{E}_x^u \left\{\chi_{\hat{x}_1\notin C_1} \hat{E}_{x_1}^u \left\{\exp\left\{\sum_{i=0}^{\tau_{C_1}} \gamma c(x_i^1, u(x_i^1)) - \lambda^\gamma(u)\right\}\right\}\right\} = \hat{E}_x^u \{\chi_{\hat{x}_1\in C_1}$$

$$\exp\left\{\gamma c(x_1^1, u(x_1^1)) - \lambda^\gamma(u)\right\}\} + \hat{E}_x^u \left\{\chi_{\hat{x}_1\notin C_1} \exp\left\{\sum_{i=1}^{\tau_{C_1}} \gamma c(x_i^1, u(x_i^1)) - \lambda^\gamma(u)\right\}\right\}$$

$$= \hat{E}_x^u \left\{\exp\left\{\sum_{i=0}^{\tau_{C_1}} \gamma c(x_i^1, u(x_i^1)) - \lambda^\gamma(u)\right\}\right\} \exp\left\{-(\gamma c(x^1, u(x^1)) - \lambda^\gamma(u))\right\}$$

from which (8) follows. If $\hat{w}^u$ is a solution to the equation (8) then by iteration

$$e^{\hat{w}^u(x)} = \hat{E}_x^u \left\{\exp\left\{\sum_{i=0}^{\tau_{C_1}} \left(\gamma c(x_i^1, u(x_i^1)) - \lambda^\gamma(u)\right)\right\} \hat{E}_{\hat{x}_{\tau_{C_1}}}^u \{\exp\{\hat{w}^u(\hat{x}(1))\}\}\right\}$$

and since by the construction of the split Markov process $\hat{E}_{\hat{x}_{\tau_{C_1}}}^u \{\exp\{\hat{w}^u(\hat{x}(1))\}\}$ is a positive constant $\hat{w}^u$ differs from $\hat{w}^u$ defined in (7) by an additive constant. Similarly if $\hat{w}$ and λ are solutions to (9) then $\hat{w}$ differs from

$$e^{\tilde{w}(x)} = \hat{E}_x^u \left\{\exp\left\{\sum_{i=0}^{\tau_{C_1}} \left(\gamma c(x_i^1, u(x_i^1)) - \lambda\right)\right\}\right\}$$

by an additive constant $\ln \hat{E}_z^u \{\exp\{\hat{w}(\hat{x}_1)\}\}$ with $z \in C_1$. Since $\tilde{w}$ is also a solution to (9) we have that $\hat{E}_z^u \{\exp\{\tilde{w}(\hat{x}_1)\}\} = 1$. Therefore

$$1 = \hat{E}_z^u \left\{\chi_{C_1}(\hat{x}_1) \exp\left\{\gamma c(x_1^1, u(x_1^1)) - \lambda\right\} \hat{E}_{\hat{x}_1}^u \left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}} \left(\gamma c(x_i^1, u(x_i^1)) - \lambda\right)\right\}\right\} +\right.$$

$$\left.(1 - \chi_{C_1}(\hat{x}_1)) \exp\left\{\sum_{i=1}^{\tau_{C_1}} \left(\gamma c(x_i^1, u(x_i^1)) - \lambda\right)\right\}\right\} =$$

$$\hat{E}_z^u \left\{\chi_{C_1}(\hat{x}_1) \exp\left\{\gamma c(x_1^1, u(x_1^1)) - \lambda\right\}\right\} \eta + \eta - \hat{E}_z^u \left\{\chi_{C_1}(\hat{x}_1) \exp\left\{\gamma c(x_1^1, u(x_1^1)) - \lambda\right\}\right\}$$

where $\eta = \hat{E}_z^u \left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}} (\gamma c(x_i^1, u(x_i^1)) - \lambda)\right\}\right\}$ for $z \in C_1$. Hence

$$\left(\hat{E}_z^u \left\{\chi_{C_1}(\hat{x}_1) \exp\left\{\gamma c(x_1^1, u(x_1^1)) - \lambda\right\}\right\} + 1\right) \eta = \hat{E}_z^u \left\{\chi_{C_1}(\hat{x}_1) \exp\left\{\gamma c(x_1^1, u(x_1^1)) - \lambda\right\}\right\} + 1$$

and $\eta = 1$, which by Lemma 3 implies that $\lambda = \lambda^{\gamma}(u)$.

□

Corollary 2 *For $x \in E$ and solution to MPE (6) $\hat{w}^u : \hat{E} \mapsto R$ we have that w^u defined by*

$$e^{w^u(x)} := e^{\hat{w}^u(x,0)} + 1_C(x)\beta(e^{\hat{w}^u(x,1)} - e^{\hat{w}^u(x,0)}) \tag{10}$$

is a solution to MPE for the original Markov process $(x(n))$

$$e^{w^u(x)} = e^{\gamma c(x,u(x)) - \lambda^{\gamma}(u)} \int_E e^{w^u(y)} P^{u(x)}(x, dy) \tag{11}$$

Furthermore if w^u is a solution to (11) then $\hat{w}^u$ defined by

$$e^{\hat{w}^u(x^1,x^2)} = e^{\gamma c(x^1,u(x^1)) - \lambda^{\gamma}(u)} \hat{E}_{x^1,x^2} \left\{ e^{w^u(x_1^1)} \right\} \tag{12}$$

is a solution to (8).

Proof. By Lemma 1 we have

$$\begin{aligned} &\hat{E}_x^u \left\{ e^{\hat{w}^u(\hat{x}_1)} \right\} = \hat{E}_x^u \left\{ \hat{E}_x^u \left\{ e^{\hat{w}^u(\hat{x}_1)} | x_1^1 \right\} \right\} \\ &= \hat{E}_x^u \left\{ \chi_C(x_1^1)((1-\beta) e^{\hat{w}^u(x_1^1,0)} + \beta e^{\hat{w}^u(x_1^1,1)}) \right. \\ &\left. + \chi_{E \setminus C}(x_1^1) e^{\hat{w}^u(x_1^1,0)} \right\} = \hat{E}_x^u \left\{ e^{w^u(x_1^1)} \right\} \end{aligned} \tag{13}$$

Therefore by (8) we obtain that w^u defined in (10) is a solution to (11). Assume now that w^u is a solution to (11). Then

$$\hat{E}_{\delta_x^*}^u \left\{ e^{w^u(x_1^1)} \right\} = E_x^u \left\{ e^{w^u(x_1^1)} \right\}$$

and for $\hat{w}^u$ given in (12) we obtain (10). From (10) we obtain (13) which in turn by (12) shows that $\hat{w}^u$ is a solution to (8).

□

Proposition 1 *If for Borel measurable $u : E \mapsto U$*

(B1) $\exists_{d(u)}$ such that for $x \in \hat{E}$, $N = 1, 2, \ldots$

$$\hat{E}_x^u \left\{ \exp \left\{ \sum_{i=1}^{\tau_{C_1} \wedge N} \left(\gamma c(x_i^1, u(x_i^1)) - d(u) \right) \right\} \right\} < \infty \tag{14}$$

and for $z \in C_1$

$$\hat{E}_z^u \left\{ \exp \left\{ \sum_{i=1}^{\tau_{C_1}} \left(\gamma c(x_i^1, u(x_i^1)) - d(u) \right) \right\} \right\} > 1 \tag{15}$$

(B2) for $x \in \hat{E}$

$$\inf_N \hat{E}_x^u \left\{ \exp\left\{ \sum_{i=1}^{\sigma_{C_1} \wedge N-1} \Big(\gamma c(x_i^1, u(x_i^1)) - \lambda^\gamma(u) \Big) \right\} \right\} > 0 \tag{16}$$

(B3) for $x \in \hat{E}$

$$\sup_N \hat{E}_x^u \left\{ \exp\left\{ \sum_{i=1}^{\sigma_{C_1} \wedge N-1} \Big(\gamma c(x_i^1, u(x_i^1)) - \lambda^\gamma(u) \Big) \right\} \right\} < \infty \tag{17}$$

with $\sigma_{C_1} = \inf\{i \geq 0 : \hat{x}(i) \in C_1\}$

then for $x \in E$

$$\lambda^\gamma(u) = \lim_{n\to\infty} \frac{1}{n} \ln E_x^u \left\{ \exp\left\{ \sum_{i=0}^{n-1} \gamma c(x_i, u(x_i))) \right\} \right\} \tag{18}$$

Proof. Let $\lambda > \lambda^\gamma(u)$. For $z \in C_1$ we have

$$\hat{E}_z^u \left\{ \exp\left\{ \sum_{i=1}^{\tau_{C_1}} \Big(\gamma c(x_i^1, u(x_i^1)) - \lambda \Big) \right\} \right\} < 1$$

and consequently for $N \geq N_0$

$$\hat{E}_z^u \left\{ \exp\left\{ \sum_{i=1}^{\tau_{C_1} \wedge N} \Big(\gamma c(x_i^1, u(x_i^1)) - \lambda \Big) \right\} \right\} \leq 1. \tag{19}$$

Let

$$e^{w_N^u(x)} = \hat{E}_x^u \left\{ \exp\left\{ \sum_{i=0}^{\sigma_{C_1} \wedge N-1} \Big(\gamma c(x_i^1, u(x_i^1)) - \lambda \Big) \right\} \right\}. \tag{20}$$

By (B3) w_N^u is well defined. For $x \notin C_1$

$$e^{w_{N+1}^u(x)} = \hat{E}_x^u \left\{ e^{\gamma c(x_0^1, u(x_0^1)) - \lambda} \hat{E}_{\hat{x}_1}^u \left\{ \exp\left\{ \sum_{i=0}^{\sigma_{C_1} \wedge N-1} \Big(\gamma c(x_i^1, u(x_i^1)) - \lambda \Big) \right\} \right\} \right\}$$
$$= \hat{E}_x^u \left\{ e^{\gamma c(x_0^1, u(x_0^1)) - \lambda} e^{w_N^u(\hat{x}_1)} \right\} \tag{21}$$

and $x \in C_1$ by (19) we have

$$
\begin{aligned}
&e^{w^u_{N+1}(x)} = e^{\gamma c(x^1_0,u(x^1_0))-\lambda} \geq \\
&\geq \hat{E}^u_x \left\{ e^{\gamma c(x^1_0,u(x^1_0))-\lambda} \hat{E}^u_{\hat{x}_1} \left\{ \exp\left\{ \sum_{i=0}^{\sigma_{C_1}\wedge N-1} \Big(\gamma c(x^1_i, u(x^1_i)) - \lambda\Big) \right\} \right\} \right\} \\
&= \hat{E}^u_x \left\{ e^{\gamma c(x^1_0,u(x^1_0))-\lambda} e^{w^u_N(\hat{x}_1)} \right\} \qquad (22)
\end{aligned}
$$

Consequently

$$
e^{w^u_{N+1}(x)} \geq \hat{E}^u_x \left\{ e^{\gamma c(x^1_0,u(x^1_0))-\lambda} e^{w^u_N(\hat{x}_1)} \right\}
$$

and by iteration for $N \geq N_0$

$$
\begin{aligned}
e^{w^u_{N+k}(x)} &\geq \hat{E}^u_x \left\{ e^{\sum_{i=0}^{k-1}(\gamma c(x^1_i,u(x^1_i))-\lambda)} e^{w^u_N(\hat{x}_k)} \right\} \\
&\geq \hat{E}^u_x \left\{ e^{\sum_{i=0}^{k-1}(\gamma c(x^1_i,u(x^1_i))-\lambda)} e^{-\gamma\|c\|N} \right\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{1}{k} \ln \hat{E}^u_x \left\{ e^{\gamma \sum_{i=0}^{k-1} c(x^1_i,u(x^1_i))} \right\} \leq \frac{1}{k}\gamma\|c\|N \\
&+\frac{1}{k}\sup_N \ln \hat{E}^u_x \left\{ \exp\left\{ \sum_{i=1}^{\sigma_{C_1}\wedge N-1} \Big(\gamma c(x^1_i, u(x^1_i)) - \lambda^\gamma(u)\Big) \right\} \right\} + \lambda
\end{aligned}
$$

and by (B3)

$$
\limsup_{k\to\infty} \frac{1}{k} \ln \hat{E}^u_x \left\{ e^{\gamma \sum_{i=0}^{k-1} c(x^1_i,u(x^1_i))} \right\} \leq \lambda
$$

Consequently letting λ decreasing to $\lambda^\gamma(u)$ we obtain

$$
\limsup_{k\to\infty} \frac{1}{k} \ln \hat{E}^u_x \left\{ e^{\gamma \sum_{i=0}^{k-1} c(x^1_i,u(x^1_i))} \right\} \leq \lambda^\gamma(u) \qquad (23)
$$

Assume now that $\lambda < \lambda^\gamma(u)$. Then by (B1) for $z \in C_1$ and λ close to $\lambda^\gamma(u)$ we have

$$
\hat{E}^u_z \left\{ \exp\left\{ \sum_{i=1}^{\tau_{C_1}} \Big(\gamma c(x^1_i, u(x^1_i)) - \lambda\Big) \right\} \right\} > 1
$$

and consequently for $N \geq N_0$

$$
\hat{E}^u_z \left\{ \exp\left\{ \sum_{i=1}^{\tau_{C_1}\wedge N} \Big(\gamma c(x^1_i, u(x^1_i)) - \lambda\Big) \right\} \right\} \geq 1. \qquad (24)
$$

Therefore by (B1) w_N^u given by (20) (now for λ as above) is well defined and similarly as in (21) (22) we have

$$e^{w_{N+1}^u(x)} \leq \hat{E}_x^u \left\{ e^{\gamma c(x_0^1, u(x_0^1)) - \lambda} e^{w_N^u(\hat{x}_1)} \right\} \tag{25}$$

and by iteration for $N \geq N_0$

$$e^{w_{N+k}^u(x)} \leq \hat{E}_x^u \left\{ e^{\sum_{i=0}^{k-1} (\gamma c(x_i^1, u(x_i^1)) - \lambda)} e^{w_N^u(\hat{x}_k)} \right\}$$
$$\leq \hat{E}_x^u \left\{ e^{\sum_{i=0}^{k-1} (\gamma c(x_i^1, u(x_i^1)) - \lambda)} e^{\gamma \|c\| N} \right\}.$$

Therefore

$$\frac{1}{k} \ln \hat{E}_x^u \left\{ e^{\gamma \sum_{i=0}^{k-1} c(x_i^1, u(x_i^1))} \right\} \geq -\frac{1}{k} \gamma \|c\| N$$
$$+ \frac{1}{k} \inf_N \ln \hat{E}_x^u \left\{ \exp \left\{ \sum_{i=1}^{\sigma_{C_1} \wedge N - 1} \left(\gamma c(x_i^1, u(x_i^1)) - \lambda^\gamma(u) \right) \right\} \right\} + \lambda$$

and by (B2)

$$\liminf_{k \to \infty} \frac{1}{k} \ln \hat{E}_x^u \left\{ e^{\gamma \sum_{i=0}^{k-1} c(x_i^1, u(x_i^1))} \right\} \geq \lambda$$

and finally

$$\liminf_{k \to \infty} \frac{1}{k} \ln \hat{E}_x^u \left\{ e^{\gamma \sum_{i=0}^{k-1} c(x_i^1, u(x_i^1))} \right\} \geq \lambda^\gamma(u) \tag{26}$$

which together with (23) using (3) completes the proof.

□

Remark 2 *Notice that under (D1) the conclusion of Proposition 1 i.e. (18) holds. In fact, under (D1) condition (14) is satisfied with* $d(u) = \inf_{x \in E, a \in U} \gamma c(x, a)$ *and (15) holds provided that the process* $(\hat{x}_n)$ *starting from* C_1 *enters the set*

$$\left\{ x = (x^1, x^2) \in \hat{E} : \gamma c(x^1, u(x^1)) > \inf_{z \in E, a \in U} \gamma c(z, a) \right\}$$

before hitting C_1 *with positive probability. In the case* $(\hat{x}_n)$ *does not enter the above set we have equality in (15) and then also (24) is satisfied so that under (B2) and (B3) equality (18) holds. By Hölder inequality using the fact that* $\lambda^\gamma(u) \leq \sup_{x \in E, a \in U} \gamma c(x, a)$ *we have that*

$$1 \leq \left(\hat{E}_x^u \left\{ \exp \left\{ \sum_{i=1}^{\sigma_{C_1} \wedge N - 1} \left(-\gamma c(x_i^1, u(x_i^1)) + \lambda^\gamma(u) \right) \right\} \right\} \right)^{\frac{1}{2}}$$

$$\left(\hat{E}_x^u\left\{\exp\left\{\sum_{i=1}^{\sigma_{C_1}\wedge N-1}\Big(\gamma c(x_i^1,u(x_i^1))-\lambda^\gamma(u)\Big)\right\}\right\}\right)^{\frac{1}{2}} \le \left(\hat{E}_x^u\left\{e^{\gamma\|c\|_{sp}\tau_{C_1}}\right\}\right)^{\frac{1}{2}}$$
$$\left(\hat{E}_x^u\left\{\exp\left\{\sum_{i=1}^{\sigma_{C_1}\wedge N-1}\Big(\gamma c(x_i^1,u(x_i^1))-\lambda^\gamma(u)\Big)\right\}\right\}\right)^{\frac{1}{2}}$$

from which under (D1) inequality (16) holds. Since $\lambda^\gamma(u) \ge \inf_{x\in E,a\in U}\gamma c(x,a)$, under (D1) (17) also holds.

4 Uniformly ergodic approximation of controlled Markov processes

We shall now assume that

(A4) for $x \in E$, $A \in \mathcal{E}$
$$P^a(x,A) = \int_A p(x,a,y)\nu(dy) \tag{27}$$

where p is a positive continuous function of its coordinates.
Denote by $|x|$ the value of $\rho(x,\theta)$, where ρ is a metric on E compatible with the topology of E and $\theta \in E$ is a fixed point.
Let
$$\tilde{p}_N(x,a,y) = \begin{cases} \frac{p(x,a,y)}{\Delta_N^a(x)} & \text{for} \quad |y| \le N \\ \frac{p(\theta,\bar{a},y)}{\Delta_N^a(x)} & \text{for} \quad |y| \ge N+1 \\ \frac{p(x,a,y)(N+1-|y|)+p(\theta,\bar{a},y)(|y|-N)}{\Delta_N^a(x)} & \text{elsewhere} \end{cases}$$

with $\Delta_N^a(x) = P^a(x,B_N) + P^{\bar{a}}(\theta, B_{N+1}^c) + \int_{B_{N+1}\setminus B_N}[p(x,a,y)(N+1-|y|)+p(\theta,\bar{a},y)(|y|-N)]\nu(dy)$, where $B_N = \{x \in E : |x| \le N\}$ and $\bar{a}$ is a fixed element of U.
Let

$$p_N(x,a,y) = \tilde{p}_N(x,a,y) \text{ if } |x| \le N$$
$$p_N(x,a,y) = \tilde{p}_N\left(\frac{x}{|x|}N,a,y\right) \text{ for } |x| > N.$$

and define
$$P_N^a(x,dy) = p_N(x,a,y)\nu(dy) \tag{28}$$

We clearly have that

Lemma 5
$$\sup_{a\in U} \|P_N^a(x,\cdot) - P^a(x,\cdot)\|_{var} \to 0 \tag{29}$$
as $N\to\infty$, uniformly in x from compact sets. Furthermore for each N

$$\sup_{a,a'\in U} \sup_{x,x'\in E} \sup_{y\in E} \frac{p_N(x,a,y)}{p_N(x',a',y)} < \infty \tag{30}$$

Let for $(a_n)\in\mathcal{U}_s$

$$F_{Nx}^{(a_n)}(\lambda) = \hat{E}_x^{(a_n),N}\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}}(\gamma c(x_i^1, a_i) - \lambda)\right\}\right\} \tag{31}$$

and
$$F_x^{(a_n)}(\lambda) = \hat{E}_x^{(a_n)}\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}}(\gamma c(x_i^1, a_i) - \lambda)\right\}\right\}, \tag{32}$$

where $\hat{P}_x^{(a_n),N}$ is a conditional probability of the split Markov process $(\hat{x}_n)$ corresponding to Markov process $(x_n)$with transition probability $P_N^{a_n}$ at time n.
Notice that whenever $x\in C_1$ the functions in (31) and (32) do not depend on x and will be denoted by $F_N^{(a_n)}$ and $F^{(a_n)}$. We have

Proposition 2 *Assume that there is N_0 such that for $N\geq N_0$ there exist $d_1<d_2$ such that for $(a_n)\in\mathcal{U}_s$, $x\in C_1$ we have*

$$F_N^{(a_n)}(d_2) = F_{Nx}^{(a_n)}(d_2) \leq 1 \leq F_{Nx}^{(a_n)}(d_1) < \infty, \tag{33}$$

$F_{Nx}^{(a_n)}(\lambda)\to F_x^{(a_n)}(\lambda)$ for $x\in C_1$ uniformly in $(a_n)\in\mathcal{U}_s$ and $\lambda\in[d_1,d_2]$, and furthermore

$$\sup_{(a_n)} |F^{(a_n)\prime}(d_1)| < \infty, \tag{34}$$

where $\prime$ stands for the derivative with respect to λ. Then

$$\lambda_N^\gamma((a_n)) := \left(F_N^{(a_n)}\right)^{-1}(1) \to \left(F^{(a_n)}\right)^{-1}(1) = \lambda^\gamma((a_n)) \tag{35}$$

uniformly in $(a_n)\in\mathcal{U}_s$ as $N\to\infty$.

Proof. Assume that there is $\varepsilon>0$, a sequence (a_n^k) of strategies from $\mathcal{U}_s$ and sequence $N_k\to\infty$ such that
$$|\lambda_{N_k}^\gamma((a_n^k)) - \lambda^\gamma((a_n^k))| > \varepsilon. \tag{36}$$

By assumption we have that

$$|F_{N_k x}^{(a_n^k)}(\lambda_{N_k}((a_n^k))) - F_x^{(a_n^k)}(\lambda_{N_k}((a_n^k)))| \to 0$$

and therefore

$$F_x^{(a_n^k)}(\lambda_{N_k}^{\gamma}((a_n^k))) \to 1 = F_{N_k x}^{(a_n^k)}(\lambda_{N_k}^{\gamma}((a_n^k))) \tag{37}$$

as $k \to \infty$. Since $F_x^{(a_n^k)}(\lambda^{\gamma}((a_n^k))) = 1$ and $\sup_{(a_n^k)} |F^{(a_n^k)\prime}(\lambda)|$ is bounded for $\lambda \in [d_1, d_2]$ (by (34)) we should have $|\lambda_{N_k}^{\gamma}((a_n^k)) - \lambda^{\gamma}((a_n^k))| \to 0$ as $k \to \infty$, which contradicts (36).

□

Remark 3 *Notice that the choice of* d_1 *and* d_2 *in (33) is uniform with respect to* $(a_n) \in \mathcal{U}_s$. *Natural candidate is* $d_1 = \inf_{x \in E, a \in U} \gamma c(x,a)$ *and* $d_2 = \sup_{x \in E, a \in U} \gamma c(x,a)$. *To have convergence* $F_{Nx}^{(a_n)}(\lambda) \to F_x^{(a_n)}(\lambda)$ *for* $x \in C_1$ *uniform in* $(a_n) \in \mathcal{U}_M$ *and* $\lambda \in [d_1, d_2]$ *we have to assume for* $x \in C_1$
(D2) $\sup_{(a_n) \in \mathcal{U}_s} \hat{E}_x^{(a_n)} \{\exp\{\gamma \|c\|_{sp} \tau_{C_1}\}\} < \infty$,
and
(D3) $\sup_N \sup_{(a_n) \in \mathcal{U}_s} \hat{E}_x^{(a_n),N} \{\exp\{\gamma \|c\|_{sp} \tau_{C_1}\}\} < \infty$.
Since

$$\begin{aligned} |F_x^{(a_n)\prime}(\lambda)| &= \hat{E}_x^{(a_n)} \left\{ \tau_{C_1} \exp\left\{ \sum_{i=1}^{\tau_{C_1}} (\gamma c(x_i^1, h_i) - \lambda) \right\} \right\} \\ &\le \hat{E}_x^{(a_n)} \{\tau_{C_1} \exp\{\gamma \|c\|_{sp} \tau_{C_1}\}\} \\ &\le K \hat{E}_x^{(a_n)} \{\exp\{(1+\varepsilon)\gamma \|c\|_{sp} \tau_{C_1}\}\} \end{aligned}$$

for $\varepsilon > 0$ *and* $K > 0$, *to have (34) it is sufficient to assume for* $x \in C_1$ *that*
(D4) $\sup_{(a_n) \in \mathcal{U}_s} \hat{E}_x^{(a_n)} \{\exp\{(1+\varepsilon)\gamma \|c\|_{sp} \tau_{C_1}\}\} < \infty$ *for a sufficiently small* $\varepsilon > 0$.

By Theorem 1 of [7] using (30) we have

Proposition 3 *For each* N *there are* λ_N^{γ} *and* $w_N \in C(E)$ *such that*

$$e^{w_N(x)} = \inf_{a \in U} [e^{\gamma c(x,a) - \lambda_N^{\gamma}} \int_E e^{w_N(y)} P_N^a(x, dy)] \tag{38}$$

and consequently

$$\lambda_N^{\gamma} = \inf_{a_n} J_x^{\gamma,N}((a_n)), \tag{39}$$

where

$$J_x^{\gamma,N}((a_n)) := \limsup_{t \to \infty} \frac{1}{t} \ln E_x^{(a_n),N} \left\{ \exp\left\{ \sum_{i=0}^{t-1} \gamma c(x_i, a_i) \right\} \right\}$$

and infimum is taken over all admissible controls (a_n).
Moreover the strategy $a_n = u_N(x_n)$, where $u_N : E \mapsto U$ is a Borel measurable function for which the infimum in (38) is attained, is optimal.

Corollary 3 *Under (D2), (D3) and (D4) we have that*

$$\lambda^\gamma := \inf_{(a_n)\in\mathcal{U}_s} J_x^\gamma((a_n)) = \lim_{N\to\infty} \lambda_N^\gamma \tag{40}$$

for $x \in E$.

Proof. By Remark 3 and Proposition 2

$$\sup_{(a_n)\in\mathcal{U}_s} |\lambda_N^\gamma((a_n)) - \lambda^\gamma((a_n))| \to 0.$$

Since by Remark 2, Proposition 1 and Proposition 2, $\lambda_N^\gamma((a_n)) = \inf_{(a_n)\in\mathcal{U}_s} \lambda_N^\gamma((a_n)) = \lambda_N^\gamma$ we obtain (40).

□

5 Risk sensitive Bellman equation

Let u_N be an optimal control function corresponding to $P_N^a(x, dy)$. Furthermore assume that

(A5) $\exists_{\epsilon>0}$ *such that* $\forall_{K \ compact\subset\ \hat{E}}$

$$\sup_{a\in U}\sup_{x\in\hat{K}}\sup_{N} \hat{E}_x^{a,N}\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}}(\gamma c(x_i^1, u_N(x_i^1)) - \lambda_N^\gamma(u_N))(1+\epsilon)\right\}\right\} = M(K) < \infty, \tag{41}$$

where above we control the split Markov process $(\hat{x}_n)$ using at time 0 control $a_0 = a$ and then $a_n = u_N(x_n^1)$ for $n \geq 1$.

Theorem 1 *Under (A1)-(A5) there exist λ^γ and a continuous function $w : E \mapsto R$ such that*

$$e^{w(x)} = \inf_{a\in U}[e^{\gamma c(x,a)-\lambda^\gamma}\int_E e^{w(y)}P^a(x, dy)] \tag{42}$$

Moreover, under (D1) satisfied for all $(a_n) \in \mathcal{U}_s$ we have that λ^γ is an optimal value of the cost functional J_x^γ and the control $\hat{u}(x_n)$, where $\hat{u}$ is a Borel measurable function for

which the infimum in the right hand side of (42) is attained, is an optimal control within the class of controls from $\mathcal{U}_s$.
Furthermore, if for admissible control (a_n) we have that

$$\limsup_{t\to\infty} E_x^{(a_n)} \left\{ \left(E_{x_t}^{a_t} \left\{ e^{w(x_1)} \right\} \right)^\alpha \right\} < \infty$$

for every $\alpha > 1$, then $\lambda^\gamma \leq J_x^\gamma((a_n))$.

Proof. The proof consists of several steps:
Step 1. We prove first that $\sup_N \hat{E}_x^{a,N} \{\exp\{\hat{w}_N^{u_N}(\hat{x}_1\}\}$ is bounded uniformly on compact subsets of $(E_0 \cup C_1) \times U$, where $\hat{w}_N^{u_N}$ is a solution to the multiplicative Poisson equation corresponding to the transition operator $P_N^a(x, dy)$ with control function u_N.
In fact,

$$\hat{E}_x^{a,N} \{\exp\{\hat{w}_N^{u_N}(\hat{x}_1\}\} = \hat{E}_x^{a,N} \left\{ \chi_{C_1}(\hat{x}_1) e^{\gamma c(x_1^1, u_N(x_1^1)) - \lambda_N^\gamma(u_N)} \right\}$$
$$+ \hat{E}_x^{a,N} \left\{ \chi_{C_1^c}(\hat{x}_1) \exp\left\{ \sum_{i=1}^{\tau_{C_1}} (\gamma c(x_i^1, u_N(x_i^1)) - \lambda_N^\gamma(u_N)) \right\} \right\} \tag{43}$$

and by (A5) follows the required boundedness.
Step 2. We show now that for $N = 1, 2 \ldots$, the functions $\hat{E}_x^{a,N} \{\exp\{\hat{w}_N^{u_N}(\hat{x}_1)\}\}$ are equicontinuous in x and a from compact subsets of $E_0 \cup C_1$ and U respectively.
Notice first that by (29) for each compact set $K \subset E_0 \cup C_1$, $\varepsilon' > 0$ there is a compact set $K_1 \supset C_0 \cup C_1$ such that

$$\sup_{a\in U} \sup_{x\in K} \sup_N \hat{P}_x^{aN} \{\hat{x}_1 \in K_1^c\} < \varepsilon' \tag{44}$$

Furthermore by Hölder inequality

$$\sup_{a\in U} \sup_{x\in K} \sup_N \hat{E}_x^{a,N} \left\{ \chi_{C_1^c}(\hat{x}_1) \chi_{K_1^c}(\hat{x}_1) \exp\left\{ \sum_{i=1}^{\tau_{C_1}} (\gamma c(x_i^1, u_N(x_i^1)) - \lambda_N^\gamma(u_N)) \right\} \right\}$$
$$\leq \sup_{a\in U} \sup_{x\in K} \sup_N \left(\hat{P}_x^{a,N} \{\hat{x}_1 \in K_1^c\} \right)^{\frac{\varepsilon}{1+\varepsilon}} \sup_{a\in U} \sup_{x\in K} \sup_N \left(\hat{E}_x^{a,N} \left\{ \exp\left\{ \sum_{i=1}^{\tau_{C_1}} (\gamma c(x_i^1, u_N(x_i^1)) \right.\right.\right.$$
$$\left.\left.\left. - \lambda_N^\gamma(u_N))(1+\epsilon) \right\}\right\}\right)^{\frac{1}{1+\varepsilon}} \leq \varepsilon'^{\frac{\varepsilon}{1+\varepsilon}} (M(K))^{\frac{1}{1+\varepsilon}} \tag{45}$$

Consequently by (43)-(45)

$$|\hat{E}_x^{a,N} \{\exp\{\hat{w}_N^{u_N}(\hat{x}_1\}\} - \hat{E}_{x'}^{a',N} \{\exp\{\hat{w}_N^{u_N}(\hat{x}_1\}\}|$$
$$\leq e^{|\gamma| \|c\|} \|\hat{P}^{aN}(x, C_1 \cap \cdot) - \hat{P}^{a'N}(x', C_1 \cap \cdot)\|_{var} + 2\varepsilon'^{\frac{\varepsilon}{1+\varepsilon}} (M(K))^{\frac{1}{1+\varepsilon}}$$
$$+ |\hat{E}_x^{a,N} \{\chi_{K_1}(\hat{x}_1) \exp\{\hat{w}_N^{u_N}(\hat{x}_1\}\} - \hat{E}_{x'}^{a',N} \{\chi_{K_1}(\hat{x}_1) \exp\{\hat{w}_N^{u_N}(\hat{x}_1'\}\}| \tag{46}$$

For $\delta > 0$ choose K_1 in (44) such that $\varepsilon'^{\frac{\varepsilon}{1+\varepsilon}} (M(K))^{\frac{1}{1+\varepsilon}} < \frac{\delta}{3}$. Since

$$|\hat{E}_x^{a,N}| \{\chi_{K_1}(\hat{x}_1) \exp\{\hat{w}_N^{u_N}(\hat{x}_1)\}\} - \hat{E}_{x'}^{a',N} \{\chi_{K_1}(\hat{x}_1) \exp\{\hat{w}_N^{u_N}(\hat{x}_1')\}\}| \\ \leq \sup_{x\in K_1} \exp\{\hat{w}_N^{u_N}(x)\} \|\hat{P}^{aN}(x, K_1 \cap \cdot) - \hat{P}^{a'N}(x', K_1 \cap \cdot)\|_{var}$$

for x, $x' \in E_0 \cup C_1$ and a, $a' \in U$ such that

$$\|\hat{P}^{aN}(x, C_1 \cap \cdot) - \hat{P}^{a'N}(x', C_1 \cap \cdot)\|_{var} \leq \frac{\delta}{3e^{\gamma\|c\|}} \tag{47}$$

and

$$\|\hat{P}^{aN}(x, K_1 \cap \cdot) - \hat{P}^{a'N}(x', K_1 \cap \cdot)\|_{var} \leq \frac{\delta}{3\sup_{z\in K_1} e^{\hat{w}_N^{u_N}(z)}} \tag{48}$$

by (46) we obtain that

$$|\hat{E}_x^{a,N} \{\exp\{\hat{w}_N^{u_N}(\hat{x}_1)\}\} - \hat{E}_{x'}^{a',N} \{\exp\{\hat{w}_N^{u_N}(\hat{x}_1)\}\}| \leq \delta.$$

Now by (A5) $\sup_{z\in K_1} e^{\hat{w}_N^{u_N}(z)}$ is bounded in N and therefore by (29) we can choose x, x' and a, a' in (47) and (48) uniformly in N, which completes the proof of equicontinuity.
Step 3. By step 1, 2 and (9) we immediately see that

$$E_x^{a,N} \{\exp\{w_N^{u_N}(x_1)\}\}$$

is uniformly (in N) bounded and equicontinuous in x and a from compact subsets of $E \times U$. Since u_N is optimal for $P_N^a(x, dy)$ we have that $w_N^{u_N} = w_N$. Therefore by Ascoli theorem (thm. 33 of [13]) there is a subsequence N_k such that

$$E_x^{a,N_k} \{\exp\{w_{N_k}(x_1)\}\}$$

converges uniformly in $a \in U$ and x from compact subsets of E and $\lambda_{N_k}^{\gamma}(u_{N_k}) \to \lambda$ (since $\lambda_N^{\gamma}(u_n) \in [\inf_{x\in E, a\in U} \gamma c(x,a), \sup_{x\in E, a\in U} \gamma c(x,a)]$). Consequently there is a continuous function w such that

$$e^{w(x)} = \inf_{a\in U} [e^{\gamma c(x,a) - \lambda} \lim_{k\to\infty} \int_E e^{w_{N_k}(y)} P_{N_k}^a(x, dy)] \tag{49}$$

Step 4. To prove that function w defined in (49) is a solution to the Bellman equation (42) it remains to show that

$$\lim_{k\to\infty} E_x^{a,N_k} \{\exp\{w_{N_k}(x_1)\}\} = E_x^a \left\{e^{w(x_1)}\right\}. \tag{50}$$

In fact, by Fatou lemma

$$E_x^a\left\{e^{w(x_1)}\right\} \leq \lim_{k\to\infty} E_x^{a,N_k}\left\{\exp\left\{w_{N_k}(x_1)\right\}\right\} < \infty \tag{51}$$

By step 1 and 2 one can find a compact set $K_1 \supset C$ such that

$$\sup_N \sup_{a\in U} E_x^{a,N}\left\{\chi_{K_1^c}(x_1)\exp\left\{w_N(x_1)\right\}\right\} \leq \frac{\varepsilon}{3} \tag{52}$$

and

$$\sup_{a\in U} E_x^a\left\{\chi_{K_1^c}(x_1)\exp\left\{w(x_1)\right\}\right\} \leq \frac{\varepsilon}{3}. \tag{53}$$

Therefore

$$\begin{aligned}
&|E_x^a\left\{\exp\left\{w(x_1)\right\}\right\} - E_x^{a,N_k}\left\{\exp\left\{w_{N_k}(x_1)\right\}\right\}| \leq \\
&|E_x^a\left\{\chi_{K_1}(x_1)\exp\left\{w(x_1)\right\}\right\} - E_x^{a,N_k}\left\{\chi_{K_1}(x_1)\exp\left\{w(x_1)\right\}\right\}| \\
&+|E_x^{a,N_k}\left\{\chi_{K_1}(x_1)\left(\exp\left\{w(x_1)\right\} - \exp\left\{w_{N_k}(x_1)\right\}\right)\right\}| \\
&+E_x^{a,N_k}\left\{\chi_{K_1^c}(x_1)\exp\left\{w_{N_k}(x_1)\right\}\right\} + E_x^a\left\{\chi_{K_1^c}(x_1)\exp\left\{w(x_1)\right\}\right\} \\
&\leq \sup_{x\in K_1} e^{w(x)}\|P^a(x, K_1\cap\cdot) - P^{aN}(x, K_1\cap\cdot)\|_{var} + \sup_{x\in K_1}|e^{w(x)} - e^{w_{N_k}}(x)| + \frac{2\varepsilon}{3}.
\end{aligned}$$

Consequently letting $k\to\infty$ and taking into account that ε may be arbitrarily small we obtain the convergence (50). By continuity on x and a of the right hand side of (42) we have the existence of a Borel measurable function $\hat{u}$ for which the infimum is attained.
Step 5. We shall show now that for Borel measurable $u: E \mapsto U$ we have we have $\lambda^\gamma(u) \geq \lambda^\gamma$.
In fact, then

$$e^{w(x)} \leq e^{\gamma c(x,u(x))-\lambda^\gamma} \int_E e^{w(y)} P^{u(x)}(x, dy). \tag{54}$$

Define following (12)

$$e^{\hat{w}^u(x^1,x^2)} = e^{\gamma c(x^1,u(x^1))-\lambda^\gamma} \hat{E}_{x^1,x^2}^{u(x^1)}\left\{e^{w(x_1^1)}\right\} \tag{55}$$

Since by (3) for $a\in U$

$$\begin{aligned}
&E_x^a\left\{e^{w(x_1)}\right\} = \hat{E}_{\delta_x^*}^a\left\{e^{w(x_1^1)}\right\} = \\
&\chi_C(x)[(1-\beta)\hat{E}_{(x,0)}^a\left\{e^{w(x_1^1)}\right\} + \beta\hat{E}_{(x,1)}^a\left\{e^{w(x_1^1)}\right\}] + \chi_{E\setminus C}(x)\hat{E}_{(x,0)}^a\left\{e^{w(x_1^1)}\right\}
\end{aligned}$$

we have from (54)

$$e^{w(x)} \leq [e^{\gamma c(x,u(x))-\lambda^\gamma} \left(\chi_C(x)[(1-\beta)\hat{E}^{u(x)}_{(x,0)}\left\{e^{w(x_1^1)}\right\}\right.$$
$$\left.+\beta\hat{E}^{u(x)}_{(x,1)}\left\{e^{w(x_1^1)}\right\}] + \chi_{E\setminus C}(x)\hat{E}^{u(x)}_{(x,0)}\left\{e^{w(x_1^1)}\right\}\right) =$$
$$\chi_C(x)\left((1-\beta)e^{\hat{w}^u(x,0)} + \beta e^{\hat{w}^u(x,1)}\right) + \chi_{E\setminus C}(x)e^{\hat{w}^u(x,0)}$$

Consequently

$$\hat{E}^{u(x^1)}_{(x^1,x^2)}\left\{e^{w(x_1^1)}\right\} \leq \hat{E}^{u(x^1)}_{(x^1,x^2)}\left\{\chi_C(x_1^1)\left((1-\beta)e^{\hat{w}^u(x_1^1,0)} + \beta e^{\hat{w}^u(x_1^1,1)}\right) + \chi_{E\setminus C}(x_1^1)e^{\hat{w}^u(x_1^1,0)}\right\}$$
$$= \hat{E}^{(u(x^1)}_{(x^1,x^2)}\left\{e^{\hat{w}^u(x_1)}\right\}. \tag{56}$$

Therefore by the definition of $\hat{w}^u$ we have that

$$e^{\hat{w}(x^1,x^2)} \leq e^{\gamma c(x^1,u(x^1))-\lambda^\gamma}\hat{E}^{u(x^1)}_{(x^1,x^2)}\left\{e^{\hat{w}^u(x_1)}\right\} \tag{57}$$

Iterating the last inequality for $z \in C_1$ we obtain

$$\hat{E}^u_z\left\{e^{\hat{w}(x_1)}\right\} \leq \hat{E}^u_z\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}}(\gamma c(x_i^1, u(x_i^1)) - \lambda^\gamma)\right\}\hat{E}^u_{x_{\tau_{C_1}}}\left\{e^{\hat{w}(x_1)}\right\}\right\} \tag{58}$$

Since by step 1 we have that $\hat{E}_z\left\{e^{\hat{w}(x_1)}\right\} < \infty$ we obtain

$$\hat{E}_z\left\{\exp\left\{\sum_{i=1}^{\tau_{C_1}}(\gamma c(x_i^1, u(x_i^1)) - \lambda^\gamma)\right\}\right\} \geq 1,$$

for $z \in C_1$, which by Lemma 3 implies that $\lambda^\gamma \leq \lambda^\gamma(u)$.

Step 6. Assuming (D1) satisfied for any $(a_n) \in \mathcal{U}_s$, by Proposition 1 and Lemma 4 and step 5 we have for any Borel measurable $u : E \mapsto U$

$$\lambda^\gamma = \lambda^\gamma(\hat{u}) = J^\gamma_x(\hat{u}(x_n)) \leq J^\gamma_x((u(x_n))),$$

which shows optimality of $(\hat{u}(x_n))$ within the class of stationary controls. If for an admissible control (a_n) we have $\limsup_{t\to\infty} E^{(a_n)}_x\left\{\left(E^{a_t}_{x_t}\left\{e^{w(x_1)}\right\}\right)^\alpha\right\} < \infty$ for every $\alpha > 1$, then by Hölder inequality we have from (42)

$$w(x) \leq \ln E^{(a_n)}_x\left\{\exp\left\{\sum_{i=0}^{t-1}(\gamma c(x_i, a_i) - \lambda^\gamma)\right\}E^{a_t}_{x_t}\left\{e^{w(x_1)}\right\}\right\}$$
$$\leq -t\lambda^\gamma + \ln\left(E^{(a_n)}_x\left\{\exp\left\{\sum_{i=0}^{t-1}\gamma c(x_i, a_i)(1+\epsilon)\right\}\right\}\right)^{\frac{1}{1+\epsilon}} +$$
$$\ln\left(E^{(a_n)}_x\left\{\left(E^{a_t}_{x_t}\left\{e^{w(x_1)}\right\}\right)^{1+\frac{1}{\epsilon}}\right\}\right)^{\frac{\epsilon}{1+\epsilon}}$$

Dividing both sides of the last inequality by t and letting t to infinity we obtain that $\frac{1}{1+\epsilon} J_x^{\gamma(1+\epsilon)}((a_n)) \geq \lambda^\gamma$ for any $\epsilon > 0$. It remains to show that the mapping $\gamma \mapsto J_x^\gamma((a_n))$ is a continuous function for $\gamma > 0$ since then letting $\epsilon \to 0$ we obtain $J_x^\gamma((a_n)) \geq \lambda^\gamma$. To prove continuity notice that for $\gamma_1, \gamma_2 > 0$, $\gamma_1 \leq \gamma_2$

$$\begin{aligned} &|J_x^{\gamma_1}((a_n)) - J_x^{\gamma_2}((a_n))| \leq \limsup_{t\to\infty} \frac{1}{t} |\ln E_x^{(a_n)} \left\{ \exp\left\{ \sum_{i=0}^{t-1} \gamma_1 c(x_i, a_i) \right\} \right\} \\ &- \ln E_x^{(a_n)} \left\{ \exp\left\{ \sum_{i=0}^{t-1} \gamma_2 c(x_i, a_i) \right\} \right\} | \leq \limsup_{t\to\infty} \frac{1}{t} |\gamma_1 - \gamma_2| \sup_{\gamma\in[\gamma_1,\gamma_2]} |g_t'(\gamma)| \\ &\leq \|c\| |\gamma_1 - \gamma_2|, \end{aligned}$$

since the derivative of the function

$$g_t(\gamma) := \ln E_x^{(a_n)} \left\{ \exp\left\{ \sum_{i=0}^{t-1} \gamma c(x_i, a_i) \right\} \right\}$$

is bounded by $t\|c\|$.

□

Remark 4 *A sufficient condition for (A5) can be formulated a follows*
(D5) there is $\varepsilon > 0$ such that for each compact set $K \subset \hat{E}$

$$\sup_{a\in U} \sup_{x\in K} \sup_{N} \hat{E}_x^{a,N} \left\{ \exp\left\{ (1+\varepsilon)|\gamma| \|c\|_{sp} \tau_{C_1} \right\} \right\} < \infty$$

where the split Markov process $(\hat{x}_n)$ after control a at time 0 is controlled using the control function u_N. Notice furthermore that since λ is defined in a unique way (as the optimal value of the cost functional (1)) we therefore have the convergence $\lambda_N^\gamma \to \lambda^\gamma$ which we obtained earlier in Corollary 3 under additional assumptions.

6 Remarks on assumptions and an example

We shall formulate first a sufficient condition for (A3).

Proposition 4 *If for $x \in E$ and $(a_n) \in \mathcal{U}_s$*

$$E_x^{(a_n)} \left\{ \exp\left\{ \gamma \|c\|_{sp} \tau_C \right\} \right\} < \infty \tag{59}$$

and

$$\sup_{x\in C} E_x^{(a_n)} \left\{ \exp\left\{ \gamma \|c\|_{sp} \tau_C \right\} \right\} - \beta \exp\left\{ \gamma \|c\|_{sp} \right\} < 1 \tag{60}$$

then (A3) holds.

Proof. Notice first that by Corollary 1 for $z \notin C$ and positive integer m we have

$$\hat{E}_{(z,0)}^{(a_n)} \left\{ \exp\left\{ \gamma \|c\|_{sp} \tau_C \right\} \chi_{\tau_C \leq m} \right\} = E_z^{(a_n)} \left\{ \exp\left\{ \gamma \|c\|_{sp} \tau_C \right\} \chi_{\tau_C \leq m} \right\}. \tag{61}$$

Letting $m \to \infty$ by (59) we obtain

$$\hat{E}_{(z,0)}^{(a_n)} \left\{ \exp\left\{ \gamma \|c\|_{sp} \tau_C \right\} \right\} = E_z^{(a_n)} \left\{ \exp\left\{ \gamma \|c\|_{sp} \tau_C \right\} \right\}. \tag{62}$$

Now for $(a_n) \in \mathcal{U}_M$ and $x \in C$ using the definition of split Markov process and (60) we have

$$\hat{E}_{(x,0)}^{(a_n)} \left\{ \exp\left\{ \gamma \|c\|_{sp} \tau_C \right\} \right\} = \hat{E}_{(x,0)}^{(a_n)} \Big\{ \chi_C(x_1^1) e^{\gamma \|c\|_{sp}} +$$
$$\chi_{C^c}(x_1^1) \hat{E}_{\hat{x}_1}^{(a_n)} \left\{ \exp\left\{ \gamma \|c\|_{sp} \tau_C \right\} \right\} \Big\} = e^{\gamma \|c\|_{sp}} \frac{P^{a_0}(x,C) - \beta}{1-\beta}$$
$$+ e^{\gamma \|c\|_{sp}} \int_{C^c} E_z^{(a_n)} \left\{ \exp\left\{ \gamma \|c\|_{sp} \tau_C \right\} \right\} \frac{P^{a_0}(x,dz)}{1-\beta} = \frac{1}{1-\beta} \Big[e^{\gamma \|c\|_{sp}} (P^{a_0}(x,C) - \beta)$$
$$- E_x^{(a_n)} \Big\{ \chi_C(x_1) e^{\gamma \|c\|_{sp}} \Big\} + E_x \Big\{ e^{\gamma \|c\|_{sp} \tau_C} \Big\} \Big] < \frac{1}{1-\beta}. \tag{63}$$

Let $\tau_1 = \tau_C := \{ i \geq 0 : x_i^1 \in C \}$, $\tau_{n+1} = \tau_n + \tau_1 \circ \theta_{\tau_n}$.
For $x \in \hat{E}$ and $L = \sup_{z \in C} \hat{E}_{(z,0)}^{(a_n)} \Big\{ e^{\gamma \|c\|_{sp} \tau_C} \Big\}$ using Lemma 1 we have

$$\hat{E}_x^{(a_n)} \Big\{ e^{\gamma \|c\|_{sp} \tau_{C_1}} \Big\} = \hat{E}_x \left\{ \sum_{i=1}^{\infty} \chi_{C_1^c}(\hat{x}_{\tau_1}) \ldots \chi_{C_1^c}(x_{\tau_{i-1}}) \chi_{C_1}(\hat{x}_{\tau_i}) e^{\gamma \|c\|_{sp} \tau_i} \right\}$$
$$\leq \sum_{i=1}^{\infty} \hat{E}_x^{(a_n)} \Big\{ \chi_{C_1^c}(\hat{x}_{\tau_1}) \ldots \chi_{C_1^c}(\hat{x}_{\tau_{i-1}}) e^{\gamma \|c\|_{sp} \tau_{i-1}} \Big\} L\beta \leq$$
$$\hat{E}_x^{(a_n)} \Big\{ e^{\gamma \|c\|_{sp} \tau_C} \Big\} \sum_{i=1}^{\infty} (1-\beta)^{i-1} \beta L^{i-1} = \hat{E}_x^{(a_n)} \Big\{ e^{\gamma \|c\|_{sp} \tau_C} \Big\} \frac{\beta}{1-(1-\beta)L},$$

and taking into account (62), (59), (60) we obtain (D1) which completes the proof.

□

Taking into account Remark 4, by the proof of Proposition 4 we easily obtain a sufficient condition for (A5)

Corollary 4 *If there is an $\varepsilon > 0$ such that for any compact set $K \subset E$ we have*

$$\sup_{a \in U} \sup_{x \in K} \sup_{N} E_x^{a,N} \left\{ \exp\left\{ (1+\varepsilon) \gamma \|c\|_{sp} \tau_C \right\} \right\} < \infty \tag{64}$$

and

$$\sup_{a\in U}\sup_{x\in C}\sup_{N} E_x^{a,N}\left\{\exp\left\{(1+\varepsilon)\gamma\|c\|_{sp}\tau_C\right\}\right\} - \beta\exp\left\{\varepsilon\gamma\|c\|_{sp}\right\} < 1, \tag{65}$$

where Markov process (x_n) *is controlled using constant* a *at time* 0 *and* $a_n = u_N(x_n)$ *afterwards, then (A5) holds.*

Consequently we see that assumptions imposed in the paper are satisfied for a class of processes for which $f(\gamma) := E_x\{e^{\gamma\tau_C}\}$ is finite provided that we choose γ sufficiently small (to guarantee (60) and (65)). As an example one can consider a discretized ergodic diffusion (x_n) in R^d given by the following equation:

$$x_{n+1} = x_n + Ax_n + b(x_n, a_n) + D(x_n, a_n)w_n, \tag{66}$$

where (w_n) is a sequence of i.i.d. standard normal random vectors in R^d, A is a stable matrix, $b(x,a)$ is a continuous bounded vector function of $x \in R^d$ and $a \in U$, and $D(x,a)$ is a continuous bounded matrix valued function which is uniformly elliptic i.e. $\inf_{x\in R^d}\inf_{a\in U} trD(x,a)D(x,a)^T > 0$. Notice that by nondegeneracy of D the minorization property (A1) is satisfied for any closed ball C in R^d. The stability of the matrix A and boundendess of b implies that if C is sufficiently large the controlled process no matter what control is used is pushed to C. For completeness we add the following Lyapunov type criterion

Lemma 6 *If for* $(a_n) \in \mathcal{U}_s$

$$\sup_{x\notin C} E_x^{(a_n)}\left\{\|x_{\tau_C}\|^{-1}\right\} < \infty \tag{67}$$

and for $\gamma > 0$

$$\sup_{x\notin C}\sup_{a\in U} E_x^a\{\|x_1\|\} \le e^{-4\gamma}\|x\| \tag{68}$$

then

$$E_x^{(a_n)}\{e^{\gamma\tau_C}\} < \infty \tag{69}$$

Proof. Define a Lyapunov function $V(s,x) := e^{2\gamma(s+1)}\|x\|$. For $x \notin C$ by (68) we have

$$E_x^{(a_n)}\{V(s+1,x_1)\} - V(s,x) \le e^{2\gamma(s+2)}E_x^{(a_n)}\{\|x_1\|\} - e^{2\gamma(s+1)}\|x\| \le \\ -(e^{2\gamma(s+1)} - e^{2\gamma s})\|x\|$$

Consequently

$$E_{x_m}^{(a_n)}\{V(m+1,x_1)\} - V(m,x_m) \le -(e^{2\gamma(m+1)} - e^{2\gamma m})\|x_m\|,$$

$$E_{x_{m-1}}^{(a_n)}\left\{V(m,x_1)\right\} - V(m-1,x_{m-1}) \le -(e^{2\gamma m} - e^{2\gamma(m-1)})\|x_{m-1}\|,$$

and summing up the above inequalities till the process (x_n) enters the set C and taking the expected value we obtain

$$E_{x_{\tau_C}}^{(a_n)}\left\{V(\tau,x_\tau)\right\} - V(0,x) \le -E_x^{(a_n)}\left\{\|x_{\tau_C}\|\right\}(e^{2\gamma\tau_C} - 1), \tag{70}$$

from which by nonegativity of V we have that

$$E_x^{(a_n)}\left\{e^{2\gamma\tau_C}\|x_{\tau_C}\|\right\} \le V(0,x). \tag{71}$$

By Hölder inequality

$$E_x^{(a_n)}\left\{e^{\gamma\tau_C}\right\} \le \left(E_x^{(a_n)}\left\{e^{2\gamma\tau_C}\|x_{\tau_C}\|\right\}\right)^{\frac{1}{2}} \left(E_x^{(a_n)}\left\{\|x_{\tau_C}\|^{-1}\right\}\right)^{\frac{1}{2}}$$

and from (67) we obtain (69).

□

References

[1] G. Avila-Goday, E. Fernández-Gaucherand, "Controlled Markov chains with exponential risk sensitive criteria: modularity, structured policies and applications", Proc. 37th *IEEE CDC*, Tampa, Florida, December 1998, pp. 778–783.

[2] T.R. Bielecki, S. Pliska, "Risk sensitive dynamic asset management", *JAMO*, **39** (1999), pp. 337-360.

[3] V.S. Borkar, S.P. Meyn, "Risk-Sensitive Optimal Control for Markov Decision Processes with Monotone Cost", *Math. Oper. Res.*, **27** (2002), pp. 192-209.

[4] A. Bran-Rojas, R. Cavazos-Cadena, E. Fernández-Gaucherand, "Controlled Markov chains with risk sensitive criteria: some (counter) examples", Proc. 37th *IEEE CDC*, Tampa, Florida, December 1998, pp. 1853–1858.

[5] R. Cavazos-Cadena, "Solution to the risk-sensitive average cost optimality in a class of Markov decision processes with finite state space", *Math. Meth. Oper. Res.*, **57** (2003), pp. 263-285.

[6] R. Cavazos-Cadena, D. Hernandez-Hernandez, "Solution to the risk-sensitive average optimality equation in communicating Markov decision chains with finite state space: An alternative approach", *Math. Meth. Oper. Res.*, **56** (2002), pp. 473-479.

[7] G.B. Di Masi, L. Stettner, "Risk sensitive control of discrete time Markov processes with infinite horizon", *SIAM J. Control Optimiz.*, **38** (2000), pp. 61-78.

[8] G.B. Di Masi, L. Stettner, "Infinite horizon risk sensitive control of discrete time Markov processes with small risk, *Sys. Control Lett*, **40** (2000), pp. 305-321.

[9] W.H. Fleming, D. Hernandez-Hernandez, "Risk sensitive control of finite state machines on an infinite horizon", *SIAM J. Control Optimiz.*, **35** (1997), pp. 1790–1810.

[10] D. Hernandez-Hernandez, S.J. Marcus, "Risk sensitive control of Markov processes in countable state space", *Sys. Control Letters*, **29** (1996), pp. 147–155.

[11] I. Kontoyiannis, S.P. Meyn, "Spectral Theory and Limit Theorems for Geometrically Ergodic Markov Processes", *Ann. Appl. Prob.* **13** (2003), 304-362.

[12] H. Pham, "Large Deviations Approach to Optimal Long Term Investment", *Finance Stoch.* **7** (2003), 169-195.

[13] H.L. Royden, "Real Analysis", Mac Millan, New York, 1968.

[14] L. Stettner, "Duality and Risk Sensitive Portfolio Optimization", in Mathematics of Finance, Proc. AMS-IMS-SIAM Joint Summer Res. Conf., ed. G. Yin and Q. Zhang, AMS 2004, 333-347.