4 Infinite horizon with ergodic cost

4.1 Introduction

This chapter considers the problem of determining nearly optimal controls for infinite horizon problems with an ergodic cost functional as given in (1.6) or, equivalently, (1.12). As in the previous chapters, the construction of nearly optimal controls is based on an approximation approach. Paralleling chapter 3, we shall first consider the problem of the construction of nearly optimal control functions which, when applied to the true filter values, yield nearly optimal controls. This will make up most of the chapter, namely sections 4.2 to 4.5. Since the true filter process takes its values in an infinite dimensional spaces of measures, it can in general not be computed explicitly. In a second part, namely section 4.6 below that is analogous to section 3.5, we shall therefore construct a computable approximating filter process and show that the nearly optimal control functions, provided they are continuous, still yield nearly optimal controls when applied to the approximating filter.

Again as in chapter 3 we shall work with normalized filters so that, without the benefits of the measure transformation exploited in chapter 2, for the construction of nearly optimal control functions we shall make use of some compactness arguments. One such argument relies on assuming the state space E to be compact. In section 4.2 we show that in such a case a solution to the problem of determining a nearly optimal control function for an ergodic cost functional can be obtained by solving the corresponding problem for the case of a discounted cost criterion where the discount factor β is close to 1. In particuliar, in section 4.2 such a result is obtained with the use of Bellman's equation thus justifying also our choice of admissible controls that are obtained by applying a control function to the filter values. Similar results are obtained also in subsection 4.5.5 without use of Bellman's equation or the compactness of E; there, following a direct approach, we restrict the control functions to be continuous and use ergodic properties of an embedded Markov chain studied in section 4.4. All the rest of the chapter exploits compactness arguments related to assuming the control functions to be continuous.

With controls that are (continuous) functions of the filter, the filter process itself is Markov and, under additional assumptions, also Feller. We may then consider invariant measures of the filter process and express the ergodic cost functional (1.12) as integral with respect to an invariant measure. It is then natural for our main purpose to study approximations, convergence, as well as uniqueness of invariant measures. Since the general problem of the uniqueness of invariant measures for the filtering process associated to a partial observation stochastic control problem is hard, in sections 4.3 and 4.4 we consider general situations for which a unique invariant measure can be shown to exist. In particular, in section 4.3 we study controlled Markov chains with a mixed observation structure: on a given recurrent subset of the state space E the state process is completely observed, but outside only partially. Since this mixed observation model is of interest in its own, in section 4.3 we obtain for it also results related to the ergodic Bellman equation that is the subject of section 4.2. In section 4.4 we consider in particular possible situations when the filter process admits an embedded i.i.d. process to which the law of large numbers applies. Both for this latter as well as for the mixed observation case, in section 4.4 we not only obtain existence and uniqueness of invariant measures but also explicit representations, which are useful to obtain convergence results.

After these various preliminary results, in section 4.5 we concentrate on the actual construction of nearly optimal control functions. The approach is analogous to that already followed in chapter 3: the original problem is approximated by simpler problems for which the associated filter process, based on fictitious observations, takes values in a finite dimensional space of measures so that for these problems the construction of a nearly optimal control function becomes feasible. Extending then these functions to functions on the space of measures where the original filtering process takes its values, one obtains the desired nearly optimal control functions for the original problem.

Recalling that the ergodic cost functional (1.12) can be expressed as integral with respect to an invariant measure of the filtering process, in subsection 4.5.1 we first establish general results on convergence of invariant measures. These general results are then applied in subsection 4.5.2 to more specific approximations resulting from approximations of the set of admissible control functions and from discretizations of the state and observation spaces. After these specific approximations, one is left with a complete observation problem where the state is the filter that now evolves on a simplex. Although a simplex is a finite dimensional space of measures, it is still infinite valued. Since for practical computations one has to deal with finite valued quantities, in subsection 4.5.3 a discretization of the simplex is introduced allowing finally the actual construction of a nearly optimal control function for this last step approximating problem that, after a suitable extension, is nearly optimal also for the original problem. Some computational considerations along with numerical results are reported in subsection 4.5.4.

4.2 Bellman equation (Discounted cost approximations)

In this section we study the Bellman equation corresponding to the infinite horizon ergodic cost problem defined in (1.6). The starting point is the Bellman equation for infinite horizon with discounting considered in section 3.2. Recalling Theorem 3.1 we know that, under (A1)–(A5), there is a unique solution $v^\beta \in C(P(E))$ to the following discounted Bellman equation (see 3.2)

$$v^\beta(\mu) = \inf_{a\in U} \Big\{ \int_E c(x,a)\mu(dx) + \beta \prod^a(\mu, v^\beta) \Big\}$$

Assume first that E is compact. Then by the continuity of v^β there exists $\mu_\beta = \arg\min v^\beta(\mu)$, and we can define the function

$$w^\beta(\nu) = v^\beta(\nu) - v^\beta(\mu_\beta) \tag{4.1}$$

By an algebraic transformation of the discounted Bellman equation we obtain

$$w^\beta(\nu) + (1-\beta)v^\beta(\mu_\beta) = \inf_{a\in U} \Big[\int_E c(x,a)\nu(dx) + \beta \prod^a(\nu, w^\beta) \Big] \tag{4.2}$$

Clearly $w^\beta(\nu) \geq 0$ and $w^\beta \in C(P(E))$. If $\{w^\beta, \beta \in (0,1)\}$ forms a relatively compact family in $C(P(E))$, since $\|(1-\beta)v^\beta\| \leq \|c\|$, one can find a subsequence $\beta_n \uparrow 1$, a constant γ and a function $w \in C(P(E))$ such that

$$\begin{aligned} & w^{\beta_n}(\nu) \to w(\nu) \quad \text{in } C(P(E)) \text{ as } n \to \infty \\ & (1-\beta_n)v^{\beta_n}(\mu_{\beta_n}) \to \gamma \end{aligned} \tag{4.3}$$

Consequently letting $\beta_n \uparrow 1$ in (4.2) and noticing that, by (4.3),

$$|\prod^a(\nu, w^{\beta_n}) - \prod^a(\nu, w)| \to 0$$

uniformly in $a \in U$ as $n \to \infty$, we obtain

$$w(\nu) + \gamma = \inf_{a\in U} \Big[\int_E c(x,a)\nu(dx) + \prod^a(\nu, w)\Big] \tag{4.4}$$

Equation (4.4) is called ergodic Bellman equation. The existence of solutions to (4.4), in particular the relative compactness of $\{w^\beta, \beta \in (0,1)\}$, the construction of nearly optimal control functions for the cost (1.6) and the study of the case when E is only locally compact will be the subject of the following subsections.

4.2.1 Existence of solutions to the ergodic Bellman equation for the case of a compact state space E

In addition to (A1)–(A5) in this section we make the following assumption

(A6) $\inf_{z,z'\in E} \ \inf_{a,a'\in U} \ \inf_{C\in\mathcal{B}(E), P^a(z,C)>0} \ \frac{P^{a'}(z',C)}{P^a(z,C)} := \lambda > 0$

Remark 4.1 *In the case of a finite state space E, the assumption (A6) has the form*

$$\min_{z,z'\in E} \ \inf_{a,a'\in U} \ \min_{x\in E, P^a(z,x)>0} \ \frac{P^{a'}(z',x)}{P^a(z,x)} > 0 \tag{4.5}$$

and its meaning is that if we enter the state x with a positive probability from a state z with control a, then we enter x starting from any $z' \in E$, and using any control $a' \in U$, with probability bounded away from 0.

Remark 4.2 *If E is nonfinite the assumption (A6) says that the transition probabilities for different controls and initial states are mutually equivalent with the Radon-Nikodym density bounded away from 0. In particular, if for some $\eta \in P(E)$, $P^a(z,C) = \int_C g^a(z,x)\eta(dx)$, for each $a \in U$, $z \in E$, then the assumption*

$$\inf_{z,z'\in E} \ \inf_{a,a'\in U} \ \inf_{x\in E, g^a(z,x)>0} \ \frac{g^{a'}(z',x)}{g^a(z,x)} > 0 \tag{4.6}$$

is sufficient for (A6) to be satisfied.

We start proving two preliminary results. First we have

Proposition 4.3 *Under (A1)–(A6) assuming additionally that E is compact, for $\beta \in (0,1)$, $\nu \in P(E)$ we have*

$$0 \le w^\beta(\nu) \le \frac{\|c\|}{\lambda^2} \tag{4.7}$$

P r o o f. Let, for $n = 0, 1, 2, \ldots$,

$$w_n^\beta(\nu) = v_n^\beta(\nu) - v_n^\beta(\mu_\beta^n) \tag{4.8}$$

where v_n^β is defined in (3.5) and $\mu_\beta^n = \arg\min v_n^\beta$. By (3.6), for any $\beta \in (0,1)$, w_n^β converges uniformly as $n \to \infty$ to w^β.

Therefore it suffices to show the inequality (4.7) for w_n^β. Clearly, by the very definition, $w_n^\beta(\nu) \ge 0$. We show by induction that, for $n = 0, 1, 2, \ldots$,

$$w_n^\beta(\nu) \le \frac{\|c\|}{\lambda^2} \quad \text{for } \nu \in P(E),\ \beta \in (0,1) \tag{4.9}$$

For $n = 0$, we have $w_n^\beta(\nu) \equiv 0$ and (4.9) holds. Assume now that for a generic $n > 0$, (4.9) is satisfied. Fix $\nu \in P(E)$. By (3.7) there exist $a, a' \in U$ such that

$$\begin{aligned} w_{n+1}^\beta(\nu) = &\int_E c(x,a)\nu(dx) - \int_E c(x,a')\mu_\beta^{n+1}(dx) + \\ &+\beta[\Pi^a(\nu, v_n^\beta) - \Pi^{a'}(\mu_\beta^{n+1}, v_n^\beta)] \end{aligned} \tag{4.10}$$

For $y \in R^d$ and $B \in \mathcal{B}(E)$ define

$$m(y)(B) = M^{a'}(y, \mu_\beta^{n+1})(B) - \lambda^2 M^a(y,\nu)(B) \tag{4.11}$$

with $M^a(y,\nu)$ as in (1.8).

Since by (A6)

$$\begin{aligned} &\int_B r(z,y) \int_E P^{a'}(z_1', dz)\mu_\beta^{n+1}(dz_1') \ge \\ &\lambda \int_B r(z,y) \int_E P^a(z_1, dz)\nu(dz_1) = \\ &= \lambda M^a(y,\nu)(B) \cdot \int_E r(z,y) \int_E P^a(z_1,dz)\nu(dz_1) \ge \\ &\ge \lambda^2 M^a(y,\nu)(B) \int_E r(z,y) \int_E P^{a'}(z_1', dz)\mu_\beta^{n+1}(dz_1') \end{aligned}$$

we have $m(y)(B) \geq 0$ for $B \in \mathcal{B}(E)$, $y \in R^d$.

From (A6) it is clear that $0 \leq \lambda \leq 1$. If $\lambda = 1$ we have a stationary, noncontrolled Markov chain with $P^a(z, C) = \eta(C)$, for $a \in U$, $z \in E$, and fixed $\eta \in P(E)$, and consequently $w_n^\beta \equiv 0$ for $n = 0, 1, 2, \ldots$. Therefore it remains to consider the case $\lambda < 1$ for which

$$(1-\lambda^2)^{-1} m(y) \in P(E).$$

Since

$$M^{a'}(y, \mu_\beta^{n+1}) = \lambda^2 M^a(y,\nu) + (1-\lambda^2)[(1-\lambda^2)^{-1} m(y)]$$

by the concavity of v_n^β (see (3.8)), we obtain

$$v_n^\beta(M^{a'}(y, \mu_\beta^{n+1})) \geq \lambda^2 v_n^\beta(M^a(y,\nu)) + (1-\lambda^2) v_n^\beta((1-\lambda^2)^{-1} m(y)) \quad (4.12)$$

Substituting (4.12) into (4.10), by the definition of $\prod^a(\nu, F)$ in (1.14) and taking into account the induction hypothesis (4.9) as well as the fact that $c \geq 0$ and $\mu_\beta^n = \arg\min v_n^\beta$, we obtain

$$\begin{aligned}
& w_{n+1}^\beta(\nu) \leq \|c\| + \beta \int_E \int_{R^d} v_n^\beta(M^a(y,\nu)) r(z,y) dy \\
& \Big(\int_E P^a(z_1, dz)\nu(dz_1) - \lambda^2 \int_E P^{a'}(z_1, dz)\mu_\beta^{n+1}(dz_1) \Big) \\
& -\beta(1-\lambda^2) \int_E \int_{R^d} v_n^\beta((1-\lambda^2)^{-1} m(y)) r(z,y) dy \int_E P^{a'}(z_1, dz)\mu_\beta^{n+1}(dz_1) \\
& = \|c\| + \beta \int_E \int_{R^d} (v_n^\beta(M^a(y,\nu)) - v_n^\beta(\mu_\beta^n)) r(z,y) dy \\
& \Big(\int_E P^a(z_1, dz)\nu(dz_1) - \lambda^2 \int_E P^{a'}(z_1, dz)\mu_\beta^{n+1}(dz_1) \Big) \\
& -\beta(1-\lambda^2) \int_E \int_{R^d} (v_n^\beta((1-\lambda^2)^{-1} m(y)) - v_n^\beta(\mu_\beta^n)) r(z,y) dy \\
& \int_E P^{a'}(z_1, dz)\mu_\beta^{n+1}(dz_1) \leq \|c\| + \beta \frac{\|c\|}{\lambda^2}(1-\lambda^2) \leq \frac{\|c\|}{\lambda^2}
\end{aligned}$$

which is exactly the induction hypothesis (4.9) for $n+1$. Thus by induction (4.9) holds and consequently we have (4.7).

■

To obtain our existence result, as we mentioned at the beginning of this section, we shall need the family $\{w^\beta, \beta \in (0,1)\}$ to be compact in $C(P(E))$ for E compact. From Proposition 4.3 we have the uniform boundedness of the family $\{w^\beta, \beta \in (0,1)\}$. To have the above family compact in $C(P(E))$, by the Ascoli-Arzela theorem (Theorem 9.33 of [28]), we shall need the equicontinuity of w^β with $\beta \in (0,1)$.

For this purpose we assume additionally

(A7) if $P(E) \ni \mu_n \Rightarrow \mu$ then $\sup_{a\in U} \sup_{C\in\mathcal{B}(E)} |P^a(\mu_n, C) - P^a(\mu, C)| \to 0$

Remark 4.4 *In the case of a finite state space $E = \{1, 2, \ldots, N\}$ a version of (A7) can be written as*

$$\sup_{a\in U} \sum_{k=1}^{N} \Big| \sum_{i=1}^{N} (s_i^n - s_i) P^a(i,k) \Big| \to 0 \quad \text{as } n \to \infty \tag{4.13}$$

for $s^n = (s_1^n, \ldots, s_N^n) \to s = (s_1, \ldots, s_N)$, $0 \le s_i^n \le 1$, $0 \le s_i \le 1$, $\sum s_i^n = 1$, $\sum s_i = 1$, and is clearly satisfied.

Remark 4.5 *In the case when $P^a(z, C) = \int_C g^a(z,x)\eta(dx)$ for $z \in E$, $a \in U$, $C \in \mathcal{B}(E)$ with $\eta \in P(E)$, the assumption that the mapping*

$$U \times E \times E \ni (a, z, x) \mapsto g^a(z,x) \quad \text{is continuous} \tag{4.14}$$

is sufficient for (A7) to be satisfied. In fact, by the compactness of E and the Stone Weierstrass theorem (Thm. 9.28 and Prob. 9.32 of [28]), $g^a(z,x)$ can be uniformly approximated by continuous functions of the form $\sum_{i=1}^{k} b_i(a)c_i(z)d_i(x)$, and then

$$\begin{aligned}
&\sup_{a\in U} \sup_{C\in\mathcal{B}(E)} |P^a(\mu_n, C) - P^a(\mu, C)| \le \\
&\le \sup_{a\in U} \int_E \Big| \int_E g^a(z,x)(\mu_n(dz) - \mu(dz)) \Big| \eta(dx) \\
&\le 2 \sup_{a\in U} \sup_{z,x\in E} \Big| g^a(z,x) - \sum_{i=1}^{k} b_i(a)c_i(z)d_i(x) \Big| + \\
&+ \sum_{i=1}^{k} \sup_{a\in U} |b_i(a)| \int_E |d_i(x)|\eta(dx) \Big| \int_E c_i(z)(\mu_n(dz) - \mu(dz)) \Big|
\end{aligned}$$

Therefore by the weak convergence of μ_n to μ we obtain (A7).

We can now formulate the result on the equicontinuity of w^β

Proposition 4.6 *Assume (A1)–(A7) and the compactness of E. Then the family of functions $\{w^\beta,\ \beta \in (0,1)\}$ is equicontinuous, i.e.*

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{\mu,\mu'\in P(E)}\ \rho_w(\mu,\mu') < \delta \Rightarrow \sup_{\beta\in(0,1)} |w^\beta(\mu) - w^\beta(\mu')| < \varepsilon \tag{4.15}$$

with ρ_w standing for a metric compatible with the weak topology on $P(E)$.

P r o o f. For $\nu, \mu \in P(E)$ define

$$\lambda(\nu,\mu) := \inf_{a\in U} \inf_{C\in\mathcal{B}(E), P^a(\mu,C)>0} \frac{P^a(\nu,C)}{P^a(\mu,C)} \tag{4.16}$$

By (A7) and using also (A6) we have that $\nu \Rightarrow \mu$ implies

$$\lambda(\nu,\mu) \to 1 \quad \text{as} \quad \lambda(\mu,\nu) \to 1 \tag{4.17}$$

Define for $a \in U$, $y \in R^d$, $\mu, \nu \in P(E)$, $B \in \mathcal{B}(E)$

$$m^a(y,\mu,\nu)(B) = M^a(y,\mu)(B) - \lambda(\mu,\nu)\lambda(\nu,\mu)M^a(y,\nu)(B) \tag{4.18}$$

Similarly as for $m(y)(B)$ in Proposition 4.3, we have that $m^a(y,\mu,\nu)(B) \geq 0$ for $B \in \mathcal{B}(E)$. Notice from (4.16) that $\lambda(\nu,\mu) \leq 1$ and $\lambda(\mu,\nu) \leq 1$. If $\lambda(\nu,\mu)\cdot\lambda(\mu,\nu) = 1$, then $\lambda(\nu,\mu) = 1$, $\lambda(\mu,\nu) = 1$ implying $P^a(\mu,\cdot) = P^a(\nu,\cdot)$ for $a \in U$, and so $P^a(\mu,w^\beta) = P^a(\nu,w^\beta)$ for $\beta \in (0,1)$, and consequently by (4.2)

$$|w^\beta(\mu) - w^\beta(\nu)| \leq \sup_{a\in U} \Big| \int_E c(x,a)(\mu(dx) - \nu(dx)) \Big| \tag{4.19}$$

Consider then $\overline{\lambda}^2 = \lambda(\mu,\nu)\cdot\lambda(\nu,\mu) < 1$. By the concavity of w^β we have

$$w^\beta(M^a(y,\mu)) \geq \overline{\lambda}^2 w^\beta(M^a(y,\nu)) + (1-\overline{\lambda}^2) w^\beta((1-\lambda^2)^{-1} m^a(y,\mu,\nu)) \tag{4.20}$$

From (4.2) we obtain

$$w^\beta(\nu) - w^\beta(\mu) \le \sup_{a\in U}\Big|\int_E c(x,a)(\nu(dx)-\mu(dx))\Big| +$$
$$+\beta\sup_{a\in U}(\textstyle\prod^a(\nu,w^\beta) - \prod^a(\mu,w^\beta) =$$
$$= \sup_{a\in U}\Big|\int_E c(x,a)(\nu(dx)-\mu(dx))\Big| +$$
$$+\beta\sup_{a\in U}\Big\{\int_E\int_{R^d} w^\beta(M^a(y,\nu))r(z,y)dy(P^a(\nu,dz) - \overline{\lambda}^2 P^a(\mu,dz))$$
$$+\int_E\int_{R^d}(\overline{\lambda}^2 w^\beta(M^a(y,\nu)) - w^\beta(M^a(y,\mu)))r(z,y)dyP^a(\mu,dz)\Big\}$$
$$= I + II + III \tag{4.21}$$

Now, by (4.7) and (4.16)

$$\begin{aligned} II &\le \beta\frac{\|c\|}{\lambda^2}\sup_{a\in U}\sup_{C\in\mathcal{B}(E)}(P^a(\nu,C) - \overline{\lambda}^2 P^a(\mu,C)) \le \\ &\le \beta\frac{\|c\|}{\lambda^2}\sup_{a\in U}\sup_{C\in\mathcal{B}(E)} P^a(\nu,C)(1-\overline{\lambda}^2\lambda(\mu,\nu)) = \\ &= \frac{\beta\|c\|}{\lambda^2}(1-\lambda^2(\mu,\nu)\lambda(\nu,\mu)) \end{aligned} \tag{4.22}$$

By (4.20) and the nonnegativity of w^β

$$III \le \sup_{a\in U}\beta\int_E\int_{R^d}(\overline{\lambda}^2 - 1)w^\beta((1-\lambda^2)^{-1}m^a(y,\mu,\nu))r(z,y)dyP^a(\mu,dz) \le 0 \tag{4.23}$$

Therefore substituting (4.22) and (4.23) into (4.21) we obtain

$$w^\beta(\nu) - w^\beta(\mu) \le \sup_{a\in U}\Big|\int_E c(x,a)(\nu(dx)-\mu(dx))\Big| +$$
$$+\frac{\beta\|c\|}{\lambda^2}(1-\lambda^2(\mu,\nu)\lambda(\nu,\mu))$$

Interchanging μ with ν in the previous inequality we finally have

$$|w^\beta(\nu) - w^\beta(\mu)| \leq \sup_{a\in U}\Big|\int_E c(x,a)(\nu(dx)-\mu(dx))\Big| + \\ + \frac{\beta\|c\|}{\lambda^2}(1-\lambda^2(\mu,\nu)\lambda^2(\nu,\mu)) \tag{4.24}$$

Recall from (4.19) that for $\overline{\lambda}^2 = 1$, (4.24) is trivially satisfied. Therefore by (4.17), (A5) and (4.24) we obtain (4.15).

■

We are now ready for our main existence result

Theorem 4.7 *Assume (A1)–(A7) and the compactness of E. Then there exist a function $w \in C(P(E))$ and a unique constant γ such that for $\nu \in P(E)$*

$$w(\nu) + \gamma = \inf_{a\in U}\Big[\int_E c(x,a)\nu(dx) + \prod^a(\nu,w)\Big] \tag{4.25}$$

Moreover there exists $u \in \mathcal{B}(P(E),U)$ for which the infimum on the right hand side of (4.25) is attained. The strategy $a_n = u(\pi_n)$ is optimal for J_μ and

$$J_\mu(u(\pi_n)) = \gamma \tag{4.26}$$

P r o o f. By Propositions 4.3, 4.6, using the Ascoli-Arzelá theorem we have that the family $\{w^\beta, \beta \in (0,1)\}$ is relatively compact in $C(P(E))$. Therefore by (4.2) and (4.3) we obtain the existence of $w \in C(P(E))$ and of γ for which (4.25) is satisfied. Since under the given assumptions the mapping

$$U \ni a \mapsto \int_E c(x,a)\nu(dx) + \prod^a(\nu,w)$$

is continuous, there exists a Borel selector $u \in \mathcal{B}(P(E),U)$. In a standard way from (4.25) and the definition of $J_\mu(u)$ in (1.6) using the boundedness of w we obtain then (4.26), from which the uniqueness of γ follows.

■

4.2.2 Control function approximations

The fact that the solutions to the ergodic Bellman equation can be obtained as suitable limits of discounted Bellman equations will be used below to the construction of nearly optimal control functions for ergodic cost problems.

Let us first notice that we have

Corollary 4.8 *Under (A1)–(A7) and the compactness of E we have*

$$\lim_{\beta \uparrow 1} \sup_{\mu \in P(E)} |(1-\beta)v^\beta(\mu) - \gamma| = 0 \tag{4.27}$$

P r o o f. By Proposition 4.6 and the fact that $\|(1-\beta)v^\beta\| \le \|c\|$ the family $\{(1-\beta)v^\beta, \beta \in (0,1)\}$ is relatively compact in $C(P(E))$. From any sequence $\beta_n \to 1$ one can then extract a subsequence β_{n_k} such that

$$(1-\beta_{n_k})v^{\beta_{n_k}} \to \gamma \quad \text{in } C(P(E))$$

By (4.3) and Theorem 4.7, γ is unique. Therefore we have (4.27).

■

Before we formulate our next result, recall first (see section 3.2) that if for given $\varepsilon \ge 0$ the function $u_\varepsilon^\beta \in \mathcal{B}(P(E), U)$ is such that

$$v^\beta(\nu) \ge \int_E c(x, u_\varepsilon^\beta(\nu))\nu(dx) + \beta \prod^{u_\varepsilon^\beta(\nu)}(\nu, v^\beta) - \varepsilon \tag{4.28}$$

for $\nu \in P(E)$, then the strategy $a_n = u_\varepsilon^\beta(\pi_n)$ is $\dfrac{\varepsilon}{1-\beta}$ optimal for the cost functional J_μ^β.

Corollary 4.9 *Under the assumptions of Corollary 4.8, if β is so large that*

$$\sup_{\mu \in P(E)} |(1-\beta)v^\beta(\mu) - \gamma| \le \varepsilon \tag{4.29}$$

and the function $u_\varepsilon^\beta \in \mathcal{B}(P(E), U)$ satisfies (4.28), then the strategy $a_n = u_\varepsilon^\beta(\pi_n)$ is $2\varepsilon + (1-\beta)\dfrac{\|c\|}{\lambda^2}$ optimal for the cost functional J_μ.

P r o o f. By (4.29) and (4.1) we have

$$w^\beta(\nu) + (1-\beta)v^\beta(\mu_\beta) \geq \int_E c(x, u_\varepsilon^\beta(\nu))\nu(dx) + \beta \prod^{u_\varepsilon^\beta(\nu)}(\nu, w^\beta) - \varepsilon$$

Using (4.29) and (4.7) we then obtain

$$w^\beta(\nu) + \gamma \geq \int_E c(x, u_\varepsilon^\beta(\nu))\nu(dx) + \prod^{u_\varepsilon^\beta(\nu)}(\nu, w^\beta) - 2\varepsilon - (1-\beta)\frac{\|c\|}{\lambda^2}$$

from which the required near optimality of the strategy $a_n = u_\varepsilon^\beta(\pi_n)$ follows, by considerations similar to the end of the proof of Theorem 4.7.

■

Remark 4.10 *As a consequence of the results of this subsection we now have that, in order to obtain a nearly optimal control function for J_μ^β, it suffices to determine a function u_ε^β satisfying (4.28). At this point we are in the same situation as pointed out at the end of Remark 3.3, so that, to practically determine the function u_ε^β, the further approximations discussed in Chapter 3 are required.*

4.2.3 The case of a locally compact state space

This subsection is devoted to the study of the ergodic Bellman equation for a locally compact state space E. When E is noncompact, several difficulties appear. As a starting point, by section 3.2, we still have the discounted Bellman equation (3.2). However a μ_β as minimizing argument of v^β may not exist. Therefore for a fixed $\delta > 0$ define μ_β as a δ-minimizer i.e.

$$v^\beta(\mu_\beta) - \delta \leq v^\beta(\nu) \quad \text{for } \nu \in P(E) \tag{4.30}$$

Let

$$w^\beta(\nu) = v^\beta(\nu) - v^\beta(\mu_\beta) \tag{4.31}$$

Then

$$w^\beta(\nu) \geq -\delta \quad \text{for } \nu \in P(E) \tag{4.32}$$

and

$$w^\beta(\nu) + (1-\beta)v^\beta(\mu_\beta) = \inf_{a\in U}\Big[\int_E c(x,a)\nu(dx) + \beta \prod^a(\nu, w^\beta)\Big] \tag{4.33}$$

One would again like to let $\beta \uparrow 1$ in (4.33). Since E is now locally compact, we cannot expect to find a subsequence β_n, for which w^{β_n} converges uniformly. However, the uniform convergence on compact subsets of $P(E)$, as will be clear later on, will guarantee the existence of solutions to a local version of the ergodic Bellman equation. We shall continue to assume also (A6) and (A7) which, for a locally compact state space, somewhat limit the class of models that can be considered.

However, if

$$x_{i+1} = F(x_i, a_i) + G(x_i)v_i \tag{4.34}$$

with $x_i \in R^n$, $F: R^n \times U \mapsto R$, $G^{-1}: R^n \mapsto R^n \times R^n$, continuous bounded, where $G^{-1}(x)$ is the inverse matrix of $G(x)$, and where v_i are n-dimensional i.i.d. standard Gaussian random variables, then (A6) and (A7) are satisfied. From the proofs of Propositions 4.3 and 4.6 we obtain

Corollary 4.11 *Under (A1)–(A7), for w^β defined by (4.31) we have*

$$-\delta \leq w^\beta(\nu) \leq \frac{\|c\|}{\lambda^2} + \frac{(1-\lambda^2)\delta}{\lambda^2} \tag{4.35}$$

P r o o f. By the same reasons as in the proof of Proposition 4.3 it suffices to show that for $n = 0, 1, 2, \ldots$,

$$w_n^\beta(\nu) \leq \frac{\|c\|}{\lambda^2} + \frac{(1-\lambda^2)\delta}{\lambda^2} \tag{4.36}$$

for $\nu \in P(E)$, $\beta \in (0,1)$, with $w_n^\beta(\nu) = v_n^\beta(\nu) - v_n^\beta(\mu_\beta^n)$, where this time μ_β^n is a δ-minimizer of v^β i.e. $v_n^\beta(\mu_\beta^n) - \delta \leq v_n^\beta(\nu)$ for $\nu \in P(E)$.

The inequality (4.36) is again shown by induction. The step $n = 0$ is clearly satisfied. Repeating the considerations of the proof of Proposition 4.3, assuming (4.36) for n, we obtain for $n+1$

$$\begin{aligned} w_{n+1}^\beta(\nu) &\leq \|c\| + \beta(1-\lambda^2)\Big(\frac{\|c\|}{\lambda^2} + \frac{(1-\lambda^2)\delta}{\lambda^2}\Big) + \beta(1-\lambda^2)\delta \\ &\leq \frac{\|c\|}{\lambda^2} + \frac{(1-\lambda^2)\delta}{\lambda^2} \end{aligned}$$

where the last inequality corresponds to putting $\beta = 1$ in the previous expression. Therefore (4.35) holds.

∎

Corollary 4.12 *Under (A1)–(A7), for any compact set $\Gamma \subset P(E)$ the family $\{w^\beta, \beta \in (0,1)\}$ is equicontinuous on Γ.*

P r o o f. By considerations similar to those of (4.16)–(4.24) we obtain

$$|w^\beta(\nu) - w^\beta(\mu)| \leq \sup_{a \in U} \Big| \int_E c(x,a)(\nu(dx) - \mu(dx)) \Big| +$$
$$+\beta\Big(\frac{\|c\|}{\lambda^2} + \frac{(1-\lambda^2)\delta}{\lambda^2}\Big)(1 - \lambda^2(\mu,\nu)\lambda^2(\nu,\mu)) + \beta\delta(1 - \lambda(\mu,\nu)\lambda(\nu,\mu)) \tag{4.37}$$

with $\lambda(\mu,\nu)$ and $\lambda(\nu,\mu)$ defined in (4.16).

Applying (4.17) to (4.37) and using (A5) we obtain the conclusion of Corollary 4.12.

■

Consider now a fixed compact set $\Gamma \subset P(E)$. By (1.19) the family of measures $\{\prod^a(\nu,\cdot), \nu \in \Gamma, a \in U\}$ is compact in $P(P(E))$. Therefore there exists (notice that $P(E)$ is a complete, separable, metric space, so we can use the Prokhorov theorem) a sequence (Γ_n^1) of compact subsets of $P(E)$, such that

$$\sup_a \sup_{\nu \in \Gamma} \prod^a(\nu, (\Gamma_n^1)^c) < \frac{1}{n} \tag{4.38}$$

Moreover the family of measures

$$\{\prod^{a_1}\prod^{a_2}\dots\prod^{a_i}(\nu,\cdot), \nu \in \Gamma, a_1, \dots, a_i \in U\}$$

is also compact in $P(P(E))$. Therefore again there exists a sequence (Γ_n^i) of compact subsets of $P(E)$ such that

$$\sup_{a_1,\dots,a_i \in U} \sup_{\nu \in \Gamma} \prod^{a_1}\prod^{a_2}\dots\prod^{a_i}(\nu, (\Gamma_n^i)^c) < \frac{1}{n} \tag{4.39}$$

Denote by

$$I(\Gamma) = \{\Gamma_n^i, i = 1,2,\dots,\ n = 1,2,\dots\}$$

the family of compact subsets defined above.

Let

$$I(I(\Gamma)) = \bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty} I(\Gamma_n^i)$$

and inductively

$$I^{n+1}(\Gamma) = I(I^n(\Gamma)) = \cup I(D)$$

with union over all D from $I^n(\Gamma)$.

Denote by Z the union of all elements of families $I(\Gamma), I^2(\Gamma), \ldots, I^n(\Gamma), \ldots$, and the set Γ.

By the very construction, for $\nu \in Z$, $n = 1, 2, \ldots$,

$$P_\nu\{\pi_n \notin Z\} = 0 \tag{4.40}$$

no matter what control strategy (a_n) was used.

We have

Theorem 4.13 *Under (A1)–(A7) there exist $w \in b\mathcal{B}(P(E))$ and a unique $\gamma \in R$ such that*

$$w(\nu) + \gamma = \inf_{a \in U} \Big[\int_E c(x,a)\nu(dx) + \prod^a(\nu, w) \Big] \tag{4.41}$$

for $\nu \in Z$.

Moreover the restriction of w to Z is continuous and there exists a function $u \in \mathcal{B}(P(E), U)$, such that for $\nu \in Z$

$$w(\nu) + \gamma = \int_E c(x, u(\nu))\nu(dx) + \prod^{u(\nu)}(\nu, w) \tag{4.42}$$

The strategy $a_n = u(\pi_n)$ is optimal for J_μ with $\mu \in Z$, and

$$J_\mu(u(\pi_n)) = \gamma \tag{4.43}$$

Furthermore γ is the optimal value of J_μ for all $\mu \in P(E)$.

P r o o f. Clearly Z is locally compact since it is the sum of an increasing sequence compact subsets in $P(E)$. By Corollaries 4.11 and 4.12 there exists a constant γ, a function $w \in C(Z)$ and a subsequence $\beta_n \uparrow 1$ such that

$$(1 - \beta_n)v^{\beta_n}(\mu_{\beta_n}) \to \gamma \tag{4.44}$$

and

$$w^{\beta_n} \to w$$

as $n \to \infty$ uniformly on compact subsets of Z.

For $\nu \in P(E) \setminus Z$ define

$$w(\nu) = \liminf_{\beta \to 1} w^\beta(\nu) \tag{4.45}$$

Letting $\beta_n \to 1$ as $n \to \infty$ in (4.33), by (4.44) and the construction of Z, we obtain (4.41) for $\nu \in Z$.

Since under the given assumptions the mapping

$$U \ni a \to \prod^a(\nu, w)$$

is continuous for $\nu \in Z$, there exists a Borel measurable function $u: Z \mapsto U$ such that (4.42) is satisfied. Extend now u to all of $P(E)$, by letting on $P(E) \setminus Z$, $u = a$ a fixed element of U. By (4.40) the strategy $a_n = u(\pi_n)$ is optimal for J_μ with $\mu \in Z$. Clearly, by (4.42), the equality (4.43) holds. Since for any $\mu_1, \mu_2 \in P(E)$ one can find a compact set $\Gamma \subset P(E)$, such that $\mu_1, \mu_2 \in \Gamma$ and so by (4.43) the optimal values of J_{μ_1} and J_{μ_2} coincide, we see that γ is in fact an optimal value for J_μ also with $\mu \in P(E)$.

■

4.3 Mixed observation model

Section 4.3 is devoted to the study of controlled Markov chains with a particular observation structure. Namely, we assume that the state space E is a closed subset of R^d and there exists a compact set $\Gamma \subset E$ in which the process (x_n) is completely observed. Outside of Γ, the observation is partial with a known density r of its distribution. To be more precise we have, with $y \in R^d$

$$P\{y_{n+1} \in A | x_0, x_1, \ldots, x_{n+1}, Y^n\} = \chi_{A \cap \Gamma}(x_{n+1}) + \chi_{\Gamma^c}(x_{n+1}) \int_{A \cap \Gamma^c} r(x_{n+1}, y) dy \tag{4.46}$$

for $n = 0, 1, \ldots$, and $A \in \mathcal{B}(E)$.

An observation of the form (4.46) arises when we are monitoring the system through a "window" Γ in which, due to the accuracy of measurements, we have complete observation of the state while outside of Γ the observations are noisy. In particular, one may consider the following example, that will be referred to also in the following sections.

Example. Let

$$x_{n+1} = Ax_n + f(x_n, a_n) + B(x_n)w_n \tag{4.47}$$

where w_n are independent standard Gaussian vectors on R^d, A is an asymptotically stable matrix, the functions f, B are continuous in their arguments and bounded with B possessing furthermore a bounded inverse (matrix). Assume Γ is a Cartesian product of intervals of the form $[-b_k, b_k], k = 1, 2, ..., d$, with $b_k > 0$.

Let the generic k-th coordinate y^k_{n+1} of y_{n+1} be equal to the k-th coordinate x^k_{n+1} of x_{n+1} when $x^l_{n+1} \in [-b_l, b_l]$, for each $l = 1, 2, ..., d$. Otherwise, y^k_{n+1} is equal to

$$\begin{aligned}
&max\{x^k_{n+1} + g(x_{n+1})v^k_{n+1}, x^k_{n+1}\} && \text{for } x^k_{n+1} \geq b_k,\\
&min\{max\{x^k_{n+1} + g(x_{n+1})v^k_{n+1}, -b_k\}, b_k\} && \text{for } -b_k \leq x^k_{n+1} \leq b_k,\\
&min\{x^k_{n+1} + g(x_{n+1})v^k_{n+1}, x^k_{n+1}\} && \text{for } x^k_{n+1} \leq -b_k,
\end{aligned} \tag{4.48}$$

where v^k_n for $k = 1, 2, ..., d$, and $n = 1, 2, ...$ are i.i.d. standard Gaussian, independent of (w_n), and g is a continuous bounded function with support equal to the closure of the complement of the Cartesian product of the intervals $[-b_k, b_k], k = 1, 2, ..., d$.

Notice that by the form of observation structure defined above, if x_n is in Γ^c, the observation y_n is also in Γ^c.

Clearly, under (4.48), we have an observation of the form (4.46). Moreover, as we shall see later on, the assumptions that we are going to make in the following section 4.3.1 are also satisfied by this example.

4.3.1 Study of the associated filter process

Given a controlled Markov process (x_n) with observations (y_n) satisfying (4.46) and admissible controls $u = (a_n)$ with a_n adapted to Y^n, we can define the filter process (π^u_n) as in (1.7).

With some abuse of notation, below we shall use the same symbols $M^a(y, \nu)$, $\prod^a(\nu, \cdot)$, $R(x, \cdot)$ as in chapter 3 to denote the corresponding quantities of this section with the particular observation structure (4.46).

By analogy to Lemma 1.1 and Lemma 1.3 we have

Lemma 4.14 *Under the observation structure (4.46) the associated filtering process* (π_n^u) *has the following representation*

$$\pi_{n+1}^u(A) = \chi_{A\cap\Gamma}(y_{n+1}) +$$

$$\chi_{\Gamma^c}(y_{n+1}) \int_{A\cap\Gamma^c} r(z, y_{n+1}) P^{a_n}(\pi_n^u, dz) \Big(\int_{\Gamma^c} r(z, y_{n+1}) P^{a_n}(\pi_n^u, dz) \Big)^{-1} \tag{4.49}$$

$$:= M^{a_n}(y_{n+1}, \pi_n^u)(A) := \chi_{A\cap\Gamma}(y_{n+1}) + \chi_{\Gamma^c}(y_{n+1}) N^{a_n}(y_{n+1}, \pi_n^u)(A)$$

P_μ *a.e., for* $A \in \mathcal{B}(E)$*, with the mappings* $M^a(y,\pi)$ *and* $N^a(y,\pi)$ *defined implicitly.*

Moreover, if $a_n = u(\pi_n)$ *for* $u \in \mathcal{B}(P(E), U)$*, then the filter process* (π_n^u) *is Markov with respect to the* σ*-field* Y^n *and has transition operator*

$$\begin{aligned} \textstyle\prod^{u(\nu)}(\nu, F) &= \int_\Gamma F(\delta_z) P^{u(\nu)}(\nu, dz) + \\ &+ \int_{\Gamma^c} \int_{\Gamma^c} F(M^u(y,\nu)) r(z,y) dy P^{u(\nu)}(\nu, dz) \end{aligned} \tag{4.50}$$

where $\nu \in P(E)$*,* $F \in b\mathcal{B}(P(E))$*, and* δ_z *denotes the Dirac measure at the point* z*.*

P r o o f. The proof is based on considerations similar to those of the proofs of Lemma 1.1 and Lemma 1.3, and is therefore left to the reader.

■

In what follows we shall also need the Feller property of the transition operators $\prod^{u(\nu)}(\nu, \cdot)$ with $u \in C(P(E), U)$ as well as of $\prod^a(\nu, \cdot)$ with $a \in U$ where

$$\begin{aligned} \textstyle\prod^{a}(\nu, F) &= \int_\Gamma F(\delta_z) P^{a}(\nu, dz) + \\ &+ \int_{\Gamma^c} \int_{\Gamma^c} F(M^a(y,\nu)) r(z,y) dy P^{a}(\nu, dz) \end{aligned} \tag{4.51}$$

for $F \in b\mathcal{B}(P(E))$.

For this purpose we make the following assumptions, where (A8) and (A9) are adaptations of (A3) and (A4) respectively

(A8) $r(z,y)$ is continuous for $z,y \in \Gamma^c$, and is bounded on the set $\Gamma^c_\delta = \{(z,y) \in \Gamma^c \times \Gamma^c, \rho_E(z,\Gamma) \geq \delta\} \cup \{(z,y) \in \Gamma^c \times \Gamma^c, \rho_E(z,y) \geq \delta\}$ for $\delta > 0$, with ρ_E standing for the metric on E. Moreover, if $\Gamma^c \ni y_m \to y \in \Gamma$ and $B(y,\delta) := \{z \in \Gamma^c : \rho_E(z,y) \leq \delta\}$ for $\delta > 0$, then

$$\inf_{a\in U} \inf_{x\in K} \int_{B(y,\delta)} r(z,y_m) P^a(x,dz) \to \infty \tag{4.52}$$

as $m \to \infty$, for any compact set $K \subset E$.

(A9) if $\Gamma^c \ni z_m \to z$, then

$$R(z_m,\cdot) \Rightarrow R(z,\cdot) \quad \text{as } m \to \infty$$

with

$$R(z,A) := \begin{cases} \int_{A\cap\Gamma^c} r(z,y)dy & \text{for } z \in \Gamma^c \\ \chi_A(z) & \text{for } z \in \Gamma \end{cases}$$

for $A \in \mathcal{B}(E)$.

(A10) for $x \in E$, $a \in U$

$$P^a(x,\partial\Gamma) = 0$$

with $\partial\Gamma$ standing for the boundary of Γ.

Let us notice that by (A9) the observation measure $R(z,\cdot)$ depends continuously on the state z and therefore the observation density $r(x,y)$ has to converge to infinity as y comes close to x, with x close to $\partial\Gamma$.

We have

Proposition 4.15 *Under (A1), (A2), (A8) and (A10) the mapping*

$$U \times R^d \times P(E) \ni (a,y,\nu) \mapsto M^a(y,\nu) \tag{4.53}$$

is continuous.

Assuming additionally (A9), for $F \in C(P(E))$ the mapping

$$U \times P(E) \ni (a,\nu) \mapsto \prod\nolimits^a(\nu,F) \tag{4.54}$$

is also continuous and for $u \in C(P(E), U)$

$$\prod^{u(\nu)}(\nu, \cdot) \quad \text{is Feller} \tag{4.55}$$

Finally if $v\colon P(E) \to R$ *is continuous and concave then for* $a \in U$ *the function*

$$P(E) \ni \nu \mapsto \prod^{a}(\nu, v) \tag{4.56}$$

is also continuous and concave.

P r o o f. To prove the continuity of the mapping (4.53) assume $U \ni a_m \to a$, $R^d \ni y_m \to y$, $P(E) \ni \nu_m \Rightarrow \nu$, as $m \to \infty$. Consider now three cases: $y \in \mathrm{Int}\Gamma$, i.e. y is in the interior of Γ; $y \in \partial\Gamma$; $y \in \Gamma^c$. In the first case the continuity of the mapping (4.53) is immediate. If $y \in \partial\Gamma$ it suffices to consider the case when $y_m \in \Gamma^c$ for large values of m, since otherwise, as in the first case, the continuity of the mapping (4.53) is immediate. For $y \in \partial\Gamma$ and $y_m \in \Gamma^c$ we have by (4.52) for $\delta > 0$

$$\int_{B(y,\delta)} r(z, y_m) P^{a_m}(\nu_m, dz) \to \infty$$

as $m \to \infty$.

Consequently, by (A8) and (4.49)

$$N^{a_m}(y_m, \nu_m) \Rightarrow \delta_y$$

as $m \to \infty$, and we have the continuity of the mapping (4.53) also for $y \in \partial\Gamma$.

The last case $y \in \Gamma^c$ can be shown similarly as in the proof of Proposition 1.4, taking into account (A10) and the fact that, by (A8), for fixed $y \in \Gamma^c$, $r(x, y)$ and $r(x, y_m)$ are bounded in $x \in \Gamma^c$ and $m = 1, 2, \ldots$.

Summarizing, $M^a(y, \nu)$ is therefore a continuous function of its arguments.

Consider now the mapping (4.54). Let $a_m \to a$ and $\nu_m \Rightarrow \nu$. Clearly it is sufficient to show the convergence

$$\begin{aligned} &\Big|\int_{\Gamma^c}\int_{\Gamma^c} r(z, y) F(M^{a_m}(y, \nu_m)) dy P^{a_m}(\nu_m, dz) \\ &- \int_{\Gamma^c}\int_{\Gamma^c} r(z, y) F(M^{a}(y, \nu)) dy P^{a}(\nu, dz)\Big| \to 0 \end{aligned} \tag{4.57}$$

for $F \in C(P(E))$, as $m \to \infty$.

By the continuity of the mapping (1.20), for any $\varepsilon > 0$ one can find a compact set $K \subset E$ such that

$$P^{a_m}(\nu_m, K) \geq 1 - \varepsilon \quad \text{for } m = 1, 2, \dots, \tag{4.58}$$

Given the compact set K, by (A9) the family of measures $\{R(z, \cdot)\ z \in K\}$ is compact and so there exists a set $L \subset E$ such that

$$\int\limits_{L \cap \Gamma^c} r(z, y) dy \geq 1 - \varepsilon \quad \text{for } z \in K \cap \Gamma^c \tag{4.59}$$

By (4.58) and (4.59) we now have

$$\begin{aligned}
&\Big|\int\limits_{\Gamma^c}\int\limits_{\Gamma^c} r(z,y)F(M^{a_m}(y,\nu_m))dyP^{a_m}(\nu_m,dz) \\
&-\int\limits_{\Gamma^c}\int\limits_{\Gamma^c} r(z,y)F(M^{a}(y,\nu))dyP^{a}(\nu,dz)\Big| \\
&\leq \int\limits_{\Gamma^c}\int\limits_{\Gamma^c} r(z,y)|F(M^{a_m}(y,\nu_m)) - F(M^{a}(y,\nu))|dyP^{a_m}(\nu_m,dz) \\
&+\Big|\int\limits_{\Gamma^c}\int\limits_{\Gamma^c} r(z,y)F(M^{a}(y,\nu))dy(P^{a_m}(\nu_m,dz) - P^{a}(\nu,dz))\Big| \\
&\leq \int\limits_{\Gamma^c \cap K}\int\limits_{\Gamma^c \cap L} r(z,y)|F(M^{a_m}(y,\nu_m)) - F(M^{a}(y,\nu))dyP^{a_m}(\nu_m,dz) \\
&+4\|F\|\varepsilon+ \\
&+\Big|\int\limits_{E} \chi_{\Gamma^c}(z)\int\limits_{\Gamma^c} r(z,y)F(M^{a}(y,\nu))dy(P^{a_m}(\nu_m,dz) - P^{a}(\nu,dz))\Big| \\
&= I_m + 4\|F\|\varepsilon + II_m
\end{aligned} \tag{4.60}$$

Since

$$I_m \leq \sup_{y \in L} |F(M^{a_m}(y,\nu_m)) - F(M^{a}(y,\nu))| \to 0$$

and by (A10) and the continuity of the mapping (1.20), $II_m \to 0$, we obtain (4.57). The mapping (4.54) is therefore continuous.

The Feller property of $\prod^{u(\nu)}(\nu, \cdot)$ with $u \in C(P(E), U)$ follows immediately from the continuity of the mapping (4.54). It therefore remains to show the concavity of the mapping

$$P(E) \ni \nu \mapsto \prod^a(\nu, v)$$

with v continuous concave.

An analysis of the proof of Proposition 1.7 shows that we can adapt the same considerations to the mixed observation model and consequently the mapping (4.56) is concave.

The proof of Proposition 4.15 is complete.

■

Remark 4.16 *If instead of (A8) we assume*

(A8') $r(x, y)$ is continuous and bounded on Γ^c

then we have the mapping (4.53) continuous for $y \notin \partial\Gamma$. This case corresponds to a discontinuous observation structure. Namely, at the boundary $\partial\Gamma$ we may have an abrupt change in the precision of the observation. Replacing (A9) by

(A9') for $x_n \to x \in \Gamma^c$ we have $R(x_n, \cdot) \Rightarrow R(x, \cdot)$, and for any compact set $K \subset E$, the family $\{R(x, \cdot)\ x \in K \cap \Gamma^c\}$ is tight,

then by considerations similar to those of (4.58)–(4.60) we obtain the convergence (4.57). Consequently, the mapping (4.54) is continuous and the Feller property (4.55) and the concavity of (4.56) hold as well.

Remark 4.17 *When the state space E is finite or countable i.e. $E = \{1, 2, \ldots, m\}$ or $E = \{1, 2, \ldots, m, \ldots\}$ and also the observation process takes its values in E, we can consider an analog of (4.46) as follows*

$$P\{y_{n+1} = i | x_0, \ldots, x_n, x_{n+1} = j,\ Y^n\} = \chi_{\Gamma}(j)\delta_j(i) + \chi_{\Gamma^c}(j) r(j, i) \quad (4.61)$$

with $\Gamma \subset E$ a finite set corresponding to the complete observation subset of the state process. Assuming

$$U \ni a \to p^a(i, j) \quad \text{continuous for } i, j \in E \quad (4.62)$$

where $p^a(i, j)$ is the transition matrix of the controlled Markov chain (x_n), all assertions of Proposition 4.15 hold also for this case.

4.3.2 Solution to the Bellman equation

By analogy to the case of partial observation i.e. the case when the observations (y_n) of (x_n) satisfy (1.1), for the mixed observation problem i.e. the problem with the observation structure (4.46) we can consider partially observed control problems with cost functionals (1.4)–(1.6) or in equivalent form (1.10)–(1.12), where (π_n^u) is now given by (4.49).

To study the Bellman equation for the ergodic cost problem with mixed observation structure, following the approach of section 4.2, we start from the Bellman equation for the discounted cost problem. Notice first that, for the discounted cost functional J_μ^β, from Theorem 3.1 we obtain the following

Corollary 4.18 *Under (A1), (A2), (A5), (A8)–(A10) all assertions of Theorem 3.1 hold also for operators* $\prod^a$ *and* $\prod^{u(\nu)}(\nu, \cdot)$ *as defined in (4.51) and (4.50) respectively.*

P r o o f. For the proof of Theorem 3.1 we needed Proposition 1.4 and Proposition 1.7, the analog of which for the mixed observation case is shown in Proposition 4.15. Therefore, repeating the same considerations as in the proof of Theorem 3.1 we obtain Corollary 4.18.

■

Starting from the solution $v^\beta \in C(P(E))$ of the discounted Bellman equation corresponding to the mixed observation problem, namely

$$v^\beta(\mu) = \inf_{a \in U} \Big\{ \int_E c(x,a)\mu(dx) + \beta \prod^a (\mu, v^\beta) \Big\} \tag{4.63}$$

in the case when E is compact we can define w^β as in (4.1) i.e.

$$w^\beta(\nu) = v^\beta(\nu) - v^\beta(\mu_\beta) \tag{4.64}$$

with $\mu_\beta = \arg\min v^\beta$, and we may expect that for a certain subsequence $\beta_n \uparrow 1$ we have

$$w^{\beta_n}(\nu) \to w(\nu) \quad \text{in } C(P(E))$$

$$(1-\beta_n) v^{\beta_n}(\mu_{\beta_n}) \to \gamma$$

where $w \in C(P(E))$ and γ are the solutions to the ergodic Bellman equation

$$w(\nu) + \gamma = \inf_{a \in U} \Big[\int_E c(x,a)\nu(dx) + \prod^a(\nu, w) \Big] \tag{4.65}$$

In fact, treating first the case when the state space E is compact we have by analogy to Theorem 4.7 and Corollary 4.8

Theorem 4.19 *Under (A1), (A2), (A5)–(A10) and the compactness of E, there exist a function $w \in C(P(E))$ and a unique constant γ for which (4.65) is satisfied.*

Moreover there exists $u \in \mathcal{B}(P(E), U)$ such that the infimum on the right hand side of (4.65) is attained. Furthermore the strategy $a_n = u(\pi_n)$ is optimal for J_μ and

$$J_\mu(u(\pi_n)) = \gamma \tag{4.66}$$

Finally we have

$$\lim_{\beta \uparrow 1} \sup_{\mu \in P(E)} |(1-\beta)v^\beta(\mu) - \gamma| = 0 \tag{4.67}$$

P r o o f. From an analysis of the proofs of Theorem 4.7 and Corollary 4.8 it follows that it is sufficient to show the uniform boundedness (4.7) and the equicontinuity (4.15) of w^β, $\beta \in (0,1)$ also in the present case.

We therefore adapt the proofs of Proposition 4.3 and Proposition 4.6 to the mixed observation case. Define w_n^β as in (4.8). In view of the proof of Proposition 4.3, to have the boundedness of the family $\{w^\beta, \beta \in (0,1)\}$, it suffices to show that

$$\sup_{\nu \in P(E)} w_n^\beta(\nu) \le \frac{\|c\|}{\lambda^2} \quad \text{for } \beta \in (0,1) \tag{4.68}$$

implies

$$\sup_{\nu \in P(E)} w_{n+1}^\beta(\nu) \le \frac{\|c\|}{\lambda^2} \quad \text{for } \beta \in (0,1) \tag{4.69}$$

By analogous reasons as in Proposition 4.3 we can restrict ourselves to the case $\lambda^2 < 1$. For fixed $\nu \in P(E)$, let then $a, a' \in U$ be such that

$$\begin{aligned} w_{n+1}^\beta(\nu) \; &= \int_E c(x,a)\nu(dx) - \int_E c(x,a')\mu_\beta^{n+1}(dx) + \\ &+\beta[\textstyle\prod^a(\nu, v_n^\beta) - \prod^{a'}(\mu_\beta^{n+1}, v_n^\beta)] \end{aligned} \tag{4.70}$$

and define by analogy to (4.11)

$$m(y)(B) = M^{a'}(y, \mu_\beta^{n+1})(B) - \lambda^2 M^a(y,\nu)(B) \quad \text{for } B \in \mathcal{B}(E)$$

from which, based on (4.49) and letting $y \in \Gamma^c$, we obtain

$$N^{a'}(y, \mu_\beta^{n+1}) = \lambda^2 N^a(y,\nu) + (1-\lambda^2)[(1-\lambda^2)^{-1} m(y)]$$

By the concavity of v_n^β (see Corollary 4.18) we then have

$$v_n^\beta(N^{a'}(y,\mu_\beta^{n+1})) \geq \lambda^2 v_n^\beta(N^a(y,\nu)) + (1-\lambda^2) v_n^\beta((1-\lambda^2)^{-1} m(y))$$

Therefore, using (4.49) and the definition of $\prod^a(\nu,\cdot)$ in (4.51) we obtain

$$\begin{aligned}
&w_{n+1}^\beta(\nu) \leq \|c\| + \beta\Big[\int\limits_\Gamma v_n^\beta(\delta_z)(P^a(\nu,dz) - P^{a'}(\mu_\beta^{n+1} dz))\Big]\\
&+\beta\Big[\int\limits_{\Gamma^c}\int\limits_{\Gamma^c} r(z,y) v_n^\beta(N^a(y,\nu)) P^a(\nu,dz)dy\\
&-\int\limits_{\Gamma^c}\int\limits_{\Gamma^c} r(z,y) v_n^\beta(N^{a'}(y,\mu_\beta^{n+1})) P^{a'}(\mu_\beta^{n+1},dz)dy\Big]\\
&\leq \|c\| + \beta\Big[\int\limits_\Gamma v_n^\beta(\delta_z)(P^a(\nu,dz) - P^{a'}(\mu_\beta^{n+1} dz))\Big]\\
&+\beta\Big[\int\limits_{\Gamma^c}\int\limits_{\Gamma^c} r(z,y) v_n^\beta(N^a(y,\nu))dy(P^a(\nu,dz) - \lambda^2 P^{a'}(\mu_\beta^{n+1},dz))\Big]\\
&-\beta(1-\lambda^2)\int\limits_{\Gamma^c}\int\limits_{\Gamma^c} v_n^\beta((1-\lambda^2)^{-1} m(y)) r(z,y) dy P^{a'}(\mu_\beta^{n+1},dz)\\
&= \|c\| + \beta\Big[\int\limits_\Gamma (v_n^\beta(\delta_z) - v_n^\beta(\mu_\beta^n))(P^a(\nu,dz) - \lambda^2 P^{a'}(\mu_\beta^{n+1} dz))\Big]\\
&-\beta(1-\lambda^2)\int\limits_\Gamma (v_n^\beta(\delta_z) - v_n^\beta(\mu_\beta^n)) P^{a'}(\mu_\beta^{n+1},dz)\\
&+\beta\Big[\int\limits_{\Gamma^c}\int\limits_{\Gamma^c} r(z,y)(v_n^\beta(N^a(y,\nu)) - v_n^\beta(\mu_\beta^n))dy(P^a(\nu,dz)\\
&-\lambda^2 P^{a'}(\mu_\beta^{n+1},dz))\Big] - \beta(1-\lambda^2)\int\limits_{\Gamma^c}\int\limits_{\Gamma^c}(v_n^\beta((1-\lambda^2)^{-1} m(y))\\
&-v_n^\beta(\mu_\beta^n)) r(z,y) dy P^{a'}(\mu_\beta^{n+1},dz) = I + II + III + IV + V
\end{aligned}$$

where the last equality follows from an appropriate adding and subtracting of terms.

Since μ_β^n is a minimizer of v_n^β, we have $III \leq 0$ and $V \leq 0$. Recalling the definition (4.8) and applying (4.68) to II and IV we finally obtain

$$w_{n+1}^\beta(\nu) \leq \|c\| + \frac{\|c\|}{\lambda^2}(1-\lambda^2) = \frac{\|c\|}{\lambda^2}$$

i.e. (4.69) is satisfied.

The family $\{w^\beta, \beta \in (0,1)\}$ is thus uniformly bounded by $\dfrac{\|c\|}{\lambda^2}$ and it remains now to show the equicontinuity of w^β, with $\beta \in (0,1)$.

For $\mu, \nu \in P(E)$ let $\lambda(\nu, \mu)$ be given by (4.16). Define the measure $m^a(y, \mu, \nu)$ analogously to (4.18).

Similarly, as in the proof of Proposition 4.6 it suffices to consider the case when $\overline{\lambda}^2 := \lambda(\mu,\nu)\lambda(\nu,\mu) < 1$. Notice that from (4.49) we then obtain

$$M^a(y,\mu) = \overline{\lambda}^2 M^a(y,\nu) + (1-\overline{\lambda}^2)[(1-\overline{\lambda}^2)^{-1} m^a(y,\mu,\nu)]$$

so that, by the concavity for w^β defined by (4.64), the inequality (4.19) holds.

From (4.63) and (4.64) we then have (compare to (4.21))

$$\begin{aligned}
&w^\beta(\nu) - w^\beta(\mu) \leq \sup_{a\in U} \Big| \int_E c(x,a)(\nu(dx) - \mu(dx)) \Big| \\
&+ \sup_{a\in U} \Big\{ \beta \int_\Gamma w^\beta(\delta_z)(P^a(\nu,dz) - \overline{\lambda}^2 P^a(\mu,dz) \Big\} \\
&+ \sup_{a\in U} \Big\{ \beta(\overline{\lambda}^2 - 1) \int_\Gamma w^\beta(\delta_z) P^a(\mu,dz) \Big\} \\
&+ \sup_{a\in U} \Big\{ \beta \int_{\Gamma^c}\int_{\Gamma^c} w^\beta(M^a(y,\nu)) r(z,y) dy (P^a(\nu,dz) - \overline{\lambda}^2 P^a(\mu,dz)) \Big\} \\
&+ \sup_{a\in U} \Big\{ \beta \int_{\Gamma^c}\int_{\Gamma^c} (\overline{\lambda}^2 w^\beta(M^a(y,\nu)) - w^\beta(M^a(y,\mu))) r(z,y) dy P^a(\mu,dz) \Big\} \\
&= I + II + III + IV + V
\end{aligned}$$

Clearly $III \leq 0$, and, applying (4.20) to V, also $V \leq 0$. By analogy to (4.22) we obtain

$$II + IV \leq 2\frac{\|c\|}{\lambda^2}(1 - \lambda^2(\mu,\nu)\lambda(\nu,\mu))$$

Therefore finally

$$|w^\beta(\nu) - w^\beta(\mu)| \le \sup_{a\in U} \Big| \int_E c(x,a)(\nu(dx) - \mu(dx))$$
$$+\frac{2\|c\|}{\lambda^2}(1 - \lambda^2(\mu,\nu)\lambda^2(\nu,\mu))$$

from which the equicontinuity of $\{w^\beta, \beta \in (0,1)\}$ follows. This finishes the proof of Theorem 4.19.

■

Remark 4.20 *Under the assumptions of Theorem 4.19, also the statement of Corollary 4.9 holds and Remark 4.10 applies. Moreover, we can repeat the considerations of section 4.2.3 and obtain results analogous to those of Corollary 4.11, Corollary 4.12 and Theorem 4.13 also for the case of mixed observations with locally compact state space E.*

Remark 4.21 *If, as in Remark 4.17 the state space is finite, the statement of Theorem 4.19 holds true requiring only assumption (A6), together with the continuity condition (4.62).*

4.4 Invariant measures for controlled filtering processes

From Theorems 4.7 and 4.19 it is clear that, under certain assumptions, among the optimal controls for the ergodic cost functional (1.12) there are controls of the form $a_n = u(\pi_n)$, with $u \in \mathcal{B}(P(E), U)$ a fixed mapping. The filtering process (π_n^u) that corresponds to a control of this form is by Lemma 1.3 and Lemma 4.14 (in the case of mixed observations) Markov, with transition operator $\prod^{u(\nu)}(\nu, \cdot)$ given by (1.14) or (4.50) respectively. It is well known that the limit behaviour of iterations of transition operators of Markov processes can be described in terms of integrals over an invariant measure $\Phi^u \in P(P(E))$, which by definition satisfies the following condition

$$\int_{P(E)} F(\nu)\Phi^u(d\nu) = \int_{P(E)} \prod^{u(\nu)}(\nu, F)\Phi^u(d\nu) \tag{4.71}$$

for any $F \in b\mathcal{B}(P(E))$.

For the purpose of studying limit theorems as well as approximations, one would like to have conditions guaranteeing the existence of a unique invariant measure Φ^u for $u \in \mathcal{B}(P(E), U)$, or at least $u \in C(P(E), U)$. As we shall see below, the general problem of the uniqueness of Φ^u seems to be hard, and we are so far able to solve it only in the particular cases studied in sections 4.4.2 and 4.4.3 below.

4.4.1 A counterexample

To better see the nature of the problems that may arise with the existence and uniqueness of invariant measures for controlled partially observed Markov processes, we begin with a fundamental result concerning uncontrolled Markov processes (x_n) with transition operator $P(x, dz)$ and observations y_n satisfying (1.1), or (4.46) in the case of mixed observations.

Theorem 4.22 *Assume the Markov process (x_n) is Feller, has a unique invariant measure μ and the filtering process (π_n), that corresponds to observations y_n satisfying (1.1) or (4.46), is also Feller. Then (π_n) has a unique invariant measure Φ if and only if for each $f \in C(E)$*

$$\overline{\lim_{n\to\infty}} \int_E |P^n f(x) - \mu(f)| \mu(dx) = 0 \tag{4.72}$$

P r o o f . For a compact state space E and an observation model with additive noise we may follow the considerations of [19], applying suitable versions of Proposition 1.7. To extend the result to more general cases of state and observation models one may exploit the approach of [35].

■

Remark 4.23 *Notice that if (x_n) is ergodic and aperiodic, condition (4.72) is automatically satisfied. Furthermore Theorem 4.22 completely characterizes the situation, when there exists a unique invariant measure of the filtering process, in terms of ergodic properties of the underlying state process.*

Coming now to controlled Markov processes we find that the situation is much more complicated:

Even a nice behaviour of the controlled state process (x_n) does not necessarily imply the existence of a unique invariant measure for the associated filtering process (π_n).

To clarify this point we now present an example for which in the uncontrolled case there exists a unique invariant measure, while in the controlled case uniqueness fails.

Example. Assume we can find measures $\eta_1, \eta_2 \in P(E)$ and a function $r(x,y)$ for which (A3) and (A4) holds, such that the sets $S_1 = \{Q(y,\eta_1)(\cdot), y \in D\}$ and $S_2 = \{Q(y,\eta_2)(\cdot), y \in D\}$, with the operator Q defined as in (3.200) and $D = R^d$, have disjoint closures in $P(E)$, that can be separated. In other words we require the distance (in the metric of $P(E)$) between S_1 and S_2 to be positive. Such a requirement is e.g. easily seen to be satisfied in the somewhat "pathological" example when $r(x,y)$ is a function of y only and $\eta_1 \neq \eta_2$. Moreover, it is also satisfied in the case when $E = \{1,\ldots,m\}$ and $D = \{d_1,\ldots,d_s\}$ for which one can easily find $\eta_1, \eta_2 \in P(E)$ and the matrix $r(i,d_j)$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,s$, such that the separation of S_1 and S_2 holds.

Next choose a Markov kernel $P^a(x,\cdot)$ such that for certain $a_1 \neq a_2 \in U$, and all $x \in E$

$$P^{a_1}(x,\cdot) = \eta_1(\cdot), \quad P^{a_2}(x,\cdot) = \eta_2(\cdot)$$

and, furthermore for $a \in U$ the assumptions (A1) and (A2) are satisfied.

Due to the separation of S_1 and S_2, we can now choose $u \in C(P(E),U)$ such that

$$u(\nu) = a_1 \text{ if } \nu \in S_1 \quad \text{and} \quad u(\nu) = a_2 \text{ if } \nu \in S_2$$

By (1.8) and their definition, the sets S_1 and S_2 are then clearly invariant for the filtering processes (π_n^u) starting from S_1 or S_2 respectively. Assuming additionally that E is compact, we know that so is $P(E)$ and therefore the Cesaro averages

$$\frac{1}{n}\sum_{i=0}^{n-1}(\prod\nolimits^{u(\nu_1)})^i(\nu_1,\cdot), \quad \frac{1}{n}\sum_{i=0}^{n-1}(\prod\nolimits^{u(\nu_2)})^i(\nu_2,\cdot),$$

with $\nu_1 \in S_1$ and $\nu_2 \in S_2$ are tight for $n = 1,2,\ldots$.

By the Feller property of $\prod^u$ (see Corollary 1.5), one can choose subsequences (n_k), $(n_{k'})$ such that

$$\frac{1}{n_k}\sum_{i=0}^{n_k-1}(\prod^{u(\nu_1)})^i(\nu_1,\cdot)\Rightarrow\Phi_1(\cdot)$$
$$\frac{1}{n'_k}\sum_{i=0}^{n_{k'}-1}(\prod^{u(\nu_2)})^i(\nu_2,\cdot)\Rightarrow\Phi_2(\cdot)$$

as $k,k'\to\infty$, where Φ_1, Φ_2 are two invariant measures of (π_n^u) with supports contained in the disjoint closures of the sets S_1 and S_2 respectively.

Assuming finally that for each fixed $a\in U$, there exists a unique invariant measure $\mu^a\in P(E)$ corresponding to the transition operator $P^a(x,\cdot)$, and that for this measure the condition (4.72) is satisfied, by Theorem 4.22, we have that for each constant control $u\equiv a$, $a\in U$ (the uncontrolled case), there exists a unique invariant measure Φ^u for the filtering process (π_n^u). If however we consider the case of a control u that takes values a_1 in S_1 and a_2 in S_2 we have already seen that there exist at least two invariant measures for the filtering process (π_n^u). Notice that in the case when $E=\{1,2,\dots,m\}$, $D=\{d_1,\dots,d_s\}$, a condition sufficient for (4.72) to be satisfied is, by Remark 4.23, the assumption that for each $a\in U$, the transition probabilities $P^a(i,k)$ are strictly positive for all i,k in E.

Remark 4.24 *It is worth pointing out that we may have two disjoint invariant sets but one invariant measure for the filtering processes. In fact, if in an uncontrolled case with a finite state space $E=\{1,2,\dots,m\}$ and an observation space $D=\{d_1,\dots,d_s\}$, the transition matrix $P(i,j)$ is such that $P(k,1)=0$ for $k=2,\dots,m$, $P(k,m)=0$ for $k=1,2,\dots,m-1$ and $P(i,j)>0$ elsewhere, then, provided $r(i,d_j)>0$ for all $i\in E$ and all $d_j\in D$, the sets $S_1=\{\mu\in P(E),\mu(1)=0 \text{ and } \mu(m)>0\}$, $S_2=\{\mu\in P(E),\mu(1)>0 \text{ and } \mu(m)=0\}$ are disjoint and invariant for π_n. However, since there exists a unique invariant measure μ for the Markov process (x_n), and condition (4.72) is satisfied, by Theorem 4.22 there exists a unique invariant measure for (π_n).*

4.4.2 Embedded i.i.d. case

In this subsection we study the situation when we may assume that, for a given control function $u\in\mathcal{B}(P(E),U)$, there exists a sequence of (Y^n)-

adapted Markov times (τ_n) such that

$$\begin{aligned} &\tau_1 = \tau \\ &\dots \\ &\tau_{n+1} = \tau_n + \tau \circ \Theta_{\tau_n} \end{aligned} \tag{4.73}$$

with τ satisfying

$$\sup_{\mu \in P(E)} E^u_\mu\{\tau\} < \infty \tag{4.74}$$

Θ_{τ_n} standing for the Markov shift operator corresponding to the controlled filtering process π^u_n, and with the random variables $\pi^u_{\tau_n}$ i.i.d. having the distribution independent of the initial law of π^u_n.

Proposition 4.25 *Under the above assumptions, for any $F \in b\mathcal{B}(P(E))$ we have*

$$\frac{1}{n}\sum_{i=0}^{n-1} F(\pi^u_i) \to E\Big\{E_{\pi_{\tau_1}}\Big\{\sum_{i=0}^{\tau-1} F(\pi^u_i)\Big\}\Big\}(E\{E_{\pi_{\tau_1}}\{\tau\}\})^{-1} \quad \text{P.a.e.} \tag{4.75}$$

as $n \to \infty$.

Furthermore the measure $\Phi^u \in P(P(E))$ defined for $F \in b\mathcal{B}(P(E))$ as

$$\Phi^u(F) = E\Big\{E_{\pi_{\tau_1}}\Big\{\sum_{i=0}^{\tau-1} F(\pi^u_i)\Big\}\Big\}(E\{E_{\pi_{\tau_1}}\{\tau\}\})^{-1} \tag{4.76}$$

is the unique invariant measure for the controlled filtering process (π^u_n).

P r o o f. Since for any $F \in b\mathcal{B}(P(E))$ the random variables

$$\sum_{i=\tau_n}^{\tau_{n+1}-1} F(\pi^u_i) \quad n = 1, 2, \dots,$$

are i.i.d., by the law of large numbers we have

$$\frac{1}{n}\sum_{i=0}^{\tau_n-1} F(\pi^u_i) \to E\Big\{E_{\pi_{\tau_1}}\Big\{\sum_{i=0}^{\tau-1} F(\pi^u_i)\Big\}\Big\} \tag{4.77}$$

and

$$\frac{1}{n}\tau_n \to E\{E_{\pi_{\tau_1}}\{\tau\}\} \quad P \text{ a.e.}$$

as $n \to \infty$.

Consequently

$$\frac{1}{\tau_n}\sum_{i=0}^{\tau_n-1} F(\pi_i^u) = \frac{n}{\tau_n} \cdot \frac{1}{n}\sum_{i=0}^{\tau_n-1} F(\pi_i^u) \to \Phi^u(F)$$

P a.e. as $n \to \infty$.

Let $N(n) = \inf\{i\colon \tau_i \geq n\}$ for $n = 1, 2, \ldots$, then

$$\begin{aligned}
&\Big|\frac{1}{n}\sum_{i=0}^{n-1} F(\pi_i^u) - \frac{1}{\tau_{N(n)}}\sum_{i=0}^{\tau_{N(n)}-1} F(\pi_i^u)\Big| \\
&\leq \Big|\Big(\frac{1}{n} - \frac{1}{\tau_{N(n)}}\Big)\sum_{i=0}^{n-1} F(\pi_i^u)\Big| + \frac{1}{\tau_{N(n)}}\Big|\sum_{i=n}^{\tau_{N(n)}-1} F(\pi_i^u)\Big| \\
&\leq 2\frac{\tau_{N(n)} - \tau_{N(n)-1}}{\tau_{N(n)}} \cdot \|F\|
\end{aligned}$$

which, using (4.77), can easily be shown to converge to 0, P a.e. as $n \to \infty$.

Therefore (4.75) holds and, since the distribution of $\pi_{\tau_1}^u$ does not depend on the initial law of (π_n^u), the measure Φ^u is the unique invariant measure for (π_n^u).

■

Below we introduce two examples for which the assumptions made in this section are satisfied.

Example 1. This example presents an entire class of models, characterized in terms of their transition kernel $P^a(x, dz)$ and their observation density $r(z, y)$, for which the above assumptions hold. More precisely, we assume

(E1.1) there exists $D_1 \subset R^d$ and $K \subset E$ such that $r(z, y) = 0$ for $y \in D_1$ and $z \notin K$

(E1.2) there exists $\lambda \in P(E)$ such that, for all $a \in U$ and all $x \in K$

$$P^a(x, \cdot) = \lambda(\cdot)$$

Assumption (E1.1) corresponds to a partial detectability of (x_n): if $y_n \in D_1$, then we know that $x_n \in K$. However it may happen that $x_n \in K$ and $y_n \notin D_1$. Identifying K with the set of "failure states" of the Markov process (x_n), assumption (E1.2) can be understood as a selfregeneration property: when a failure occurs, the Markov chain (x_n) continues to evolve starting from a same measure λ, independent of the position of the state process in K and of the chosen control.

We shall also need the following two technical assumptions

(E1.3) $\inf\limits_{z\in K} \int\limits_{D_1} r(z,y)dy = \beta_1 > 0$

and

(E1.4) $\inf\limits_{a\in U} \inf\limits_{z\in E\setminus K} P^a(z,K) = \beta_2 > 0$

Before proving that, under (E1.1)–(E1.4), the assumptions made in this subsection are satisfied, let us show the following:

Putting

$$\sigma = \inf\{i > 0 \colon y_i \in D_1\} \tag{4.78}$$

we have

Lemma 4.26 *Under (E1.3) and (E1.4)*

$$\sup_{u=(a_n)} \sup_{x\in E} E_x^u\{\sigma^2\} < \infty \tag{4.79}$$

where the first supremum is taken over all admissible controls (a_n), namely those for which a_n is Y^n adapted.

P r o o f. Let

$$\tau = \inf\{i \geq 0 \colon x_i \in K\} \tag{4.80}$$

We have

$$\sup_{u=(a_n)} \sup_{x\in E} E_x \tau^2 = M < \infty \tag{4.81}$$

In fact, by (E1.4) for $a \in U$, $z \in E \setminus K$,

$$P^a(z, E\setminus K) \leq 1 - \beta_2$$

and therefore

$$E_x\tau^2 \le 1 + 2^2 + 3^2(1-\beta_2) + 4^2(1-\beta_2)^2 + \ldots =$$
$$= 1 + \sum_{n=0}^{\infty}(n+2)^2(1-\beta_2)^n < \infty$$

Let

$$\tau_1 = \tau$$
$$\ldots$$
$$\tau_{n+1} = \tau_n + \tau \circ \Theta_{\tau_n}$$

Then

$$\begin{aligned}
&E_x\sigma^2 = E_x\{\tau_1^2\chi_{y_{\tau_1}\in D_1}\} + E_x\{\tau_2^2\chi_{y_{\tau_2}\in D_1}\chi_{y_{\tau_1}\notin D_1}\} \\
&+ \ldots + E_x\{\tau_n^2\chi_{y_{\tau_n}\in D_1}\chi_{y_{\tau_{n-1}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\ldots \le \\
&\le E_x\{\tau^2\} + E_x\{\tau_2^2\chi_{y_{\tau_1}\notin D_1}\} + \ldots + E_x\{\tau_n^2\chi_{y_{\tau_{n-1}}\notin D_1}\cdots \\
&\chi_{y_{\tau_1}\notin D_1}\} + \ldots
\end{aligned} \tag{4.82}$$

By (E1.3)

$$\begin{aligned}
&E_x\{\chi_{y_{\tau_{n-1}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\} \\
&= E_x\{P\{y_{\tau_{n-1}\notin D_1}|x_0, x_1, \ldots, x_{\tau_{n-1}}, Y^{\tau_{n-1}-1}\} \\
&\chi_{y_{\tau_{n-2}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\} \le (1-\beta_1)E_x\{\chi_{y_{\tau_{n-2}}\notin D_1} \\
&\cdots\chi_{y_{\tau_1}\notin D_1}\} \le \ldots \le (1-\beta_1)^{n-1}
\end{aligned} \tag{4.83}$$

and applying (4.81) we have

$$\begin{aligned}
&E_x\{\tau_{n-1}\chi_{y_{\tau_{n-1}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\\
&\leq(1-\beta_1)E_x\{\tau_{n-1}\chi_{y_{\tau_{n-2}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\\
&\leq(1-\beta_1)E_x\{\tau_{n-2}\chi_{y_{\tau_{n-2}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\\
&+(1-\beta_1)M^{1/2}E_x\{\chi_{y_{\tau_{n-2}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\\
&\leq(1-\beta_1)E_x\{\tau_{n-2}\chi_{y_{\tau_{n-2}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}+(1-\beta_1)^{n-1}M^{1/2}
\end{aligned}\tag{4.84}$$

Iterating (4.84) we obtain

$$E_x\{\tau_{n-1}\chi_{y_{\tau_{n-1}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\leq(n-1)(1-\beta_1)^{n-1}M^{1/2}\tag{4.85}$$

Now, by (4.81), (4.83) and (4.85)

$$\begin{aligned}
&E_x\{\tau_n^2\chi_{y_{\tau_{n-1}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\\
&\leq E_x\{\tau_{n-1}^2\chi_{y_{\tau_{n-1}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\\
&+2M^{1/2}E_x\{\tau_{n-1}\chi_{y_{\tau_{n-1}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\\
&+ME_x\{\chi_{y_{\tau_{n-1}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\\
&\leq(1-\beta_1)E_x\{\tau_{n-1}^2\chi_{y_{\tau_{n-2}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\\
&+2(n-1)(1-\beta_1)^{n-1}M+(1-\beta_1)^{n-1}M
\end{aligned}$$

and by iteration

$$\begin{aligned}
&E_x\{\tau_n^2\chi_{y_{\tau_{n-1}}\notin D_1}\cdots\chi_{y_{\tau_1}\notin D_1}\}\leq(1-\beta_1)^{n-1}E_x\{\tau_1^2\}\\
&+2(n-1)^2(1-\beta_1)^{n-1}M+(n-1)(1-\beta_1)^{n-1}M\\
&=n(1-\beta_1)^{n-1}M+2(n-1)^2(1-\beta_1)^{n-1}M
\end{aligned}$$

Thus finally by (4.82)

$$E_x\sigma^2\leq M+\sum_{n=2}^{\infty}n(1-\beta_1)^{n-1}M+2(n-1)^2(1-\beta_1)^{n-1}M<\infty$$

Since the bounds we obtained do not depend on the initial state nor on the chosen control, we obtain (4.79).

■

Putting now

$$\begin{aligned}
&\sigma_1 = \sigma \\
&\dots \\
&\sigma_{n+1} = \sigma_n + \sigma \circ \Theta_{\sigma_n}
\end{aligned} \tag{4.86}$$

which clearly form a sequence of Y^n-adapted Markov times, we can show

Proposition 4.27 *Under (E1.1)–(E1.4) for a control function $u \in \mathcal{B}(P(E), U)$ we have*

(a) (x_{σ_n+1}) *is a sequence of i.i.d.'s with common distribution* λ

(b) (y_{σ_n+1}) *is a sequence of i.i.d.'s with common distribution* $\int_E R(z, \cdot)\lambda(dz)$

(c) *the filtering process* $\pi^u_{\sigma_n+1}$ *has the form*

$$\pi^u_{\sigma_n+1} = Q(y_{\sigma_n+1}, \lambda)$$

with Q defined in (3.200), and $\pi^u_{\sigma_n+1}$ are i.i.d. with distribution independent of the initial law of (π^u_n)

(d) *the convergence (4.75) holds with $\tau = \sigma + 1$; furthermore there exists a unique invariant measure Φ^u for the filtering process (π^u_n), and it has the form (4.76).*

P r o o f. Notice that, by the definition of σ, we have $y_{\sigma_n} \in D_1$ and therefore by (E1.1), $x_{\sigma_n} \in K$. Using (E1.2) we see that x_{σ_n+1} is independent of $x_0, \dots, x_{\sigma_n}, y_1, \dots, y_{\sigma_n}$ and has law λ. Consequently by (1.1), y_{σ_n+1} is independent of $x_0, \dots, x_{\sigma_n}, y_1, \dots, y_{\sigma_n}$ and has law $\int_E R(z, \cdot)\lambda(dz)$. The statement (c) follows immediately from the fact that $x_{\sigma_n} \in K$ and (E1.2) as well as from (b). Since all the assumptions of Proposition 4.25 are satisfied we also have (d).

■

Example 2. Also in this second example we consider an entire class of models, characterized by their transition operators and observation structure, for which the assumptions made in this subsection are satisfied.

First we assume that the set of control parameters U is such that $U \subset R^l$, $l \geq 1$, and $\{0\} \in U$. Furthermore let the observation process $y_n \in R^d$ be of the form

$$y_n = h(x_n) + w_n, \tag{4.87}$$

where $h \in C(E, R^d)$ and (w_n) are i.i.d. d-dimensional standard Gaussian random variables, independent of x_k for $k \leq n$.

Moreover assume

(E2.1) There exists $j \in \{1, 2, \ldots, d\}$ such that the j-th component $h^j(x)$ of $h(x)$ has a limit at "∞" and attains at "∞" its strong maximum or strong minimum; more precisely, letting

$$B_n = \{x \in E \colon \rho_E(x, \overline{x}) \leq n\} \tag{4.88}$$

where ρ_E is a metric on E and $\overline{x}$ a fixed element of E we either have

$$\sup_{x \in B_n} h^j(x) < \sup_{x \in E} h^j(x) \quad \text{for } n = 1, 2, \ldots$$

or (4.89)

$$\inf_{x \in B_n} h^j(x) > \inf_{x \in E} h^j(x) \quad \text{for } n = 1, 2, \ldots$$

(E2.2) There exists $\lambda \in P(E)$ such that

$$P^a(x, \cdot) = \lambda(\cdot)$$

for $a = 0$ and all $x \in E$,

(E2.3) For any compact set $K \subset E$ there exists $\alpha > 0$ such that

$$\inf_{a \in U} \inf_{x \in E} P^a(x, K^c) \geq \alpha$$

Notice that the assumptions (E2.2) and (E2.3) are satisfied for any model of the form

$$x_{n+1} = F(x_n, a_n) + G(x_n, a_n)v_n$$

where $F(x, a)$ is bounded uniformly in (x, a) with $F(x, 0)$ being a constant vector, and where for positive constants c_1, c_2 the matrix $G(x, a)$ satisfies

$$G(x, a)G^T(x, a) \geq c_1 I$$

for all (x, a) and $G(x, 0) = c_2 I$. Furthermore v_n are i.i.d. standard Gaussian vectors.

Finally let $r: [0, 1] \to [0, 1]$ be a continuous nondecreasing function such that, with $0 < b < c < 1$,

$$r(x) = \begin{cases} 0 & \text{for } x \leq b \\ 1 & \text{for } x \geq c \end{cases} \tag{4.90}$$

Additionally let $\psi_m \in C(E)$ with values in $[0, 1]$ be a given function satisfying (compare to (3.36))

$$\psi_m(x) = \begin{cases} 1 & \text{for } x \in B_m \\ 0 & \text{for } x \in E \setminus B_{m+1} \end{cases} \tag{4.91}$$

with B_m defined in (4.88).

We shall show below that for a control function $u \in \mathcal{B}(P(E), U)$ of the form

$$u(\nu) = \overline{u}(\nu) r(\nu(\psi_{\overline{m}})) \tag{4.92}$$

for some $\overline{m}$, and $\overline{u} \in \mathcal{B}(P(E), U)$, the assumptions of this subsection are satisfied and therefore by Proposition 4.25, there exists a unique invariant measure for the filtering process (π_n^u).

We need first an auxiliary result

Lemma 4.28 *Under (E2.1) and (E2.3), for any $\gamma \in (0, 1)$, $C > 0$, $m = 1, 2, \ldots$, one can find y_0 such that, if (y^j is the j-th component of $y \in R^d$, for* *which one of inequalities (4.89) holds)* $y^j > y_0$, $|y^i| \leq C$ *for* $i \neq j$ *in the case of the first inequality (4.89),* $y^j < y_0$, $|y^i| \leq C$ *for* $i \neq j$ *in the case of the second inequality (4.89), we have for any* $\nu \in P(E)$ *and any* $u \in \mathcal{B}(P(E), U)$

$$M^u(y,\nu)(B^c_m) \geq \gamma \tag{4.93}$$

where B_m *is as in (4.88) and* $M^u(y,\nu)$ *as in (1.8) with* $r(x,y)$ *given by (1.2).*

P r o o f. Assume that the first alternative in (4.89) holds. Let $\|h^j\|_m = \sup_{x\in B_m} |h^j(x)|$ and put $y_0 > 0$. Then for $y^j > y_0$ and $|y^i| \leq C$ if $i \neq j$, $m_1 > m$ such that $h^j(z) \geq \|h^j\|_m + \varepsilon$ for some $\varepsilon > 0$ and all $z \notin B_{m_1}$, we have

$$\begin{aligned} M^u(y,\nu)(B_{m_1}) &\leq e^{2\|h\|C}\Big(\int_{B^c_{m_1}} \exp[(y, h(z) - \|h^j\|_m) \\ &-\frac{1}{2}(h(z),h(z))]P^{u(\nu)}(\nu,dz)\Big)^{-1} \leq \\ &\leq e^{2\|h\|C} e^{\frac{1}{2}\|h\|^2} e^{-\varepsilon y_0} e^{2\|h\|C} (P^{u(\nu)}(\nu, B^c_{m_1}))^{-1} \\ &\leq e^{4\|h\|C} e^{\frac{1}{2}\|h\|^2} \frac{1}{\alpha} e^{-\varepsilon y_0} \end{aligned}$$

and therefore choosing y_0 sufficiently large we obtain

$$M^u(y,\nu)(B_{m_1}) \leq 1-\gamma$$

Consequently (4.93) holds. The proof in the case when the second alternative in (4.89) holds is similar.

■

Let now, for $u \in \mathcal{B}(P(E), U)$ and fixed positive integer $\overline{m}$

$$\overline{\sigma} = \inf\{i > 0, \pi^u_i(\psi_{\overline{m}}) \leq b\} \tag{4.94}$$

with ψ_m and b as in (4.91), (4.90) respectively. From Lemma 4.28 we easily obtain

Corollary 4.29 *Under the assumptions of Lemma 4.28*

$$\sup_{u\in\mathcal{B}(P(E),U)} \sup_{\mu\in P(E)} E^u_\mu\{\overline{\sigma}^2\} < \infty \tag{4.95}$$

where the first supremum is over all controls $a_n = u(\pi_n)$ *with* $u \in \mathcal{B}(P(E), U)$*.*

P r o o f. Letting $\gamma = 1 - b$, by Lemma 4.28 we have that for $y^j > y_0$, $|y^i| \leq C$ when $i \neq j$, in the case of the first inequality in (4.89), and for $y^j < y_0$, $|y^i| < C$ when $i \neq j$, in the case of the second inequality in (4.89)

$$\sup_{u \in \mathcal{B}(P(E),U)} \sup_{\mu \in P(E)} M^u(y,\mu)(\psi_{\overline{m}}) \leq 1 - \gamma = b \tag{4.96}$$

Define

$$\Gamma = \{\nu \in P(E), \nu(\psi_{\overline{m}}) \leq b\}$$

By (4.96), taking into account the observation structure (4.87), we therefore have for some $\beta > 0$

$$\inf_{u \in \mathcal{B}(P(E),U)} \inf_{\mu \in P(E)} \prod^u (\mu, \Gamma) \geq \beta$$

from which (4.95) immediately follows.

■

Let

$$\begin{aligned} &\overline{\sigma}_1 = \overline{\sigma} \\ &\dots \\ &\overline{\sigma}_{n+1} = \overline{\sigma}_n + \overline{\sigma} \circ \Theta_{\overline{\sigma}_n} \end{aligned} \tag{4.97}$$

By analogy to Proposition 4.27 we now have

Proposition 4.30 *Under (E2.1)–(E2.3) for a control function $u \in \mathcal{B}(P(E), U)$ of the form (4.92) we have*

(a) $(x_{\overline{\sigma}_n+1})$ *is a sequence of i.i.d.'s with common distribution* λ

(b) $(y_{\overline{\sigma}_n+1})$ *is a sequence of i.i.d.'s with common distribution* $\int_E R(z,\cdot)\lambda(dz)$

(c) *the filtering process* $\pi^u_{\overline{\sigma}_n+1}$ *has the form*

$$\pi^u_{\overline{\sigma}_n+1} = Q(y_{\overline{\sigma}_n+1}, \lambda)$$

with Q as in (3.200), and $\pi^u_{\overline{\sigma}_n+1}$ are i.i.d. with distribution independent of the initial law of (π^u_n)

(d) *the convergence (4.75) holds, there exists a unique invariant measure Φ^u for the filtering process (π_n^u), and it has the form (4.76).*

P r o o f. Notice that by (E2.2), the form (4.92) of u together with the definition (4.90) of $r(x)$ and the definition (4.94) of $\overline{\sigma}$, we have that the random variable $x_{\overline{\sigma}_n+1}$ is independent of $x_0, \ldots, x_{\overline{\sigma}_n}, y_1, \ldots, y_{\overline{\sigma}_n}$ and has law λ. Therefore by considerations similar to those of Proposition 4.27, using in particular the fact that $u(\pi_{\overline{\sigma}}) = 0$, we obtain (b), (c); applying furthermore Proposition 4.25 we obtain (d) as well.

■

4.4.3 Mixed observation case

In this subsection we consider the case when the observations are of the mixed type, i.e. satisfy (4.46).

We need the following assumption, where Γ is as in section 4.3

(A11) there is a compact set $\Gamma_1 \subset \Gamma$ such that

- (i) $P^a(x, \partial\Gamma_1) = 0$ for $a \in U$, $x \in E$
- (ii) $E_\mu^u T_{\Gamma_1} < \infty$ for $\mu \in P(E)$, $u \in C(P(E), U)$ where $T_{\Gamma_1} = \inf\{s \geq 0 \colon x_s \in \Gamma_1\}$ and E_μ^u stands for the conditional expectation of the filtering process starting from μ with $u \in C(P(E), U)$.
- (iii) $\sup_{x \in \Gamma_1} \sup_{u \in C(P(E),U)} E_x^u \tau^2 < \infty$ with $\tau = T_{\Gamma^c} + T_{\Gamma_1} \circ \Theta_{T_{\Gamma^c}}$ and Θ_t standing for the Markov shift operator corresponding to the state process (x_n)
- (iv) For $\tau_1 = \tau$, $\tau_{n+1} = \tau_n + \tau \circ \Theta_{\tau_n}$, the embedded Markov chain x_{τ_n} has for $u \in C(P(E), U)$ a unique invariant measure η^u and the strong law of large numbers holds for (x_{τ_n}).

Let us comment on the assumptions (A11)(i)–(iv)

Remark 4.31 *Notice that τ is the first time of return to Γ_1 after hitting Γ^c. To guarantee (A11)(iii) one usually imposes Lyapunov type conditions. Assumption (A11)(iii) is also satisfied when*

$$\inf_{a \in U} \inf_{x \in E} P^a(x, \Gamma_1) > 0$$

and

$$\inf_{a\in U}\inf_{x\in E} P^a(x,\Gamma^c) > 0$$

Notice furthermore that since Γ_1 *is compact, if the transition operator* $P_x^u\{x_\tau \in \cdot\}$ *is Feller for* $x\in\Gamma_1$ *and* $u\in C(P(E),U)$*, then there exists an invariant measure* η^u *for the embedded Markov chain* (x_{τ_n})*. It does not mean however that* η^u *is unique.*

Finally, as follows from Theorem 6.2 in Chapter 5 of [13] and its proof, the strong law of large numbers holds for (x_{τ_n}) *if it is uniformly ergodic, or more generally if there are no invariant sets of* η^u *measure zero.*

We have

Proposition 4.32 *Under (A11) for any* $u\in C(P(E),U)$ *there exists a unique invariant measure* Φ^u *for the controlled filtering process* (π_n^u) *and it is of the form*

$$\Phi^u(F) = \int\limits_{\Gamma_1} E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{\delta_x,u})\Big\}\eta^u(dx)\Big(\int\limits_{\Gamma_1} E_x^u\{\tau\}\eta^u(dx)\Big)^{-1} \tag{4.98}$$

for $F\in b\mathcal{B}(P(E))$*, with* $(\pi_i^{\delta_x,u})$ *standing for the filtering process starting from the measure* $\mu=\delta_x$*.*

Moreover for $F\in b\mathcal{B}(P(E))$*,* $\mu\in P(E)$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} F(\pi_i^{\mu,u}) = \Phi^u(F) \quad P_\mu \text{ a.e.} \tag{4.99}$$

P r o o f. Since η^u is invariant for x_{τ_n} and therefore

$$\begin{aligned}&\int\limits_{\Gamma_1} E_x^u\{F(\pi_\tau^{\delta_x,u})\}\eta^u(dx) =\\ &= \int\limits_{\Gamma_1} E_x^u\{F(\delta_{x_\tau})\}\eta^u(dx) =\\ &= \int\limits_{\Gamma_1} F(\delta_x)\eta^u(dx),\end{aligned}$$

we have

$$\int\limits_{\Gamma_1} E_x^u\Big\{\sum_{i=0}^{\tau-1}\prod F(\pi_i^{\delta_x,u})\Big\}\eta^u(dx) =$$

$$= \int\limits_{\Gamma_1} E_x^u\Big\{\sum_{i=1}^{\tau} F(\pi_i^{\delta_x,u})\Big\}\eta^u(dx) =$$

$$= \int\limits_{\Gamma_1} E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{\delta_x,u})\Big\}\eta^u(dx)$$

and clearly Φ^u is an invariant measure for $(\pi_n^{\mu,u})$.

It remains to show the convergence (4.99) from which the uniqueness of Φ^u follows.

For $g \in b\mathcal{B}(P(E))$ by the strong law of large numbers for martingales (see Thm. III.8.2 of [15]) we have

$$\lim_{n\to\infty} n^{-1}\Big(\sum_{i=0}^{\tau_n-1} g(\pi_i^{\delta_x,u}) - \sum_{i=0}^{n-1} E_{x_{\tau_i}}\Big\{\sum_{j=0}^{\tau-1} g(\pi_j^{\delta_{x_{\tau_i}},u})\Big\}\Big) \to 0 \quad P_x \text{ a.e.}$$

for $x \in \Gamma_1$.

Now, by (A11)(iv) the strong law of large numbers holds for (x_{τ_n}), so that

$$\lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} E_{x_{\tau_i}}\Big\{\sum_{j=0}^{\tau-1} g(\pi_j^{\delta_{x_{\tau_i}},u})\Big\} = \int\limits_{\Gamma_1} E_x\Big\{\sum_{j=0}^{\tau-1} g(\pi_j^{\delta_x,u})\Big\}\eta^u(dx)$$

P_x a.e. for $x \in \Gamma_1$.

Letting $g \equiv F$ and $g \equiv 1$ in the above two equalities we obtain

$$\lim_{n\to\infty} \tau_n^{-1}\sum_{i=0}^{\tau_n-1} F(\pi_i^{\delta_x,u}) = \Phi^u(F)$$

P_x a.e. for $x \in \Gamma_1$.

Since by considerations similar to those of the proof of Proposition 4.25 we have that

$$\lim_{n\to\infty} \tau_n^{-1}\sum_{i=0}^{\tau_n-1} F(\pi_i^{\delta_x,u}) = \lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} F(\pi_i^{\delta_x,u})$$

P_x a.e. for $x \in \Gamma_1$, using (A11)(ii) we obtain (4.99).

■

Remark 4.33 *If $u : P(E) \to U$ is Borel measurable, η^u is an invariant measure of (x_{τ_n}) and*

$$\sup_{x \in \Gamma_1} E_x^u \tau \leq \infty$$

then, by the first part of the proof of Proposition 4.32, it is clear that Φ^u defined in (4.98) is an invariant measure of $\pi_n^{\mu,u}$.

Remark 4.34 *Whenever (4.75) or (4.99) hold, we can express the ergodic cost functional $J_\mu(u)$ in (1.6), ((1.12)) as*

$$J_\mu(u) = \int_{P(E)} \int_E c(x, u(\nu))\nu(dx)\Phi^u(d\nu)$$

4.5 Construction of nearly optimal continuous control functions

By analogy to the infinite horizon case with discounting, that was studied in section 3, also in the present case with the ergodic cost criterion (1.6) the nearly optimal controls are obtained along two steps: on a first step (correspoding to section 3.3) nearly optimal control functions are constructed which, when applied to the true filter values, yield nearly optimal controls; since in general the true filter values cannot be computed, on a second step (corresponding to section 3.5) a computable approximating filter process is constructed. The nearly optimal controls are then obtained by applying the nearly optimal control functions to the approximating filter values.

This section is devoted to the construction of nearly optimal control functions for the ergodic case. For this purpose, as in section 3, we have to make use of some compactness arguments by either assuming that the state space E is compact, or by approximating the class of admissible controls by a compact family of controls. For the infinite horizon with discounting problem, in section 3.3 we constructed for the case of a compact state space E nearly optimal Borel measurable control functions with the use of Bellman's equation; in the general case we restricted ourselves from the beginning to the class $\mathcal{A}$ of continuous control functions. Since in the present context of an ergodic cost, in the case when E is compact and (A6), (A7) hold, by section 4.2.2 the problem of the construction of nearly optimal Borel measurable control functions can be reduced to that for a discounted cost criterion where the factor β is sufficiently close to 1, here we shall consider only the case of a general E restricting ourselves to continuous control functions.

Paralleling partly subsection 3.3.1, as preliminary to the construction of nearly optimal control functions, in subsection 4.5.1 we shall obtain general results on approximations of invariant measures of controlled filtering processes. This subsection will be split into two further subsections: in 4.5.1.a we obtain such results for the mixed observation model; in 4.5.1.b we shall then formulate suitable versions of these results also for the embedded i.i.d. case concentrating on the two examples of section 4.4.2. In subsection 4.5.2 we shall apply these general convergence results to more specific approximations. More precisely, in the further subsection 4.5.2.a we shall first approximate the class of continuous control functions by suitable compact subclasses. Having done this, in subsection 4.5.2.b we then proceed analogously to section 3.3.2 to a discretization of the state and observation spaces that allows to reduce the original partial observation problem to a complete observation problem where the state is the filter that evolves on a simplex. This subsection is further split into 4.5.2.b_1 dealing with the mixed observation model, and 4.5.2.b_2 concerning the embedded i.i.d. case. Although a simplex is a finite - dimensional space of measures, it is still infinite - valued. In subsection 4.5.3. we therefore introduce a discretization of the simplex that allows finally the actual construction of a nearly optimal control function which, after a suitable extension, is nearly optimal also for the original problem. Computational considerations along with numerical results are reported in subection 4.5.4.

Finally, subsection 4.5.5 contains results along the line of section 4.2 stating that, under certain assumptions, a nearly optimal control function for the infinite horizon case with discounting is nearly optimal also for the infinite horizon case with ergodic cost functional if the discount factor β is close to 1. While in section 4.2 these results are obtained with the use of Bellman's equation assuming the compactnes of E, here, in line with the rest of section 4.5, we obtain such results in a direct way restricting the control functions to be continuous and exploiting ergodic properties of embedded Markov chains.

4.5.1 Approximation of invariant measures (General convergence result)

In this section, similarly as in 3.3.1, we formulate a general convergence result on the approximation of invariant measures, which later will be applied to more specific approximations.

We shall first prove the convergence result for the mixed observation model and then formulate suitable versions for the embedded i.i.d. case concentrating on the examples 1 and 2, that were considered in section 4.4.2. By (4.75) and (4.99) it is clear that the approximation of the ergodic cost functional $J_\mu(u)$, defined in (1.6)(see also (1.12)) can now be reduced to the approximation of the invariant measure Φ^u defined in (4.76) and (4.98) respectively.

4.5.1.a The mixed observation case

Consider first the case of a mixed observation structure. Assume the state process (x_n) is approximated by a process (x_n^m) corresponding to a transition kernel P_m^a for which (D1) of section 3.3.1 is satisfied as well as

(D4) for any compact set $K \subset E$, and any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$P_m^a(x, \Gamma(\delta)) < \varepsilon \quad \text{uniformly for} \quad a \in U,\ x \in K,\ m = 1, 2, \dots,$$

with $\Gamma(\delta) = \{z \in \Gamma^c \colon \rho_E(z, \Gamma) < \delta\}$,

Assumption (D4) is mainly motivated by the fact that (see also comment below (A10)) the observation density $r(x, y)$ has to converge to infinity as y comes close to x, with x close to $\partial\Gamma$.

Furthermore, assume (x_n^m) is, for each m, completely observed in the set Γ and partially outside with the observation density $r_m(x, y)$, that approximates $r(x, y)$ in the following way

(D5) (i) $r_m \in b\mathcal{B}(\Gamma^c \times \Gamma^c)$, $r_m(x, y) \to r(x, y)$ uniformly in (x, y) from compact subsets of $\Gamma^c \times \Gamma^c$, as $m \to \infty$, and $r_m(z, y)$ are uniformly in m bounded on Γ_δ^c (defined in (A8)) for $\delta > 0$,

(ii) for any compact $K \subset \Gamma^c$

$$\sup_{z \in K} \int_{\Gamma^c} |r(z,y) - r_m(z,y)|\, dy \to 0 \quad \text{as} \quad m \to \infty$$

(iii) if $\Gamma^c \ni y_m \to y \in \partial\Gamma$, then for $\delta > 0$ and any compact set $K \subset E$

$$\inf_{a \in U} \inf_{x \in K} \int_{B(y,\delta)} r_m(z, y_m) P_m^a(x, dz) \to \infty$$

as $m \to \infty$, with $B(y,\delta) = \{z \in \Gamma^c : \rho_E(z,y) < \delta\}$,

We assume moreover that (x_n^m) is controlled by control functions u_m satisfying the following two conditions below where, as in the rest of this subsection we use $E_x^{u_m,m}$ to denote the conditional expectation of (x_n^m) starting from x and controlled in the generic period n by $u_m(\pi_n^{m,u_m})$, where π_n^{m,u_m} is the filtering process corresponding to (x_n^m); similarly for $E_\mu^{u_m,m}$,

(D6) $u_m : P(E) \to U$ are Borel measurable and $u_m(\nu) \to u(\nu)$, as $m \to \infty$, uniformly in ν from compact subsets of $P(E)$, where $u \in \mathcal{A}$ is the continuous function used to control (x_n),

(D7) for the compact set Γ_1 of assumption (A11) and for the Markov times T_{Γ_1}, τ, τ_n, defined in (A11), considered now as Markov times with respect to (x_i^m) we have

- (i) $E_\mu^{u_m,m} T_{\Gamma_1} < \infty$ for $\mu \in P(E)$, and $m = 1, 2, \ldots$,
- (ii) $\sup_{x \in \Gamma_1} \sup_m E_x^{u_m,m} \tau^2 < \infty$
- (iii) for each $m = 1, 2, \ldots$, there exists an invariant measure $\eta_m^{u_m}$, of the Markov chain $(x_{\tau_n}^m)$ that is embedded in (x_n^m) with control $u_m(\pi_n^{m,u_m})$ in the generic period n.

Under the above assumptions, by Remark 4.33 there is an invariant measure Φ^{u_m} of π_n^{m,u_m} of the following form, where $F \in b\mathcal{B}(P(E))$

$$\Phi_m^{u_m}(F) = \int_{\Gamma_1} E_x^{u_m,m} \Big\{ \sum_{i=0}^{\tau-1} F(\pi_i^{m,u_m}) \Big\} \eta_m^{u_m}(dx) \Big(\int_{\Gamma_1} E_x^{u_m,m} \{\tau\} \eta_m^{u_m}(dx) \Big)^{-1} \tag{4.100}$$

The main result of this section can be formulated as follows

Theorem 4.35 *Under (A1), (A2), (A8)-(A11), and (D1), (D4)-(D7), we have*

$$\Phi_m^{u_m} \Rightarrow \Phi^u \quad \textit{weakly in} \quad P(P(E)) \quad \textit{as} \quad m \to \infty$$

P r o o f. We split the proof into a sequence of Lemmas and Corollaries.

Lemma 4.36 *Under (A1),(A2), (D1) and (D6) for $f \in C(E)$ and a set $A \in \mathcal{B}(E)$ such that*

$$\sup_{x\in E} \sup_{a\in U} P^a(x, \partial A) = 0 \tag{4.101}$$

we have

$$P_m^{u_m(\nu)}(\nu, f\chi_A) \to P^{u(\nu)}(\nu, f\chi_A) \tag{4.102}$$

as $m \to \infty$, uniformly in ν from compact subsets of $P(E)$.

P r o o f. Assume (4.102) does not hold, i.e. for $\nu_m \Rightarrow \nu$ we have

$$|P_m^{u_m(\nu_m)}(\nu_m, f\chi_A) - P^{u(\nu_m)}(\nu_m, f\chi_A)| \geq \delta > 0 \tag{4.103}$$

Taking into account the tightness of $\{\nu_m, m = 1, 2, \ldots\}$ by (D1) we have that $|P_m^{u(\nu_m)}(\nu_m, f_1) - P^{u(\nu)}(\nu_m, f_1)| \to 0$, for $f_1 \in C(E)$, as $m \to \infty$. Therefore by (A1), for $f_1 \in C(E)$

$$|P_m^{u_m(\nu_m)}(\nu_m, f_1) - P^{u(\nu)}(\nu, f_1)| \leq$$

$$|P_m^{u_m(\nu_m)}(\nu_m, f_1) - P^{u(\nu)}(\nu_m, f_1)| + |P^{u(\nu)}(\nu_m, f_1) - P^{u(\nu)}(\nu, f_1)| \to 0$$

as $m \to \infty$, and consequently and $P_m^{u_m(\nu_m)}(\nu_m, \cdot) \Rightarrow P^{u(\nu)}(\nu, \cdot)$. By Theorem 1.2.1 (v) of [6] we therefore have by (4.101)

$$P_m^{u_m(\nu_m)}(\nu_m, f\chi_A) \to P^{u(\nu)}(\nu, f\chi_A)$$

and

$$P^{u(\nu_m)}(\nu_m, f\chi_A) \to P^{u(\nu)}(\nu, f\chi_A) \text{ as } m \to \infty$$

as $m \to \infty$, a contradiction to (4.103). Thus (4.102) is holds.

■

Before we formulate the next lemma, by analogy to (4.49) define the measures

$$N_m^u(y,\nu)(A) := \int_{A\cap\Gamma^c} r_m(z,y)P_m^{u(\nu)}(\nu,dz)\{\int_{\Gamma^c} r_m(z,y)P_m^{u(\nu)}(\nu,dz)\}^{-1} \quad (4.104)$$

for $y \in \Gamma^c$ and

$$M_m^u(y,\nu)(A) := \chi_{A\cap\Gamma}(y) + \chi_{\Gamma^c}(y)N_m^u(y,\nu)(A) \quad (4.105)$$

for $y \in E, A \in \mathcal{B}(E)$.

It will be convenient also to let

$$N_m^u(y,\nu)(A) = N^u(y,\nu)(A) := \chi_A(y)$$

for $y \in \Gamma$.

Lemma 4.37 *Assume (A1),(A2), (A8)-(A10) and (D1),(D5),(D6). Then*

$$N_m^{u_m}(y,\nu) \Rightarrow N^u(y,\nu) \quad \text{as } m \to \infty \quad (4.106)$$

uniformly on compact subsets of $\overline{\Gamma^c} \times P(E)$.

P r o o f. It suffices to show for any $\varphi \in C(E)$ that if $\overline{\Gamma^c} \ni y_m \to y$ and $P(E) \ni \nu_m \Rightarrow \nu$ we have

$$|N_m^{u_m(\nu_m)}(y_m,\nu_m)(\varphi) - N^{u(\nu_m)}(y_m,\nu_m)(\varphi)| \to 0$$

Since Proposition 4.15 implies

$$N^{u(\nu_m)}(y_m,\nu_m)(\varphi) \to N^{u(\nu)}(y,\nu)(\varphi)$$

as $m \to \infty$, it remains to show that

$$N_m^{u_m(\nu_m)}(y_m,\nu_m)(\varphi) \to N^{u(\nu)}(y,\nu)(\varphi)$$

We consider two cases $y \in \Gamma^c$ and $y \in \partial\Gamma^c$. If $y \in \Gamma^c$ it is sufficient to prove the convergence of the numerator of $N_m^{u_m(\nu_m)}(y_m,\nu_m)(\varphi)$ to the numerator of $N^{u(\nu)}(y,\nu)(\varphi)$.

We have

$$\begin{aligned}
&|\int\limits_{\Gamma^c} r_m(z,y_m)\varphi(z)P_m^{u_m(\nu_m)}(\nu_m,dz) - \int\limits_{\Gamma^c} r(z,y)\varphi(z)P^{u(\nu)}(\nu,dz)| \\
&\le \int\limits_{\Gamma^c} |r_m(z,y_m) - r(z,y)|\ |\varphi(z)|P_m^{u_m(\nu_m)}(\nu_m,dz) \\
&+\Big|\int\limits_{\Gamma^c} r(z,y)\varphi(z)(P_m^{u_m(\nu_m)}(\nu_m,dz) - P^{u(\nu)}(\nu,dz))\Big| \\
&= \mathrm{I}_m + \mathrm{II}_m
\end{aligned}$$

Notice that since $y \in \Gamma^c$, by (A8) and (D5)(i) there are positive constants $m_0(y)$ and $M(y)$ such that for $m > m_0(y)$, $x \in \Gamma^c$

$$max\{r(x,y), r_m(x,y_m)\} < M(y).$$

Now, by the tightness of $\{\nu_m, m = 1,2,...\}$ and (D4), for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned}
&\int\limits_{\Gamma^c} |r_m(z,y_m) - r(z,y)|\ |\varphi(z)|P_m^{u_m(\nu_m)}(\nu_m,dz) \\
&\le 4\varepsilon\|\varphi\|M(y) + \int\limits_{\Gamma^c\cap\Gamma(\delta)} |r_m(z,y_m) - r(z,y)|\ |\varphi(z)|P_m^{u_m(\nu_m)}(\nu_m,dz)
\end{aligned}$$

for $m > m_0(y)$.

Therefore, by Lemma 3.4, there exist a compact set $K \subset E$ and positive integer m_1, such that for $m \ge max\{m_0(y), m_1\}$ we have

$$\mathrm{I}_m \le 6\varepsilon\|\varphi\|M(y) + \sup_{z\in\Gamma^c\cap\Gamma(\delta)\cap K} |r_m(z,y_m) - r(z,y)|\ \|\varphi\|$$

and consequently by (D5)(i) $\limsup_{m\to\infty} \mathrm{I}_m \le 6\varepsilon\|\varphi\|M(y)$. By Lemma 4.36, (A10) and (A1), (A2), also $\mathrm{II}_m \to 0$ as $m \to \infty$, which completes the proof in the case when $y \in \Gamma^c$.

If $y \in \partial\Gamma^c$, then by (D5)(iii)

$$N_m^{u_m(\nu_m)}(y_m,\nu_m)(\varphi) \to \varphi(y)$$

as $m \to \infty$. Summarizing and noticing that $\varphi(y) = N^{u(\nu)}(y,\nu)(\varphi)$ for $y \in \partial\Gamma$ we have $N_m^{u_m(\nu_m)}(y_m,\nu_m) \Rightarrow N^{u(\nu)}(y,\nu)$, for $y \in \overline{\Gamma^c}$ and the proof of Lemma 4.37 is completed.

■

Let by analogy to (4.50) for $F \in b\mathcal{B}(P(E))$, $\nu \in P(E)$

$$\begin{aligned}\Pi_m^u(\nu, F) &= \int_\Gamma F(\delta_z) P_m^{u(\nu)}(\nu, dz) + \\ &+ \int_{\Gamma^c}\int_{\Gamma^c} F(M_m^u(y,\nu)) r_m(z,y)\, dy P_m^{u(\nu)}(\nu, dz)\end{aligned} \tag{4.107}$$

which is the transition operator of the filter process $(\pi_n^{m,u})$. Given a set $A \in \mathcal{B}(E)$ denote by $\tilde{A}$ the set of all measures δ_x with $x \in A$.

We have

Lemma 4.38 *Assume (A1),(A2),(A8)-(A11), (D1),(D4)-(D6). Let* $F_m \in b\mathcal{B}(P(E))$, $F \in C(P(E))$, F_m *be uniformly bounded and* $F_m(\nu) \to F(\nu)$, *as* $m \to \infty$, *uniformly in* ν *from compact subsets of* $P(E)$.

Then

$$\Pi_m^{u_m}(\nu, F_m) \to \Pi^u(\nu, F) \tag{4.108}$$

and

$$\Pi_m^{u_m}(\nu, F_m\chi_{\tilde{A}}) \to \Pi^u(\nu, F\chi_{\tilde{A}}) \tag{4.109}$$

uniformly in ν *from compact subsets of* $P(E)$, *as* $m \to \infty$, *with* $A = \Gamma$ *or* $A = \Gamma_1$.

P r o o f. Let H be a compact subset of $P(E)$. We have

$$\begin{aligned}|\Pi_m^{u_m}(\nu, F_m) - \Pi^u(\nu, F)| &\le \int_\Gamma |F_m(\delta_z) - F(\delta_z)| P_m^{u_m(\nu)}(\nu, dz) \\ &+ \Big| \int_\Gamma F(\delta_z)(P_m^{u_m(\nu)}(\nu, dz) - P^{u(\nu)}(\nu, dz)) \Big| + \\ &+ \Big| \int_{\Gamma^c}\int_{\Gamma^c} (r_m(z,y) - r(z,y)) F_m(M_m^{u_m}(y,\nu))\, dy P_m^{u_m(\nu)}(\nu, dz) \Big| \\ &+ \Big| \int_{\Gamma^c}\int_{\Gamma^c} r(z,y)(F_m(M_m^{u_m}(y,\nu)) - F(M^u(y,\nu)))\, dy P_m^{u_m(\nu)}(\nu, dz) \Big|\end{aligned}$$

$$+\Big|\int_{\Gamma^c}\int_{\Gamma^c} r(z,y)F(M^u(y,\nu))\,dy(P_m^{u_m(\nu)}(\nu,dz)-P^{u(\nu)}(\nu,dz))\Big|$$
$$=\mathrm{I}_m+\mathrm{II}_m+\mathrm{III}_m+\mathrm{IV}_m+\mathrm{V}_m$$

Since, $\sup_{z\in\Gamma}|F_m(\delta_z)-F(\delta_z)|\to 0$, clearly $\mathrm{I}_m\to 0$ as $m\to\infty$, uniformly in $\nu\in H$. By (A10), (A11)(i) and Lemma 4.36, also $\mathrm{II}_m\to 0$ as $m\to\infty$ uniformly in $\nu\in H$.

Given $\varepsilon>0$, by Lemma 3.4 there is a compact set $K\subset E$ and a positive integer m_0 such that for $m\geq m_0$, (3.16) holds. Furthermore by the compactness of H there is a compact set $K_1\subset E$ such that $\nu(K_1)\geq 1-\varepsilon$ for $\nu\in H$. By (D4), for some $\delta>0$ we then have

$$P_m^a(z,\Gamma(\delta))<\varepsilon \quad \text{for } a\in U,\ z\in K_1,\ m=1,2,\ldots,$$

Therefore, for $m>m_0$

$$\begin{aligned}
\mathrm{III}_m \quad &\leq \quad \int_{\Gamma^c\cap K}\int_{\Gamma^c}|r_m(z,y)-r(z,y)|\,\|F_m\|\,dyP_m^{u_m(\nu)}(\nu,dz)\\
&+ \quad 2\varepsilon\|F_m\|\leq\\
&\leq \quad \sup_{z_1\in K_1}\int_{\Gamma^c\cap K}\int_{\Gamma^c}|r_m(z,y)-r(z,y)|\,\|F_m\|\,dyP_m^{u_m(\nu)}(z_1,dz)\\
&+ \quad 2\varepsilon\|F_m\|+2\varepsilon\|F_m\|\leq\\
&\leq \quad \sup_{z_1\in K_1}\int_{\{\Gamma^c\setminus\Gamma(\delta)\}\cap K}\int_{\Gamma^c}|r_m(z,y)-r(z,y)|\,\|F_m\|\,dyP_m^{u_m(\nu)}(z_1,dz)\\
+6\varepsilon\|F_m\| \quad &\leq \sup_{z\in\{\Gamma^c\setminus\Gamma(\delta)\}\cap K}\|F_m\|\int_{\Gamma^c}|r_m(z,y)-r(z,y)|\,dy+6\varepsilon\|F_m\|
\end{aligned}$$

Letting $m\to\infty$, since $\|F_m\|\leq C$, by (D5) (ii) we obtain that $\limsup_{m\to\infty}\mathrm{III}_m\leq 6\varepsilon C$, uniformly in $\nu\in H$.

Now, by (A9) there is a compact set $L\subset E$ such that with the compact set K as before

$$\sup_{z\in K}R(z,L^c)<\varepsilon$$

Hence

$$\begin{aligned}
\mathrm{IV}_m \;\leq\; 4\varepsilon C+\int_{\Gamma^c\cap K}\int_{\Gamma^c\cap L} r(z,y)|F_m(N_m^{u_m}(y,\nu))-\\
F(N^u(y,\nu))|\,dyP_m^{u_m(\nu)}(\nu,dz)
\end{aligned}$$

and by Lemma 4.37, $\limsup_{m\to\infty} \mathrm{IV}_m \le 4\varepsilon C$ for $\nu \in H$. Finally, by Lemma 4.36 and (A9), also $\mathrm{V}_m \to 0$ as $m \to \infty$ uniformly in $\nu \in H$.

Summarizing, $\mathrm{I}_m + \mathrm{II}_m + \mathrm{III}_m + \mathrm{IV}_m + \mathrm{V}_m \to 0$ as $m \to \infty$ uniformly in $\nu \in H$. Consequently (4.108) holds.

The proof of (4.109) is almost immediate since

$$\begin{aligned} \prod\nolimits_m^{u_m}(\nu, F_m\chi_{\tilde{A}}) &= \int_{A\cap\Gamma} F_m(\delta_z) P_m^{u_m(\nu)}(\nu, dz) \\ &\to \int_{A\cap\Gamma} F(\delta_z) P^{u(\nu)}(\nu, dz) = \prod\nolimits^u(\nu, F\chi_{\tilde{A}}) \quad \text{as } m \to \infty \end{aligned}$$

as $m \to \infty$, by Lemma 4.36, uniformly in $\nu \in H$. The proof of Lemma 4.38 is complete.

■

In the next Lemma 4.39 we extend (4.109) to the probabilities of more complex events.

Lemma 4.39 *Assume (A1),(A2),(A8)-(A11),(D1),(D4)-(D6). Let* $b\mathcal{B}(P(E)) \ni F_m \to F \in C(P(E))$ *uniformly on compact subsets of* $P(E)$ *and* F_m *be uniformly bounded. Then for each* $i = 1, 2, \ldots,$

$$\begin{aligned} &E_x^{u_m,m}\{\chi_{\Delta_1}(\pi_1^{m,u_m}) \ldots \chi_{\Delta_i}(\pi_i^{m,u_m}) F_m(\pi_i^{m,u_m})\} \\ &\to E_x^u\{\chi_{\Delta_1}(\pi_1^u) \ldots \chi_{\Delta_i}(\pi_i^u) F(\pi_i^u)\} \end{aligned} \tag{4.110}$$

as $m \to \infty$*, uniformly in* x *belonging to the compact set* Γ_1 *from (A11) with* Δ_k*, for* $k = 1, 2, \ldots, i$*, standing for any of the sets* $\tilde{\Gamma}_1$, $\tilde{\Gamma}_1^c$, $\tilde{\Gamma}$ *or* $\tilde{\Gamma}^c$*, where* ˜ *denotes the operator defined before Lemma 4.38.*

P r o o f. The proof is by induction. For $i = 1$ (4.110) holds by (4.109). Assume, now that (4.110) holds for i. Then, again by (4.109) and induction hypothesis, for $i + 1$ we have

$$\begin{aligned} &E_x^{u_m,m}\{\chi_{\Delta_1}(\pi_1^{m,u_m}) \ldots \chi_{\Delta_{i+1}}(\pi_{i+1}^{m,u_m}) F_m(\pi_{i+1}^{m,u_m})\} = \\ &E_x^{u_m,m}\{\chi_{\Delta_1}(\pi_1^{m,u_m}) \ldots \chi_{\Delta_i}(\pi_i^{m,u_m}) \prod\nolimits_m^{u_m}(\pi_i^{m,u_m}, \chi_{\Delta_{i+1}} F_m)\} \\ &\to E_x^u\{\chi_{\Delta_1}(\pi_1^u) \ldots \chi_{\Delta_i}(\pi_i^u) \prod\nolimits^u(\pi_i^u, \chi_{\Delta_{i+1}} F)\} \\ &= E_x^u\{\chi_{\Delta_1}(\pi_1^u) \ldots \chi_{\Delta_i}(\pi_i^u) \chi_{\Delta_{i+1}}(\pi_{i+1}^u) F(\pi_{i+1}^u)\} \end{aligned}$$

as $m \to \infty$, uniformly in $x \in \Gamma_1$.

■

From formula (4.110) we now obtain

Corollary 4.40 *Under (A1),(A2),(A8)-(A11),(D1),(D4)-(D7), for* $f \in C(\Gamma_1)$ *and* $F \in C(P(E))$ *we have*

$$E_x^{u_m,m}\{f(x_\tau^m)\} \to E_x^u\{f(x_\tau)\} \tag{4.111}$$

and

$$E_x^{u_m,m}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{m,u_m})\Big\} \to E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^u)\Big\} \tag{4.112}$$

uniformly in $x \in \Gamma_1$, *as* $m \to \infty$, *with* τ *on the left hand side of (4.111),(4.112) standing for the Markov time defined as in (A11) but with respect to the process* (x_n^m), *while on the right hand side it corresponds to* (x_n).

P r o o f. Recall that, by its definition in (A11), τ can equivalently be considered as stopping time of the process (x_n) and of the corresponding filter π_n. We therefore have

$$\begin{aligned} f(x_\tau) = \ & \sum_{i=2}^{\infty}\sum_{j=1}^{i-1} \chi_{\tilde\Gamma}(\pi_1)\dots\chi_{\tilde\Gamma}(\pi_{j-1})\chi_{\tilde\Gamma^c}(\pi_j)\chi_{\tilde\Gamma_1^c}(\pi_{j+1})\dots\chi_{\tilde\Gamma_1^c}(\pi_{i-1}) \\ & \chi_{\tilde\Gamma_1}(\pi_i)\pi_i(f) \end{aligned} \tag{4.113}$$

and

$$\begin{aligned} \sum_{i=0}^{\tau-1} F(\pi_i) = & \sum_{i=0}^{\infty} \chi_{\tilde\Gamma}(\pi_0)\dots\chi_{\tilde\Gamma}(\pi_i)[F(\pi_i) + \chi_{\tilde\Gamma^c}(\pi_{i+1})F(\pi_{i+1}) \\ & + \chi_{\tilde\Gamma^c}(\pi_{i+1})\sum_{k=i+2}^{\infty} \chi_{\tilde\Gamma_1^c}(\pi_{i+2})\dots\chi_{\tilde\Gamma_1^c}(\pi_k)F(\pi_k)] \end{aligned} \tag{4.114}$$

and the analogous representations hold for $f(x_\tau^m)$ and $\sum_{i=0}^{\tau-1} F(\pi_i^m)$.

By (A11) (iii) and (D7) (ii) to prove (4.111) and (4.112) it is therefore sufficient to show that for each $i, j, k = 1, 2, \ldots$ $(j < i < k-1)$

$$\begin{aligned} & E_x^{u_m,m}\{\chi_{\tilde{\Gamma}}(\pi_1^{m,u_m})\ldots\chi_{\tilde{\Gamma}}(\pi_{j-1}^{m,u_m})\chi_{\tilde{\Gamma}^c}(\pi_j^{m,u_m})\chi_{\tilde{\Gamma}_1^c}(\pi_{j+1}^{m,u_m})\ldots \\ & \chi_{\tilde{\Gamma}_1^c}(\pi_{i-1}^{m,u_m})\chi_{\tilde{\Gamma}_1}(\pi_i^{m,u_m})\pi_i^{m,u_m}(f)\} \to \\ & E_x^u\{\chi_{\tilde{\Gamma}}(\pi_1^u)\ldots\chi_{\tilde{\Gamma}}(\pi_{j-1}^u)\chi_{\tilde{\Gamma}^c}(\pi_j^u)\chi_{\tilde{\Gamma}_1^c}(\pi_{j+1}^u)\ldots \\ & \chi_{\tilde{\Gamma}_1^c}(\pi_{i-1}^u)\chi_{\tilde{\Gamma}_1}(\pi_i^u)\pi_i^u(f)\} \end{aligned} \tag{4.115}$$

and

$$\begin{aligned} & E_x^{u_m,m}\{\chi_{\tilde{\Gamma}}(\pi_1^{m,u_m})\ldots\chi_{\tilde{\Gamma}}(\pi_i^{m,u_m})[F(\pi_i^{m,u_m}) + \chi_{\tilde{\Gamma}^c}(\pi_{i+1}^{m,u_m}) \\ & F(\pi_{i+1}^{m,u_m}) + \chi_{\tilde{\Gamma}^c}(\pi_{i+1}^{m,u_m})\chi_{\tilde{\Gamma}_1^c}(\pi_{i+2}^{m,u_m})\ldots\chi_{\tilde{\Gamma}_1^c}(\pi_k^{m,u_m})F(\pi_k^{m,u_m})]\} \\ & \to E_x^u\{\chi_{\tilde{\Gamma}}(\pi_1^u)\ldots\chi_{\tilde{\Gamma}}(\pi_i^u)[F(\pi_i^u) + \chi_{\tilde{\Gamma}^c}(\pi_{i+1}^u)F(\pi_{i+1}^u) + \chi_{\tilde{\Gamma}^c}(\pi_{i+1}^u) \\ & \chi_{\tilde{\Gamma}_1^c}(\pi_{i+2}^u)\ldots\chi_{\tilde{\Gamma}_1^c}(\pi_k^u)F(\pi_k^u)]\} \end{aligned} \tag{4.116}$$

uniformly in $x \in \Gamma_1$, as $m \to \infty$. Since by Lemma 4.39, the convergences (4.115) and (4.116) hold, we obtain (4.111) and (4.112).

■

With the use of the representations (4.113) and (4.114) we prove now the following

Lemma 4.41 *Under (A1), (A2), (A8)-(A11), for* $f \in C(\Gamma_1)$, $F \in C(P(E))$, $u \in \mathcal{A} = C(P(E), U)$ *the mappings*

$$\Gamma_1 \ni x \mapsto E_x^u\{f(x_\tau)\} \tag{4.117}$$

$$\Gamma_1 \ni x \mapsto E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^u)\Big\} \tag{4.118}$$

are continuous and bounded.

P r o o f. By (4.113), (4.114) and (A11) (iii) it suffices to show the continuity of the mappings

$$\Gamma_1 \ni x \mapsto \quad E_x^u\{\chi_{\tilde{\Gamma}}(\pi_1^u)\ldots\chi_{\tilde{\Gamma}}(\pi_{j-1}^u)\chi_{\tilde{\Gamma}^c}(\pi_j^u)$$

$$\chi_{\tilde{\Gamma}_1^c}(\pi_{j+1}^u)\ldots\chi_{\tilde{\Gamma}_1^c}(\pi_{i-1}^u)\chi_{\tilde{\Gamma}_1}(\pi_i^u)\pi_i^u(f)\}$$

and

$$\Gamma_1 \ni x \mapsto \quad E_x^u\{\chi_{\tilde{\Gamma}}(\pi_1^u)\ldots\chi_{\tilde{\Gamma}}(\pi_i^u)[F(\pi_i^u) + \chi_{\tilde{\Gamma}^c}(\pi_{i+1}^u)F(\pi_{i+1}^u)$$

$$+\chi_{\tilde{\Gamma}^c}(\pi_{i+1}^u)\chi_{\tilde{\Gamma}_1^c}(\pi_{i+2}^u)\ldots\chi_{\tilde{\Gamma}_1^c}(\pi_k^u)F(\pi_k^u)]\}$$

for $i, j, k = 1, 2, \ldots$ $(j < i < k-1)$.

For this purpose it is sufficient in turn to show by induction the following property:

for $F \in C(P(E))$, $i = 1, 2, \ldots$, the mapping

$$\Gamma_1 \ni x \mapsto E_x^u\{\chi_{\Delta_1}(\pi_1^u)\ldots\chi_{\Delta_i}(\pi_i^u)F(\pi_i^u)\} \tag{4.119}$$

with Δ_k, $k = 1, 2, \ldots, i$ standing for $\tilde{\Gamma}$, $\tilde{\Gamma}^c$, $\tilde{\Gamma}_1$ or $\tilde{\Gamma}_1^c$, is continuous.

Step $i = 1$ follows immediately from the Feller property of $\prod^u$ (see (4.55)), and (A10),(A11)(i). Given (4.119) true for i, we have for $i+1$

$$E_x^u\{\chi_{\Delta_1}(\pi_1^u)\ldots\chi_{\Delta_{i+1}}(\pi_{i+1}^u)F(\pi_{i+1}^u)\} =$$
$$= E_x^u\{\chi_{\Delta_1}(\pi_1^u)\ldots\chi_{\Delta_i}(\pi_i^u)\prod^u(\pi_i^u, \chi_{\Delta_{i+1}}F)\}$$

and by the continuity of the mapping $\nu \mapsto \prod^u(\nu, \chi_{\Delta_{i+1}}F)$ (follows from step $i = 1$) and the induction hypothesis, we obtain (4.119) for $i+1$. Therefore (4.119) holds and consequently the mappings (4.117) and (4.118) are continuous.

■

In (D7) (iii) we assume that the embedded Markov chains $(x_{\tau_n}^m)$ corresponding to control functions u_m have invariant measures $\eta_m^{u_m}$. Since Γ_1 is compact, the measures are tight, and therefore one can choose a convergent subsequence. In the next lemma we identify the limit

Lemma 4.42 *Under (A1),(A2),(A8)-(A11) and (D1),(D4)-(D7) we have*

$$\eta_m^{u_m} \Rightarrow \eta^u \qquad as \quad m \to \infty \tag{4.120}$$

where η^u is the unique invariant measure according to (A11)(iv).

P r o o f. Assume $\overline{\eta}^u$ is a weak limit of a subsequence $\eta_{m_k}^{u_{m_k}}$, for simplicity denoted by $\eta_m^{u_m}$. By Lemma 4.41 and (4.111) for $f \in C(\Gamma_1)$ we have

$$\begin{aligned}
&\Big|\int_{\Gamma_1} E_x^u\{f(x_\tau)\}\overline{\eta}^u(dx) - \int_{\Gamma_1} f(x)\overline{\eta}^u(dx)\Big| \leq \\
&\leq \Big|\int_{\Gamma_1} E_x^u\{f(x_\tau)\}(\overline{\eta}^u(dx) - \eta_m^{u_m}(dx))\Big| + \\
&+\Big|\int_{\Gamma_1}(E_x^u\{f(x_\tau)\} - E_x^{u_m,m}\{f(x_\tau^m)\})\eta_m^{u_m}(dx)\Big| \\
&+\Big|\int_{\Gamma_1} f(x)(\eta_m^{u_m}(dx) - \overline{\eta}^u(dx))\Big| \to 0
\end{aligned}$$

as $m \to \infty$.

Therefore $\overline{\eta}^u$ is invariant for (x_{τ_n}) and by (A11) (iv) $\overline{\eta}^u = \eta^u$.

■

We are now in a position to complete the proof of Theorem 4.35. By (4.98) and (4.100) it suffices to show that for $F \in C(P(E))$

$$\begin{aligned}
&\int_{\Gamma_1} E_x^{u_m,m}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{m,u_m})\Big\}\eta_m^{u_m}(dx) \\
&\to \int_{\Gamma_1} E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^u)\Big\}\eta^u(dx) \quad \text{as } m \to \infty
\end{aligned} \tag{4.121}$$

We have

$$\begin{aligned}
&\Big|\int_{\Gamma_1} E_x^{u_m,m}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{m,u_m})\Big\}\eta_m^{u_m}(dx) \\
&- \int_{\Gamma_1} E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^u)\Big\}\eta^u(dx)\Big| \leq
\end{aligned}$$

$$\leq \sup_{x\in\Gamma_1} |E_x^{u_m,m}\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^{m,u_m})\Big\} - E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^u)\Big\}|$$

$$+\int_{\Gamma_1} E_x^u\Big\{\sum_{i=0}^{\tau-1} F(\pi_i^u)\Big\}(\eta_m^{u_m}(dx) - \eta^u(dx)) \to 0$$

as $m \to \infty$ by (4.112), Lemma 4.41 and Lemma 4.42. Consequently $\Phi_m^{u_m} \Rightarrow \Phi^u$ as $m \to \infty$. The proof of Theorem 4.35 is completed.

■

Remark 4.43 *In the case when (A8) and (A9) are replaced by (A8') and (A9') the assertion of Theorem 4.35 still holds if instead of (D5) we assume just (D5)(i) with uniform in m boundedness of $r_m(x,y)$ on $\Gamma^c \times \Gamma^c$ as well as (D5)(ii). In fact, by the uniform boundedness of r_m, to prove Theorem 4.35, we need Lemma 4.37 to hold only on compact subsets of $\Gamma^c \times P(E)$. On the other hand assumption (D5)(iii), which here is not satisfied, was only used in Lemma 4.37 to treat the case when $y \in \partial\Gamma^c$.*

Remark 4.44 *In the case when E is finite (see Remark 4.17) under the continuity assumption (4.62) (that corresponds to (A2)) and (A11)(ii)-(iv), the statement of Theorem 4.35 is true if the transition and observation matrices $p^a(i,j)$, $r(i,j)$ are approximated by $p_m^a(i,j)$ and $r_m(i,j)$ respectively in such a way that*

$$p_m^{a_m}(i,j) \to p^a(i,j),$$

whenever $a_m \to a$ and $m \to \infty$,

$$r_m(i,j) \to r(i,j),$$

as $m \to \infty$, and furthermore also (D6) and (D7)(i)-(iii) are satisfied.

4.5.1.b The embedded i.i.d. case

This further subsection is devoted to the study of the embedded i.i.d. case, for which we restrict ourselves to the two particular cases considered in examples 1 and 2 of section 4.4.2.

Starting with Example 1 assume that the state and observation processes (x_n), (y_n) satisfy (A1)–(A4), (E 1.1)–(E 1.4).

Furthermore assume that

(E 1.5) the d-dimensional Lebesgue measure of ∂D_1 is zero, where D_1 is as in (E 1.1), and that the state process (x_n) is controlled by a continuous control function $u \in C(P(E), U) = \mathcal{A}$.

Let (x_n) be approximated by (x_n^m) with transition operator P_m^a, observation density $r_m(x, y)$, and control function $u_m \in \mathcal{B}(P(E), U)$ for which (D1), (D2) and (D6) are satisfied.

Assume moreover

($\mathrm{E_m}$ 1.1) $r_m(x, y) = 0$ for $y \in D_1$ and $x \notin K$, where D_1 and K are as in (E 1.1)

($\mathrm{E_m}$ 1.2) for each m there exists $\lambda_m \in P(E)$ such that for $x \in K$, $a \in U$, $P_m^a(x, \cdot) = \lambda_m(\cdot)$

($\mathrm{E_m}$ 1.3) $\inf_m \inf_{z \in K} \int_{D_1} r_m(z, y) dy = \overline{\beta}_1 > 0$

($\mathrm{E_m}$ 1.4) $\inf_m \inf_{a \in U} \inf_{z \in E \setminus K} P_m^a(z, K) = \overline{\beta}_2 > 0$

Then by Proposition 4.27 there are unique invariant measures Φ^u and $\Phi_m^{u_m}$ for the filtering processes (π_n^u) and (π_n^{m,u_m}) corresponding to (x_n) and (x_n^m) respectively, and they are of the following forms, for $F \in b\mathcal{B}(P(E))$

$$\Phi^u(F) = E_\lambda^u \Big\{ \sum_{i=0}^{\sigma} F(\pi_i^u) \Big\} (E_\lambda\{\sigma + 1\})^{-1} \tag{4.122}$$

$$\Phi_m^{u_m}(F) = E_{\lambda_m}^{u_m,m} \Big\{ \sum_{i=0}^{\sigma} F(\pi_i^{m,u_m}) \Big\} (E_{\lambda_m}^{u_m,m}\{\sigma + 1\})^{-1} \tag{4.123}$$

with σ defined in (4.78), and having here a double usage analogous to τ in (4.111) and (4.112).

We have the following convergence theorem

Theorem 4.45 *Under (A1)–(A4), (E 1.1)–(E 1.5), (D1), (D2), (D6) and ($\mathrm{E_m}$ 1.1)–($\mathrm{E_m}$ 1.4), we have*

$$\Phi_m^{u_m} \Rightarrow \Phi^u \ \textit{weakly in } P(P(E)),$$

as $m \to \infty$.

P r o o f. By analogy to the proof of Theorem 4.35 it suffices to show the convergence of the numerator of (4.123) to the numerator of (4.122) for $F \in C(P(E))$.

Using similar arguments as in the proof of Corollary 4.40, by Lemma 4.26 it is enough to prove that for any positive integer i

$$\begin{aligned} &E^{u_m,m}_{\lambda_m}\{\chi_{D_1^c}(y_1^m)\dots\chi_{D_1^c}(y_{i-1}^m)\chi_{D_1}(y_i)\sum_{k=0}^{i}F(\pi_k^{m,u_m})\} \\ &\to E^u_\lambda\{\chi_{D_1^c}(y_1)\dots\chi_{D_1^c}(y_{i-1})\chi_{D_1}(y_i)\sum_{k=0}^{i}F(\pi_k^u)\} \end{aligned} \tag{4.124}$$

as $m \to \infty$, with (y_n^m) standing for the observation process corresponding to (x_n^m).

The proof of (4.124) is a immediate consequence of the following Lemma

Lemma 4.46 *Under (A1)–(A4), (D1), (D2), (D6) and (E 1.2), (E 1.5), (E_m 1.2), if $b\mathcal{B}(P(E)) \ni F_m \to F \in C(P(E))$ uniformly on compact subsets of $P(E)$, and F_m are uniformly in m bounded we have for any positive integer i*

$$\begin{aligned} &E^{u_m,m}_{\lambda_m}\{\chi_{\Delta_1}(y_1^m)\dots\chi_{\Delta_i}(y_i^m)F_m(\pi_i^{m,u_m})\} \\ &\to E^u_\lambda\{\chi_{\Delta_1}(y_1)\dots\chi_{\Delta_i}(y_i)F(\pi_i^u)\}. \end{aligned} \tag{4.125}$$

as $m \to \infty$, where Δ_k is any of D_1, D_1^c or R^d.

P r o o f. Notice first that by (E 1.2), (E_m 1.2) and (D1), we have $\lambda_m \Rightarrow \lambda$.

Now, let for $\mu \in P(E)$

$$G_m^{u_m,i}(\mu) := E_\mu^{u_m,m}\{\chi_{\Delta_i}(y_1^m)F_m(\pi_1^{m,u_m})\}$$

and

$$G^{u,i}(\mu) := E^u_\mu\{\chi_{\Delta_i}(y_1)F(\pi_1^u)\}$$

Clearly, by the definition of r and r_m

$$G_m^{u_m,i}(\mu) = \int_E\int_{\Delta_i} F_m(M_m^{u_m}(y,\mu))r_m(z,y)dyP_m^{u_m(\mu)}(\mu,dz)$$

and

$$G^{u,i}(\mu) = \int_E \int_{\Delta_i} F(M^u(y,\mu)) r(z,y) dy P^{u(\mu)}(\mu, dz)$$

Repeating the considerations of the proof of Proposition 3.7, taking into account (E 1.5) we obtain that for $i = 1, 2, \ldots$,

$$G_m^{u_m,i}(\mu) \to G^{u,i}(\mu) \tag{4.126}$$

as $m \to \infty$, uniformly in μ from compact subsets of $P(E)$.

By (4.126) we immediately have that (4.125) holds with $i = 1$. Assuming that (4.125) holds true for i, we have for $i + 1$

$$\begin{aligned}
&E_{\lambda_m}^{u_m,m}\{\chi_{\Delta_1}(y_1^m) \ldots \chi_{\Delta_{i+1}}(y_{i+1}^m) F_m(\pi_{i+1}^{m,u_m})\} \\
&= E_{\lambda_m}^{u_m,m}\{\chi_{\Delta_1}(y_1^m) \ldots \chi_{\Delta_i}(y_i^m) G_m^{u_m,i+1}(\pi_i^{m,u_m})\} \\
&\to E_\lambda^u\{\chi_{\Delta_1}(y_1) \ldots \chi_{\Delta_i}(y_i) G^{u,i+1}(\pi_i^u)\} \\
&= E_\lambda^u\{\chi_{\Delta_1}(y_1) \ldots \chi_{\Delta_{i+1}}(y_{i+1}) F(\pi_{i+1}^u)\}
\end{aligned}$$

as $m \to \infty$, by (4.126) and the induction hypothesis. Therefore by induction we have (4.125) and the proof of Lemma 4.46 is finished.

■

With the proof of Lemma 4.46 we also completed the proof of Theorem 4.45.

■

Consider now the model corresponding to example 2 of section 4.4.2. Assume additionally to (E 2.1)–(E 2.3)

(E 2.4) for each open set O, $a \in U$ and $x \in E$ we have $P^a(x, O) > 0$

and denote by $\overline{\mathcal{A}}$ the following class of control functions that correspond to those in (4.92) when $\overline{u}$, and therefore also u, are continuous

$$\overline{\mathcal{A}} = \{u \in \mathcal{A}\colon \text{ there is } \overline{u} \in \mathcal{A} \text{ and a positive integer } \overline{m} \text{ such that for } \nu \in P(E),\ u(\nu) = \overline{u}(\nu) r(\nu(\psi_{\overline{m}}))\}$$

Assume the state process (x_n), that is controlled by a control function $u \in \overline{\mathcal{A}}$, i.e. $u(\nu) = \overline{u}(\nu) r(\nu(\psi_{\overline{m}}))$, $\overline{u} \in \mathcal{A}$, where now $\overline{m}$ is a fixed positive integer, is approximated by (x_n^m) with a transition operator P_m^a and the observations (y_n^m)

$$y_n^m = h_m(x_n^m) + w_n^m \tag{4.127}$$

where $h_m \in b\mathcal{B}(E, R^d)$ and w_n^m are i.i.d. standard Gaussian random variables, independent of x_k^m for $k \leq n$.

Moreover, assume

(D8) h_m are uniformly in m bounded, h_m converges uniformly on compact sets of E to h; moreover for the j-th coordinate, for which (E 2.1) holds, the convergence of h_m^j to h^j is uniform on E

($\mathrm{E_m}$ 2.2) For each n there is $\lambda_m \in P(E)$ such that

$$P_m^a(x, \cdot) = \lambda_m(\cdot)$$

for $a = 0$ and $x \in E$

($\mathrm{E_m}$ 2.3) For any compact set $K \subset E$ there exists $\overline{\alpha} > 0$ such that

$$\inf_m \inf_{a \in U} \inf_{x \in E} P_m^a(x, K^c) \geq \overline{\alpha}$$

Furthermore, we assume that (x_n^m) is controlled with a control function u_m that has the following properties

(D9) $u_m(\nu) \to u(\nu)$ uniformly on compact subsets of $P(E)$. Furthermore u_m is of the form

$$u_m(\nu) = \overline{u}_m(\nu) r(K_m \nu(\psi_{\overline{m}})) \tag{4.128}$$

where $\overline{u}_m \in \mathcal{B}(P(E), U)$ and, uniformly on compact subsets of $P(E)$ $\overline{u}_m(\nu) \to \overline{u}(\nu)$, as $m \to \infty$ with $\overline{u}$ being the same as for the limit control $u \in \overline{\mathcal{A}}$ that controls the process (x_n). Finally, $K_m : P(E) \mapsto P(E)$ converge uniformly on compact subsets of $P(E)$, as $m \to \infty$ to the identity transformation of $P(E)$, and if $\nu(K_{\overline{m}+1}) \leq b$, we have $K_m \nu(\psi_{\overline{m}}) \leq b$, for $m = 1, 2, \ldots$, $\nu \in P(E)$.

Denote by $(\pi_n^{m u_m})$ the filtering process corresponding to (x_n^m) and (y_n^m). Assuming m large enough so that by (D8) h_m inherits the growth property

(E 2.1) of h, by Lemma 4.28 and the proof of Proposition 4.30, there is a unique invariant measure $\Phi_m^{u_m}$ of $(\pi_n^{mu_m})$ and it is of the form

$$\Phi_m^{u_m}(F) = E_{\lambda_m}^{u_m,m}\Big\{\sum_{i=0}^{\overline{\sigma}_m} F(\pi_i^{m,u_m})\Big\}(E_{\lambda_m}^{u_m,m}\{\overline{\sigma}_m+1\})^{-1} \tag{4.129}$$

for $F \in b\mathcal{B}(P(E))$, with

$$\overline{\sigma}_m = \inf\{i > 0,\ K_m\pi^{m,u_m}(\psi_{\overline{m}}) \le b\} \tag{4.130}$$

By the same Proposition 4.30, there is also a unique invariant measure Φ^u of (π_n^u) and it has the form

$$\Phi^u(F) = E_\lambda^u\Big\{\sum_{i=0}^{\overline{\sigma}} F(\pi_i^u)\Big\}(E_\lambda^u\{\overline{\sigma}+1\})^{-1} \tag{4.131}$$

with $\overline{\sigma}$ defined in (4.94).

We have

Theorem 4.47 *Under (A1), (A2), (E 2.1)–(E 2.4), (E_{m} 2.2), (E_{m} 2.3), (D1), (D8), (D9)*

$$\Phi_m^{u_m} \Rightarrow \Phi^u \ \textit{weakly in}\ P(P(E))$$

as $m \to \infty$.

P r o o f. By Lemma 4.28 applied to $M_m^{u_m}$ (defined in (3.14) with $r_m(x,y)$ given as in (1.2) for h equal to h_m) we have that for $m > m_1$, where m_1 is such that by (D8) h_m inherits the growth property (E 2.1) of h (compare to the proof of Corollary 4.29)

$$\sup_{m>m_1}\sup_{\mu\in P(E)} E_\mu^{u_m,m}\{\overline{\sigma}_m^2\} < \infty \tag{4.132}$$

Therefore by analogy to the proofs of Theorems 4.35 and 4.45 it suffices to show the convergence of

$$E_{\lambda_m}^{u_m,m}\{\chi_{\Delta_1}(K_m\pi_1^{m,u_m})\dots\chi_{\Delta_i}(K_m\pi_i^{m,u_m})F(\pi_i^{m,u_m})\}$$

to

$$E_\lambda^u\{\chi_{\Delta_1}(\pi_1^u)\dots\chi_{\Delta_i}(\pi_i^u)F(\pi_i^u)\} \tag{4.133}$$

as $m \to \infty$ with $\Delta_k = \{\nu \in P(E)\colon \nu(\psi_{\overline{m}}) > b\}$, or $\Delta_k = \{\nu \in P(E)\colon \nu(\psi_{\overline{m}}) \le b\}$, or $\Delta_k = P(E)$, for $k = 1, 2, \dots, i$.

To prove the convergence (4.133) we need the following auxiliary result

Lemma 4.48 *Under (A1), (A2), (E 2.1), (E 2.4), if Z is a continuous R^d-valued random variable, and $0 \le \varphi \in C(E)$ has a compact support, then for any $b > 0$, $\nu \in P(E)$, $u \in \overline{\mathcal{A}}$*

$$P\{M^u(Z,\nu)(\varphi) = b\} = 0$$

P r o o f. If $P\{M^u(Z,\nu)(\varphi) = b\} > 0$, for some $b > 0$, then by the continuity of the mapping $y \mapsto M^u(y,\nu)(\varphi)$ (Proposition 1.4), we have $M^u(y,\nu)(\varphi) = b$ for y from some open set G of R^d, or, equivalently,

$$\int_E \exp[(y,h(z)) - \frac{1}{2}(h(z),h(z))](\varphi(z) - b)P^{u(\nu)}(\nu,dz) = 0 \qquad (4.134)$$

for $y \in G$.

Assume (by E 2.1) that the j-th coordinate h^j attains its strong maximum at "∞". Differentiating m times (4.134) with respect to y_j, we obtain

$$\int_E (h^j(z))^m \exp[(y,h(z)) - \frac{1}{2}(h(z),h(z))](\varphi(z) - b)P^{u(\nu)}(\nu,dz) = 0$$

for $y \in G$.

Therefore, for any continuous function $g\colon [-\|h\|, \|h\|] \mapsto R$, and $y \in G$

$$\int_E g(h^j(z)) \exp[(y,h(z)) - \frac{1}{2}(h(z),h(z))](\varphi(z) - b)P^{u(\nu)}(\nu,dz) = 0 \quad (4.135)$$

Since φ has compact support, there is n such that $\varphi(x) = 0$ for $x \notin B_n$. By (E 2.1) there is a compact set $K \supset B_n$ such that

$$\inf_{z \in K^c} h^j(z) = a_1 > \sup_{z \in B_n} h^j(z)$$

Let now $g(a) = 0$ for $a \le a_1$, and $g(a) > 0$ for $a > a_1$. Then by (4.135)

$$\int_{B_n^c} g(h^j(z)) \exp[(y,h(z)) - \frac{1}{2}(h(z),h(z))](-b)P^{u(\nu)}(\nu,dz) = 0$$

Since $g(h^j(z))$ is strictly positive on an open subset of B_n^c, by (E 2.4) we obtain $b = 0$, a contradiction.

■

To show the convergence (4.133) we now prove the following

Lemma 4.49 *Under (A1), (A2), (E 2.1)–(E 2.4), (E_m 2.2), (E_m 2.3), (D1), (D8), (D9) if* $b\mathcal{B}(P(E)) \ni F_m \to F \in C(P(E))$, *uniformly on compact subsets of* $P(E)$, *as* $m \to \infty$, *and* F_m *are uniformly bounded in* m, *we have*

$$E^{u_m,m}_{\lambda_m}\{\chi_{\Delta_1}(K_m\pi^{m,u_m}_1)\dots\chi_{\Delta_i}(K_m\pi^{m,u_m}_i)F_m(\pi^{m,u_m}_i)\} \\ \to E^u_\lambda\{\chi_{\Delta_1}(\pi^u_1)\dots\chi_{\Delta_i}(\pi^u_i)F(\pi^u_i)\} \tag{4.136}$$

as $m \to \infty$, *with* Δ_k, *for* $k = 1, 2, \dots, i$ *as in (4.133).*

P r o o f. By analogy to the proof of Lemma 4.46 we have $\lambda_m \Rightarrow \lambda$ and it suffices to show that

$$G^{u_m,i}_m(\mu) := E^{u_m,m}_\mu\{\chi_{\Delta_i}(K_m\pi^{m,u_m}_1)F_m(\pi^{m,u_m}_1)\}$$

converges as $m \to \infty$, uniformly on compact subsets of $P(E)$ to

$$G^{u,i}(\mu) := E^u_\mu\{\chi_{\Delta_i}(\pi^u_1)F(\pi^u_1)\}$$

The last convergence follows from Lemma 4.48 and considerations similar to those of the proof of Proposition 3.7.

■

Since by (4.136) clearly (4.133) holds, the proof of Theorem 4.47 is completed.

■

4.5.2 Specific approximations

The general convergence theorems 4.35, 4.45 and 4.47 of the previous subsection will now be applied to more specific approximations for the mixed observation model as well as for the models of the examples 1 and 2 described in section 4.4.2. Namely, by analogy to the approximations for the case of continuous controls in the discounted cost problem (sections 3.3.1.b and 3.3.2.b), we shall first approximate the class of admissible control functions $\mathcal{A}$ and then discretize the state and observation spaces.

Since we shall need results from section 3.3.1.b, in what follows we assume that the set of control parameters U is a compact, convex subset of R^l, $l \geq 1$, that contains the origin of R^l.

Moreover in the case of mixed observations we also assume that

(A12) for any compact set $K \subset E$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{z\in K}\sup_{a\in U} P^a(z,\Gamma(\delta)) < \varepsilon$$

with $\Gamma(\delta)$ as in (D4).

4.5.2.a Control function approximation

Let the class $\mathcal{A}(L,n)$ with $L > 0$ and positive integer n be as in (3.37).

We have (compare with Corollary 3.14)

Proposition 4.50 *Assume, for the mixed observation model that (A1), (A2), (A5), (A8)–(A12) are satisfied. Then, for $\mu \in P(E)$*

$$\lim_{L\to\infty}\lim_{n\to\infty}\inf_{u\in\mathcal{A}(L,n)} J_\mu(u) = \inf_{u\in\mathcal{A}} J_\mu(u) \tag{4.137}$$

P r o o f. By Proposition 3.12 each $u \in \mathcal{A}$ can be approximated, uniformly on compact subsets of $P(E)$ by $u_{L,n} \in \mathcal{A}(L,n)$ with $L \to \infty$, $n \to \infty$.

Let $\Phi^{u_{L,n}}$, Φ^u be invariant measures corresponding to the controlled filtering processes $(\pi_i^{L,n})$, (π_i^u) (by Proposition 4.32 these invariant measures exist and are unique). Then by Theorem 4.35

$$\Phi^{u_{L,n}} \Rightarrow \Phi^u \text{ as } L,\ n \to \infty$$

Consequently the family $\{\Phi^u, \Phi^{u_{L,n}},\ L > 0,\ n > 0\}$ is tight, and for given $\varepsilon > 0$ there is a compact set $H \subset P(E)$ such that

$$\Phi^{u_{L,n}}(H) \geq 1 - \varepsilon, \quad \text{for } L > 0,\ n > 0.$$

By (A5) we clearly have

$$\int_E c(x,u_{L,n}(\nu))\nu(dx) \to \int_E c(x,u(\nu))\nu(dx)$$

as $L,\ n \to \infty$, uniformly in $\nu \in H$.

Therefore, by (1.12) and using Remark 4.34 we obtain

$$|J_\mu(u) - J_\mu(u_{L,n})| \leq \Big| \int_{P(E)}\int_E c(z,u(\nu))\nu(dz)(\Phi^u(d\nu) - \Phi^{u_{L,n}}(d\nu))\Big|$$

$$+ \int_H \Big|\int_E c(z,u(\nu))\nu(dz) - \int_E c(z,u_{L,n}(\nu))\nu(dz)\Big|\Phi^{u_{L,n}}(d\nu) + 2\|c\|\varepsilon$$

Letting $L,\ n \to \infty$, since ε can be chosen arbitrarily small, we have

$$\lim_{L,\ n \to \infty} J_\mu(u_{L,n}) = J_\mu(u)$$

from which (4.137) easily follows.

■

Remark 4.51 *As one can see from the proof of the last Proposition, if $\mathcal{A} \ni u_m \to u \in \mathcal{A}$, uniformly on compact subsets of $P(E)$, we always have $J_\mu(u_m) \to J_\mu(u)$.*

Almost analogously, using Theorem 4.45, as well as Proposition 4.27 we obtain the following result for the model of Example 1.

Proposition 4.52 *Assume that for Example 1 the assumptions (A1)–(A5), (E 1.1)–(E 1.5) are satisfied. Then, for $\mu \in P(E)$*

$$\lim_{L\to\infty} \lim_{n\to\infty} \inf_{u \in \mathcal{A}(L,n)} J_\mu(u) = \inf_{u \in \mathcal{A}} J_\mu(u) \tag{4.138}$$

■

In the case of Example 2, recalling the definition given in section 4.5.1.b of the class $\overline{\mathcal{A}} \subset \mathcal{A}$, define a class $\overline{\mathcal{A}}(L,n)$ by

$$\begin{aligned}\overline{\mathcal{A}}(L,n) := \{u \in \overline{\mathcal{A}},\ u(\nu) = \overline{u}(\nu(\varphi_1),\ldots,\nu(\varphi_m))r(\nu(\psi_m)),\ \text{where}\\ \overline{u}: [-\|\varphi_1\|,\|\varphi_1\|] \times \ldots \times [-\|\varphi_n\|,\|\varphi_n\|] \to 0 \text{ is Lipschitz}\\ \text{with constant } L \text{ and } \psi_n \text{ is as in (4.91)}\}\end{aligned}$$

Notice that, by Proposition 3.12 and the form of the functions r and ψ_n (see (4.90), (4.91)) it is immediately seen that any $u \in \mathcal{A}$ (in particular $u \in \overline{\mathcal{A}}$) can be uniformly approximated on compact sets of $P(E)$ by functions from $\overline{\mathcal{A}}(L,n)$ for $L,\ n \to \infty$. Analogously to Propositions 4.50 and 4.52 using this time Proposition 4.30 and Theorem 4.47 we have

Proposition 4.53 *Under (A1), (A2), (A5), (E 2.1)–(E 2.4) for Example 2 we have with $\mu \in P(E)$*

$$\lim_{L\to\infty} \lim_{n\to\infty} \inf_{u \in \overline{\mathcal{A}}(L,n)} J_\mu(u) = \inf_{u \in \overline{\mathcal{A}}} J_\mu(u) \tag{4.139}$$

■

4.5.2.b Discretization of state and observations

Consider now the discretization of the state and observation spaces introduced in section 3.3.2. We shall adjust the form of the partitions of E and R^d so that successively the assumptions of Theorems 4.35, 4.45 and 4.47 are satisfied.

4.5.2.b$_1$ The mixed observation case

In the case of the mixed observation model we assume that $k_m = s_m$, $B_k^m = D_k^m$, for $k = 1, 2, \ldots, k_m$, and that there exist k_r, k_p, $k_r \leq k_p < k_m$ such that

$$\bigcup_{k=1}^{k_r} B_k^m = \Gamma_1 \qquad \bigcup_{k=1}^{k_p} B_k^m = \Gamma$$

Then define

$$\begin{aligned} r_m(x,y) = \ & \Big(\int\limits_{B_k^m} dz\Big)^{-1}\Big[\int\limits_{B_k^m} r(b_j^m, z)dz + \\ & \frac{1}{k_m - k_p - 1}\int\limits_{B_{k_m}^m} r(b_j^m, z)dz\Big] \end{aligned} \tag{4.140}$$

for $x \in B_j^m$, $y \in B_k^m$ with $k_p < k < k_m$, $j > k_p$, and $r_m(x,y) = 0$ for $y \in B_{k_m}^m$ and $y \notin \Gamma^c$.

Let now $E_m = \{1, 2, \ldots, k_m\}$, $\Gamma_1^m = \{1, 2, \ldots, k_r\}$, $\Gamma^m = \{1, 2, \ldots, k_p\}$, and for $j,\ k$, with $k_p < j,\ k \leq k_m$

$$\overline{r}_m(j,k) = \int\limits_{B_k^m} r_m(b_j^m, z)dz \tag{4.141}$$

For a fixed $u \in \mathcal{A}$ consider the following two partially observed systems

I. The unobserved process (x_i^m) evolves in E according to the transition operator $P_m^{a_i}(x, \cdot)$, defined in (3.50), with initial law μ and observations (y_i^m) such that

$$\begin{aligned} & P\{y_{i+1}^m \in A | x_0^m, \ldots, x_{i+1}^m, y_1^m, \ldots, y_i^m\} = \chi_{A\cap\Gamma}(x_{i+1}^m) + \\ & + \chi_{\Gamma^c}(x_{i+1}^m) \int\limits_{A\cap\Gamma^c} r_m(x_{i+1}^m, y)dy, \end{aligned} \tag{4.142}$$

the control in the generic period i is $a_i = \overline{\mathcal{L}}_m u(\pi_i^m)$, where the operator $\overline{\mathcal{L}}_m$ is defined above (3.71) and π_i^m stands for the filtering process corresponding to x_i^m with observations y_i^m.

II. The unobserved process $(\overline{x}_i^m)$ evolves in E_m according to the transition matrix $\overline{P}_m^{a_i}(k,p)$, defined in (3.52), with initial law $(\mu(B_1^m),\ldots,\mu(B_{k_m}^m))$ and observations $\overline{y}_i^m \in E_m$ satisfying

$$P\{\overline{y}_{i+1}^m = k|\overline{x}_0^m,\ldots,\overline{x}_i^m,\overline{x}_{i+1}^m = j,\overline{y}_1^m,\ldots,\overline{y}_i^m\} = \chi_{k\cap\Gamma^m}(j)+ \\ +\chi_{(\Gamma^m)^c}(j)\overline{r}_m(j,k) \tag{4.143}$$

for $1 \le k \le k_m$ with $\overline{r}_m(j,k)$ according to (3.54); the control in the generic period i is $a_i = \widetilde{\mathcal{L}}_m u(\overline{\pi}_i^m)$, where the operator $\widetilde{\mathcal{L}}_m$ is also defined above (3.71) and $\overline{\pi}_i^m$ is the filtering process that corresponds to $\overline{x}_i^m$ and $\overline{y}_i^m$.

By analogy to (4.49) define

$$M_m^a(y,\nu)(A) = \chi_{A\cap\Gamma}(y) + \chi_{\Gamma^c}(y)N_m^a(y,\nu)(A) \tag{4.144}$$

with

$$N_m^a(y,\nu)(A) = \int\limits_{A\cap\Gamma^c} r_m(z,y)P_m^a(\nu,dz)\Big(\int\limits_{\Gamma^c} r_m(z,y)P_m^a(\nu,dz)\Big)^{-1}$$

for $y \in E$, $\nu \in P(E)$, and

$$\overline{M}_m^a(y,\eta)(k) = \chi_{k\cap\Gamma^m}(y) + \chi_{(\Gamma^m)^c}(y)\overline{N}_m^a(y,\eta)(k) \tag{4.145}$$

with

$$\overline{N}_m^a(y,\eta)(k) = \overline{r}_m(k,y)\overline{P}_m^a(\eta,k)\Big(\sum_{j=1}^{k_m}\overline{r}_m(j,y)\overline{P}_m^a(\eta,j)\Big)^{-1}$$

for $y \in E_m$, $\eta \in P(E_m)$ and letting $\overline{P}_m^a(\eta,k) = \sum_{j=1}^{k_m} \overline{P}_m^a(j,k)\eta_j$.

We have

Proposition 4.54 *The filtering processes* (π_i^m) *and* $(\overline{\pi}_i^m)$ *have the following representations*

$$\pi_{i+1}^m(A) = M_m^{\overline{\mathcal{L}}_m u}(y_{i+1}^m, \pi_i^m)(A) \qquad \text{for } A \in \mathcal{B}(E),\ P \text{ a.e.},$$
$$\overline{\pi}_{i+1}^m(k) = \overline{M}_m^{\widetilde{\mathcal{L}}_m u}(\overline{y}_{i+1}^m, \overline{\pi}_i^m)(k) \qquad \text{for } k \in E_m,\ P \text{ a.e.},$$
$$\pi_0^m(A) = \mu(A) \qquad \overline{\pi}_0^m = \mu(B_k^m)$$

The processes (π_i^m) *and* $(\overline{\pi}_i^m)$ *are Markov with respect to the σ-fields* $Y_m^i = \sigma\{y_1^m, \dots, y_i^m\}$ *and* $\overline{Y}_m^i = \sigma\{\overline{y}_1^m, \dots, \overline{y}_i^m\}$ *respectively, with transition operators*

$$\begin{aligned}\prod\nolimits_m^{\overline{\mathcal{L}}_m u}(\nu, F) = & \int_\Gamma F(\delta_z) P_m^{\overline{\mathcal{L}}_m u(\nu)}(\nu, dz) + \\ & + \int_{\Gamma^c}\int_{\Gamma^c} F(M_m^{\overline{\mathcal{L}}_m u(\nu)}(y, \nu)) r_m(z, y) dy P_m^{\overline{\mathcal{L}}_m u(\nu)}(\nu, dz)\end{aligned} \tag{4.146}$$

and

$$\begin{aligned}\overline{\prod}_m^{\widetilde{\mathcal{L}}_m u}(\eta, f) = & \sum_{k=1}^{k_p} f(\delta_k) \overline{P}_m^{\widetilde{\mathcal{L}}_m u(\eta)}(\eta, k) + \\ & + \sum_{k=k_p+1}^{k_m} \sum_{j=k_p+1}^{k_m} f(\overline{M}_m^{\widetilde{\mathcal{L}}_m u(\eta)}(k, \eta)) r_m(j, k) \overline{P}_m^{\widetilde{\mathcal{L}}_m u(\eta)}(\eta, j)\end{aligned} \tag{4.147}$$

for $F \in b\mathcal{B}(P(E))$, $f \in b\mathcal{B}(P(E_m))$, *and denoting by* δ_k *an element of* $P(E_m)$ *with* 1 *corresponding to the* k*-th coordinate and zeros elsewhere.*

Moreover, under (A2) and (B9), for $u \in \mathcal{A}$ *the operator* $\overline{\prod}_m^{\widetilde{\mathcal{L}}_m u}$ *is Feller i.e. it transforms* $C(P(E_m))$ *into itself.*

P r o o f . The first part of the Proposition follows by considerations analogous to those of Lemma 4.14 (compare with Lemma 1.1 and 1.3). The Feller property of $\overline{\prod}_m^{\widetilde{\mathcal{L}}_m u}$ is a consequence of (B9) and the proof of Proposition 4.15. ■

The filtering process $\overline{\pi}_i^m$ will play a fundamental role in the construction of nearly optimal control functions. For this purpose we consider the following ergodic cost functional (see (3.55))

$$J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) := \limsup_{n\to\infty} n^{-1} E_{\overline{\mu}}\Big\{\sum_{i=0}^{n-1} c(\overline{x}_i^m, \widetilde{\mathcal{L}}_m u(\overline{\pi}_i^m))\Big\} \tag{4.148}$$

with $u \in \mathcal{A}(L,n)$, $c(j,a)$ identified with $c(b_j^m, a)$ as in (3.55), and $\overline{\mu} = (\mu(B_1^m), \ldots, \mu(B_{k_m}^m))$.

Since we have equivalently

$$J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) = \limsup_{n\to\infty} n^{-1} E_{\overline{\mu}}\Big\{\sum_{i=0}^{n-1}\sum_{j=1}^{k_m} c(j, \widetilde{\mathcal{L}}_m u(\overline{\pi}_i^m))\overline{\pi}_i^m(j)\Big\} \tag{4.149}$$

the evaluation of this cost functional leads to the study of the ergodic properties of $(\overline{\pi}_i^m)$. Therefore corresponding to (A11) and with the symbols having an analogous meaning as in (A11) e.g. $T_{\Gamma^m} = \inf\{i > 0 \colon \overline{x}_i^m \in \Gamma^m\}$, we assume

(C13) (i) $E_j^{u,m} T_{\Gamma_1^m} < \infty$ for any $u \in C(P(E_m), U)$, $j = 1, 2, \ldots, k_m$

(ii) $\sup_m \sup_{u\in C(P(E_m),U)} \sup_{j=1,2,\ldots,k_r} E_j^{u,m}\{\tau^2\} < \infty$ with $\tau = T_{(\Gamma^m)^c} + T_{\Gamma_1^m} \circ \Theta_{T_{(\Gamma^m)^c}}$

(iii) for any $u \in C(P(E_m), U)$ there is a unique invariant measure $\overline{\eta}_m^u$ of $\overline{x}_{\tau_n}^m$ with $\tau_1 = \tau, \ldots, \tau_{n+1} = \tau_n + \tau \circ \Theta_{\tau_n}$,

where, for given $u \in C(P(E_m), U)$, $\overline{x}_i^m$ stands for the unobserved process on E_m with transition matrix $P^{u(\overline{\pi}_i^m)}(k,p)$ and $P_j^{u,m}$ denotes a measure generated by $(\overline{x}_i^m)$ with initial state $\overline{x}_0^m = j$.

Under (C13) Proposition 4.32 applies also to the process $(\overline{\pi}_i^m)$ so that for $u \in C(P(E_m), U)$ there is a unique invariant measure $\overline{\Phi}_m^u$ of $\overline{\pi}_i^m$ and it has the form

$$\overline{\Phi}_m^u(f) = \sum_{j=1}^{k_r} E_j^{u,m}\Big\{\sum_{i=0}^{\tau-1} f(\overline{\pi}_i^m)\Big\}\overline{\eta}_m^u(j) \cdot \Big(\sum_{j=1}^{k_r} E_j^{u,m}\{\tau\}\overline{\eta}_m^u(j)\Big)^{-1} \tag{4.150}$$

for $f \in b\mathcal{B}(P(E_m))$.

Consequently, by (4.149) and (4.99) we have

$$J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) = \int_{P(E_m)} \sum_{j=1}^{k_m} c(j, \widetilde{\mathcal{L}}_m u(\eta))\eta_j \overline{\Phi}_m^{\widetilde{\mathcal{L}}_m u}(d\eta) \tag{4.151}$$

We are now in a position to formulate the theorem justifying our discretizations as leading to a correct approximation

Theorem 4.55 *Assume (A1), (A2), (A5), (A8)–(A12) and (C13). Then, for $L, n > 0$*

$$\sup_{u\in\mathcal{A}(L,n)} |J^m_{\overline{\mu}}(\widetilde{\mathcal{L}}_m u) - J_\mu(u)| \to 0 \tag{4.152}$$

as $m \to \infty$, uniformly in $\mu \in P(E)$ with $\overline{\mu} = (\mu(B^m_1), \ldots, \mu(B^m_{k_m}))$.

P r o o f. Assume, contrary to (4.152) that

$$|J^m_{\overline{\mu}_m}(\widetilde{\mathcal{L}}_m u_m) - J_{\mu_m}(u_m)| \geq \delta > 0 \tag{4.153}$$

for $m = 1, 2, \ldots$, $u_m \in \mathcal{A}(L, n)$ and $\mu_m \in P(E)$.

By the compactness of the class $\mathcal{A}(L, n)$ we may assume that $u_m(\nu) \to u(\nu)$, $u \in \mathcal{A}(L, n)$, uniformly on $P(E)$. Therefore by Lemma 3.21(ii)

$$\overline{\mathcal{L}}_m u_m(\nu) \to u(\nu) \quad \text{as } m \to \infty \tag{4.154}$$

uniformly on compact subsets of $P(E)$.

Notice now, that for the process x^m_i and $\overline{x}^m_i$, defined in I and II with controls $a_i = \overline{\mathcal{L}}_m u_m(\pi^m_i)$ and $a_i = \widetilde{\mathcal{L}} u_m(\overline{\pi}^m_i)$ respectively, the corresponding Markov times T_{Γ_1}, T_{Γ^c}, τ, τ_n have the same distributions as $T_{\Gamma^m_1}$, $T_{(\Gamma^m)^c}$, τ, τ_n, respectively. Moreover, under (C13), the measure

$$\eta^{\overline{\mathcal{L}}_m u_m}(B) := \sum_{j=1}^{k_r} P^{\overline{\mathcal{L}}_m u_m, m}_{b^m_j}\{x^m_\tau \in B\} \overline{\eta}^{\widetilde{\mathcal{L}}_m u_m}_m(j) \tag{4.155}$$

for $B \in \mathcal{B}(\Gamma_1)$, is invariant for $(x^m_{\tau_n})$.

Therefore, by Remark 4.33

$$\begin{aligned}\Phi^{\overline{\mathcal{L}}_m u_m}_m(F) := &\int_{\Gamma_1} E^{\overline{\mathcal{L}}_m u_m, m}_x \Big\{ \sum_{i=0}^{\tau-1} F(\pi^m_i) \Big\} \eta^{\overline{\mathcal{L}}_m u_m}_m(dx) \\ &\Big(\int_{\Gamma_1} E^{\overline{\mathcal{L}}_m u_m, m}_x \{\tau\} \eta^{\overline{\mathcal{L}}_m u_m}_m(dx) \Big)^{-1}\end{aligned} \tag{4.156}$$

defined for $F \in b\mathcal{B}(P(E))$ is an invariant measure of π^m_i with control $a_i = \overline{\mathcal{L}}_m u_m(\pi^m_i)$.

Furthermore, by (4.150), (4.151), (4.155) and (4.156)

$$J^m_{\overline{\mu}_m}(\widetilde{\mathcal{L}}_m u_m) = \int_{P(E)} \int_E c_m(x, \widetilde{\mathcal{L}}_m u_m(\nu)) \nu(dx) \Phi^{\overline{\mathcal{L}}_m u_m}_m(d\nu) \tag{4.157}$$

with c_m as in (3.49).

Now, we use Theorem 4.35 with (x_i^m) defined in I and controlled by $a_i = \overline{\mathcal{L}}_m u_m(\pi_i^m)$ and the observations having density $r_m(x,y)$ defined in (4.140). For its applicability we need to check assumptions (D1) and (D4)–(D7). By (A1), (A2), clearly (D1) is satisfied, and (D4) follows from (A12). By (A8) and (A9) one can show (D5) (i)–(iii). Since by (4.154) and (C13) also assumptions (D6) and (D7) hold, from Theorem 4.35 we obtain that

$$\Phi_m^{\overline{\mathcal{L}}_m u_m} \Rightarrow \Phi^u \quad \text{weakly in } P(P(E)) \text{ as } m \to \infty$$

The next part of the proof is similar to that of Proposition 4.50. Namely, taking into account that c_m given by (3.49) satisfies (D3) (see Lemma 3.15) we obtain by (4.157), as well as (1.12), (4.98), (4.99) that

$$J_{\overline{\mu}_m}^m(\widetilde{\mathcal{L}}_m u_m) \to J_\mu(u) \quad \text{as } m \to \infty \tag{4.158}$$

The last convergence together with Remark 4.51 form a contradiction to (4.153). Thus (4.152) holds.

■

Corollary 4.56 *Under the assumptions of Theorem 4.55*

(i)
$$\inf_{u \in \mathcal{A}_m(L,n)} J_{\overline{\mu}}^m(u) \to \inf_{u \in \mathcal{A}(L,n)} J_\mu(u) \tag{4.159}$$

as $m \to \infty$, uniformly in $\mu \in P(E)$, with the class $\mathcal{A}_m(L,n)$ defined in 3.3.2.b.

(ii) *if for m such that*

$$\sup_{u \in \mathcal{A}(L,n)} |J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) - J_\mu(u)| < \varepsilon \tag{4.160}$$

a control function $\widetilde{\mathcal{L}}_m u_m$ with $u_m \in \mathcal{A}(L,n)$ is ε-optimal for $J_{\overline{\mu}}^m$ over $\mathcal{A}_m(L,n)$, then the control function u_m is 3ε optimal for J_μ over $\mathcal{A}(L,n)$.

P r o o f. Since $\widetilde{\mathcal{L}}_m \mathcal{A}(L,n) = \mathcal{A}_m(L,n)$, (see (3.72)) (i) immediately follows from (4.152). To show (ii) notice that, using (4.160)

$$J_\mu(u_m) \le J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u_m) + \varepsilon \le \inf_{u \in \mathcal{A}(L,n)} J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u_m) + 2\varepsilon \le \inf_{u \in \mathcal{A}(L,n)} J_\mu(u) + 3\varepsilon \tag{4.161}$$

■

Remark 4.57 *In view of Corollary 4.56, the problem of the construction of a nearly optimal control function u is now reduced to that of constructing a nearly optimal control function for* $J^m_{\overline{\mu}}$ *with respect to the class* $\widetilde{\mathcal{L}}_m u$ *where* $u \in \mathcal{A}(L,n)$ *(or equivalently with respect to the class* $\mathcal{A}_m(L,n)$*). By (4.151) the latter problem can be viewed as a finite dimensional, complete observation control problem of the filtering process* $(\overline{\pi}^m_i)$ *with values in the simplex* $P(E_m)$ *and ergodic cost functional given in (4.151). Notice moreover that for* $u \in \mathcal{A}_m(L,n)$ *there is a unique invariant measure* $\overline{\Phi}^u_m$ *of* $\overline{\pi}^m_i$ *and it has the form (4.150).*

4.5.2.b$_2$ The embedded i.i.d. case

In the remaining part of this section we point out the changes that are required for the models of Examples 1 and 2 and then we formulate the analogs of Theorem 4.55 together with Corollary 4.56 for these two cases.

In the case of Example 1 we assume that the partitions (B^m_k) of E and (D^m_s) of R^d introduced in 3.2.2 are such that

$$\bigcup_{k=1}^{k_p} B^m_k = K \quad \text{and} \quad \bigcup_{s=1}^{s_r} D^m_s = D_1$$

for some $k_p < k_m$, $s_r < s_m$, where K and D_1 are as in (E 1.1)–(E 1.4). Then, we let P^a_m be as in (3.50), $r_m(x,y)$ as in (3.48), and $\lambda_m = \lambda$ as in (E 1.2). Clearly, in this case (E$_m$ 1.1)–(E$_m$ 1.4) as well as (D1) (D2) are satisfied.

Let E_m, D_m be as in (3.51) and

$$K_m = \{1, \ldots, k_p\} \quad D^1_m = \{1, \ldots, s_r\} \tag{4.162}$$

Consider now the processes (x^m_i) and $(\overline{x}^m_i)$ with observations (y^m_i) and $(\overline{y}^m_i)$ defined in I and II respectively, with the only change that (4.142) is replaced by (1.1) with $r = r_m$ given by (3.48) and (4.143) is replaced by (3.54). The corresponding filtering processes (π^m_i) and $(\overline{\pi}^m_i)$ are Markov where, for $u \in \mathcal{A}(L,n)$, the transition operators $\prod^{\widetilde{\mathcal{L}}_m u}_m$ and $\overline{\prod}^{\widetilde{\mathcal{L}}_m u}_m$ are given by (3.15) and (3.58) respectively. Furthermore, under (A2) and (B9), for $u \in \mathcal{A}$ the operator $\overline{\prod}^{\widetilde{\mathcal{L}}_m u}_m$ is Feller. Let

$$\sigma^m = \inf\{i > 0,\ y^m_i \in D_1\} \tag{4.163}$$

and

$$\overline{\sigma}^m = \inf\{i > 0,\ \overline{y}_i^m \in D_m^1\}$$

By construction, it is clear that the distributions of σ^m and $\overline{\sigma}^m$ coincide. By analogy to (4.148) let

$$J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) := \limsup_{n\to\infty} n^{-1} E_{\overline{\mu}}\Big\{\sum_{i=0}^{n-1} c(\overline{x}_i^m, \widetilde{\mathcal{L}}_m u(\overline{\pi}_i^m))\Big\} \tag{4.164}$$

so that we have also (4.149) for which, using Proposition 4.27, we obtain analogously to (4.151)

$$J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) = \int\limits_{P(E_m)} \sum_{j=1}^{k_m} c(j, \widetilde{\mathcal{L}}_m u(\eta))\eta_j \overline{\Phi}_m^{\widetilde{\mathcal{L}}_m u}(d\eta) \tag{4.165}$$

with

$$\overline{\Phi}_m^{\widetilde{\mathcal{L}}_m u}(F) = E_\lambda\Big\{\sum_{i=0}^{\overline{\sigma}^m} F(\overline{\pi}_i^m)\Big\}(E_\lambda\{\overline{\sigma}^m + 1\})^{-1} \tag{4.166}$$

for $F \in b\mathcal{B}(P(E_m))$.

We have

Theorem 4.58 *Assume (A1)–(A5), (E 1.1)–(E 1.5). Then, for $L, n > 0$*

$$\sup_{u\in\mathcal{A}(L,n)} |J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) - J_\mu(u)| \to 0 \tag{4.167}$$

as $m \to \infty$, uniformly in $\mu \in P(E)$. Moreover, the statements (i) and (ii) of Corollary 4.56 also hold.

P r o o f. We repeat the steps of the proof of Theorem 4.55. That is, we have to show a contradiction to (4.153). To this end we first recall (see remarks before Theorem 4.45), that under the given assumption there exists invariant measures of (π_i^m) corresponding to x_i^m with controls $a_i = \overline{\mathcal{L}}_m u_m(\pi_i^m)$ and that by Theorem 4.45 they converge to Φ^u of (4.122) as $m \to \infty$. Then, by considerations similar to the proof of Proposition 4.50 we complete the proof of the contradiction to (4.153). The proof of the statements (i) and (ii) of Corollary 4.56 is immediate.

■

In the case of Example 2 we use the discretizations of section 3.3.2 with the only change that now $r_m(x, y)$ is given by a particular d-dimensional form of (1.2) where h replaced by h_m

$$h_m(x) := h(b_j^m) \quad \text{for} \quad x \in B_j^m. \tag{4.168}$$

Moreover instead of the class $\mathcal{A}(L, n)$ we use $\overline{\mathcal{A}}(L, n)$ that is defined before Proposition 4.53.

Similarly as in the case of Example 1, the construction of a nearly optimal control function for $J_\mu(u)$ will be reduced to that for

$$J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) = \limsup_{n \to \infty} n^{-1} E_{\overline{\mu}} \Big\{ \sum_{i=0}^{n-1} \sum_{j=1}^{k_m} c(j, \widetilde{\mathcal{L}}_m u(\overline{\pi}_i^m)) \overline{\pi}_i^m(j) \Big\} \tag{4.169}$$

over $u \in \overline{\mathcal{A}}(L, n)$, where $\overline{\pi}_i^m$ is the filtering process corresponding to $(\overline{x}_i^m)$ defined in II with (4.143) replaced by (1.1) for $r = r_m$ as above. To this effect notice that, for m sufficiently large, h_m inherits the growth property (E 2.1) of h, and by Proposition 4.30 there exists then a unique invariant measure $\overline{\Phi}_m^{\widetilde{\mathcal{L}}_m u}$ of $\overline{\pi}_i^m$ and it is of the form

$$\overline{\Phi}_m^{\widetilde{\mathcal{L}}_m u}(F) = E_\lambda \Big\{ \sum_{i=0}^{\overline{\sigma}^m} F(\overline{\pi}_i^m) \Big\} (E_\lambda\{\overline{\sigma}^m + 1\})^{-1} \tag{4.170}$$

for $F \in b\mathcal{B}(P(E_m))$ and $\overline{\sigma}^m = \inf\{i > 0,\ \overline{\pi}_i^m(\psi_n) \le b\}$. Therefore, again by Proposition 4.30, for $u \in \overline{\mathcal{A}}(L, n)$ we have analogously to (4.165)

$$J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) = \int_{P(E_m)} \sum_{j=1}^{k_m} c(j, \widetilde{\mathcal{L}}_m u(\eta)) \eta_j \overline{\Phi}_m^{\widetilde{\mathcal{L}}_m u}(d\eta) \tag{4.171}$$

By analogy to Theorem 4.58 we obtain

Theorem 4.59 *Assume (A1), (A2), (A5), (E 2.1)–(E 2.4). Then for L, $n > 0$*

$$\sup_{u \in \overline{\mathcal{A}}(L,n)} |J_{\overline{\mu}}^m(\widetilde{\mathcal{L}}_m u) - J_\mu(u)| \to 0 \tag{4.172}$$

as $m \to \infty$, uniformly in $\mu \in P(E)$.

Furthermore the statements (i) and (ii) of Corollary 4.56 hold true with $\mathcal{A}_m(L, n)$ and $\mathcal{A}(L, n)$ replaced by $\overline{\mathcal{A}}_m(L, n) := \widetilde{\mathcal{L}}_m \overline{\mathcal{A}}(L, n)$ and $\overline{\mathcal{A}}(L, n)$ respectively.

P r o o f. Notice first that by (E 2.1), the functions h_m defined in (4.168) satisfy (D8). By the definition of P_m^a (see 3.50) also (E_m 2.2) with $\lambda_m = \lambda$ and (E_m 2.3) are satisfied. Now as in the proof of Theorem 4.55 it suffices to show a contradiction to (4.153). For this purpose we use here Theorem 4.47 with the process (x_i^m) defined in I, the observations according to (1.1), (1.2) with $h = h_m$ from (4.168), and the controls $a_i = \overline{\mathcal{L}}_m u_m(\pi_i^m)$, where $u_m \in \overline{\mathcal{A}}(L, n)$. Since the family $\overline{\mathcal{A}}(L, n)$ is compact in $C(P(E), U)$, we can also guarantee (D9) to be satisfied with the operator $K_m = \mathcal{L}_m$.

■

4.5.3 Discretization of the simplex $P(E_m)$

Given an ergodic control problem, either with mixed observations or with a structure corresponding to Examples 1 and 2 of section 4.4.2, from the results of the previous subsection 4.5.2 (Corollary 4.56 and Remark 4.57 for the mixed observation case and Theorems 4.58 and 4.59 for the models of Examples 1 and 2 respectively) we have that the construction of a nearly optimal control function for the given partially observed control problem can be reduced to the construction of a nearly optimal control function for a complete observation ergodic control problem, where the state is the filter process $(\overline{\pi}_i^m)$ that takes values in the simplex $P(E_m)$, the cost functional is

$$J_{\overline{\mu}}^m(u) = \limsup_{n\to\infty} n^{-1} E_{\overline{\mu}}\Big\{ \sum_{i=1}^{n} \sum_{j=1}^{k_m} c(j, u(\overline{\pi}_i^m))\overline{\pi}_i^m(j)\Big\} \tag{4.173}$$

and the control belongs either to $\mathcal{A}_m(L, n)$ or to $\overline{\mathcal{A}}_m(L, n)$.

Furthermore, for $u \in \mathcal{A}_m(L, n)$ and the assumptions of either Theorem 4.55 or Theorem 4.58, and for $u \in \overline{\mathcal{A}}_m(L, n)$ and the assumptions of Theorem 4.59 (with m sufficiently large that h_m in (4.168) inherits the growth property (E 2.1)), there exist unique invariant measures $\overline{\Phi}_m^u$ of the filter process $(\overline{\pi}_i^m)$ having the representations (4.150), (4.166) and (4.170) respectively. Correspondingly, in the first case, namely mixed observations and $u \in \mathcal{A}_m(L, n)$, the cost functional $J_{\overline{\mu}}^m(u)$ in (4.173) admits also the representation (4.151); for the second case, namely Example 1 and $u \in \mathcal{A}_m(L, n)$ and the third case of Example 2 and $u \in \overline{\mathcal{A}}_m(L, n)$, $J_{\overline{\mu}}^m(u)$ admits also the representations (4.165) and (4.171) respectively. Notice that, formally, all these latter representations are the same. Finally, if also assumption (B9) holds, the filtering processes $(\overline{\pi}_i^m)$ are Feller with transition operators $\overline{\prod}_m^u$, defined in (4.147) for the mixed observation model, and in (3.58) for the models of Examples 1 and 2.

Since the processes $(\overline{\pi}_i^m)$ still take their values in the infinite space $P(E_m)$, for the actual construction of nearly optimal control functions we consider a further approximation based on the discretization of the space $P(E_m)$ introduced in section 3.3.3.a$_1$ and already used in an analogous way in section 3.3.3.b. More precisely let $(G_k^q)_{k=1,2,\ldots,k_q}$ be a partition of $P(E_m)$ with representative elements $\{e_1^q, \ldots, e_{k_q}^q\}$.

Given a control function u in $\mathcal{A}_m(L,n)$ or $\overline{\mathcal{A}}_m(L,n)$, for the Markov process $\hat{\pi}_i$ on $\{e_1^q, \ldots, e_{k_q}^q\}$ with transition matrix

$$\hat{\prod}_m^{u(e_k^q)}(e_k^q, e_p^q) = \overline{\prod}_m^{u(e_k^q)}(e_k^q, G_p^q)$$

(see (3.126)) define analogously to (3.127) the cost functional $\hat{J}_{e_p^q}^q(u)$ by

$$\hat{J}_{e_p^q}^q(u) = \limsup_{n\to\infty} n^{-1} E_{e_p^q}^u \Big\{ \sum_{i=0}^{n-1} \sum_{j=1}^{k_m} c(j, u(\hat{\pi}_i))\hat{\pi}_i(j) \Big\} \tag{4.174}$$

We have by analogy to Theorem 3.39

Theorem 4.60 *Under (A2), (A5) and (B9) together with assumptions guaranteeing the uniqueness of the invariant measure of* $\overline{\pi}_i^m$*, for given* m *(in the case of Example 2, a sufficiently large* m*) we have*

$$\sup_{u\in\mathcal{A}_m(L,n)} \sup_{\eta\in P(E_m)} |J_\eta^m(u) - \hat{J}_{\hat{Q}_q\eta}^q(u)| \to 0 \tag{4.175}$$

as $q \to \infty$*, where* $\hat{Q}_q$ *is defined in (3.125) and where* $\mathcal{A}_m(L,n)$ *is replaced by* $\overline{\mathcal{A}}_m(L,n)$ *in the case of Example 2.*

P r o o f. For a given $u \in \mathcal{A}_m(L,n)$ or $u \in \overline{\mathcal{A}}_m(L,n)$, consider a process $(\check{\pi}_i)$ on $P(E_m)$ with transition operator

$$\check{\prod}_m^{u(\eta)}(\eta, \cdot) = \overline{\prod}_m^{u(\hat{Q}_q\eta)}(\hat{Q}_q\eta, \cdot) \tag{4.176}$$

Let $\hat{\Phi}^u_q$ be an invariant measure for $\hat{\pi}_i$. Then the measure $\check{\Phi}^u_q$

$$\check{\Phi}^u_q(\cdot) := \sum_{k=1}^{k_q} \check{\prod}_m^{u(e^q_k)}(e^q_k, \cdot)\hat{\Phi}^u_q(e^q_k) \tag{4.177}$$

is invariant for $(\check{\pi}_i)$.

In fact, we have for $A \in \mathcal{B}(P(E_m))$

$$\begin{aligned}
&\int_{P(E_m)} \check{\prod}_m^{u(\eta)}(\eta, A)\check{\Phi}^u_q(d\eta) = \\
&= \sum_{l=1}^{k_q}\sum_{k=1}^{k_q} \overline{\prod}_m^{u(e^q_l)}(e^q_l, A)\,\overline{\prod}_m^{u(e^q_k)}(e^q_k, G^q_l)\hat{\Phi}^u_q(e^q_k) \\
&= \sum_{l=1}^{k_q} \overline{\prod}_m^{u(e^q_l)}(e^q_l, A)\sum_{k=1}^{k_q} \hat{\prod}_m^{u(e^q_k)}(e^q_k, e^q_l)\hat{\Phi}^u_q(e^q_k) \\
&= \sum_{l=1}^{k_q} \check{\prod}_m^{u(e^q_l)}(e^q_l, A)\hat{\Phi}^u_q(e^q_l) = \check{\Phi}^u_q(A)
\end{aligned}$$

Now we show that if, as $q \to \infty$, $u_q \in \mathcal{A}_m(L,n)$ (or $\overline{\mathcal{A}}_m(L,n)$ with m sufficiently large in the case of Example 2), converge uniformly to $u \in \mathcal{A}_m(L,n)(\overline{\mathcal{A}}_m(L,n))$, then

$$\check{\Phi}^{u_q}_q \Rightarrow \overline{\Phi}^u_m \quad \text{as } q \to \infty \tag{4.178}$$

Notice that, since $(\hat{\pi}_i)$ is a finite state Markov chain, for any given u_q it has an invariant measure $\hat{\Phi}^{u_q}$ and therefore $\check{\Phi}^{u_q}_q$ given by (4.177) is well defined. Moreover, since $P(E_m)$ is compact, there is a subsequence $q_k \to \infty$ and a measure $\Phi \in P(E_m)$ such that $\check{\Phi}^{u_{q_k}}_{q_k} \Rightarrow \Phi$.

Furthermore, by (3.132) and the Feller property of $\overline{\prod}_m^u$ we have for $f \in C(P(E_m))$

$$\begin{aligned}
&|\Phi(f) - \Phi(\overline{\prod}_m^u f)| \le |\Phi(f) - \check{\Phi}^{u_{q_k}}_{q_k}(f)| + \\
&+|\check{\Phi}^{u_{q_k}}_{q_k}(f) - \check{\Phi}^{u_{q_k}}_{q_k}(\check{\prod}_m^{u_{q_k}} f)| + |\check{\Phi}^{u_{q_k}}_{q_k}(\check{\prod}_m^{u_{q_k}} f) \\
&-\check{\Phi}^{u_{q_k}}_{q_k}(\overline{\prod}_m^u f)| + |\check{\Phi}^{u_{q_k}}_{q_k}(\overline{\prod}_m^u f) - \Phi(\overline{\prod}_m^u f)| \\
&\le |\Phi(f) - \check{\Phi}^{u_{q_k}}_{q_k}(f)| + \sup_\eta |\check{\prod}_m^{u_{q_k}(\eta)}(\eta, f) - \overline{\prod}_m^{u(\eta)}(\eta, f)| \\
&+|\check{\Phi}^{u_{q_k}}_{q_k}(\overline{\prod}_m^u f) - \Phi(\overline{\prod}_m^u f)| \to 0 \quad \text{as } k \to \infty
\end{aligned}$$

Thus Φ is invariant for $\overline{\pi}_i^m$, and by the uniqueness of the invariant measure $\overline{\Phi}_m^u$ for $(\overline{\pi}_i^m)$ we have that $\Phi = \overline{\Phi}_m^u$.

Therefore (4.178) holds.

Assume now that (4.175) is not satisfied. Then for some sequence $u_q \in \mathcal{A}_m(L,n)$ (or $\overline{\mathcal{A}}_m(L,n)$), $\eta_q \in P(E_m)$ and $\varepsilon > 0$ we have

$$|J^m_{\eta_q}(u_q) - \hat{J}^q_{\hat{Q}_q\eta_q}(u_q)| > \varepsilon \tag{4.179}$$

By the compactness of the classes $\mathcal{A}_m(L,n)$ and $\overline{\mathcal{A}}_m(L,n)$ we may assume that $u_q \to u \in \mathcal{A}_m(L,n)$ (resp. $\overline{\mathcal{A}}_m(L,n)$), uniformly on $P(E_m)$, as $q \to \infty$. For each $q = 1, 2, \ldots$, and $\hat{Q}_q\eta_q$, there exists an invariant measure $\hat{\Phi}_q^{u_q}$ of $(\hat{\pi}_i)$, which may depend on $\hat{Q}_q\eta_q$ such that

$$\hat{J}^q_{\hat{Q}_q\eta_q}(u_q) = \sum_{k=1}^{k_q}\sum_{j=1}^{k_m} c(j, u_q(e_k^q))e_k^q(j)\hat{\Phi}_q^{u_q}(e_k^q)$$

Then with $\check{\Phi}^{u_q}$ defined in (4.177) we have

$$\hat{J}^q_{\hat{Q}_q\eta_q}(u_q) = \sum_{k=1}^{k_q}\sum_{j=1}^{k_m} c(j, u_q(e_k^q))e_k^q(j)\check{\Phi}_q^{u_q}(G_k^q) = \int_{P(E_m)} C_q(\eta)\check{\Phi}_q^{u_q}(d\eta)$$

with

$$C_q(\eta) := \sum_{j=1}^{k_m} c(j, u_q(e_k^q))e_k^q(j) \quad \text{for } \eta \in G_k^q.$$

Since $C_q(\eta) \to \sum_{j=1}^{k_m} c(j, u(\eta))\eta_j$, as $q \to \infty$, uniformly in $\eta \in P(E_m)$, by (4.178)

$$\hat{J}^q_{\hat{Q}_q\eta_q}(u_q) \to \int_{P(E_m)} \sum_{j=1}^{k_m} c(j, u(\eta))\eta_j \overline{\Phi}_m^u(d\eta) = J_\eta^m(u)$$

as $q \to \infty$, a contradiction to (4.179).

Therefore (4.175) is satisfied.

■

By analogy to Corollary 3.40 we immediately have

Corollary 4.61 *Under the assumptions of Theorem 4.60, if $u \in \mathcal{A}_m(L,n)$ (or $u \in \overline{\mathcal{A}}_m(L,n)$ with m sufficiently large in the case of Example 2) is an ε-optimal control function for the cost functional $\hat{J}^q_{\hat{Q}_q\eta}(u)$, $\eta \in P(E_m)$ and q is so large that*

$$\sup_{u\in\mathcal{A}_m(L,n)\ (\overline{\mathcal{A}}_m(L,n))} \sup_{\eta\in P(E_m)} |J^m_\eta(u) - \hat{J}^q_{\hat{Q}_q\eta}(u)| < \varepsilon \tag{4.180}$$

then u is a 3ε-optimal control function for the cost functional J^m_η.

By Theorem 4.60 and Corollary 4.61, to construct ε-optimal control functions for the cost functional $J^m_\eta(u)$, it remains to find a nearly optimal control function $u \in \mathcal{A}_m(L,n)$ ($\overline{\mathcal{A}}_m(L,n)$) for $\hat{J}^q_{\hat{Q}_q\eta}$ with q sufficiently large.

Since the Markov process $(\hat{\pi}_i)$ takes values in the finite set $\{e^q_1, \dots, e^q_{k_q}\}$, for a given $u \in \mathcal{A}_m(L,n)$ or $\overline{\mathcal{A}}_m(L,n)$, we need only a finite number of values $\overline{u}(\zeta_i)$, $i = 1, 2, \dots, k_q$, where ζ_i is as in (3.134) and $\overline{u}$ is the Lipschitz function corresponding to u in the definition of the class $\mathcal{A}_m(L,n)$, or $\overline{\mathcal{A}}_m(L,n)$ respectively.

Therefore, one only needs to consider the values $a^1, \dots, a^{k_q} \in U$, where $a^i = \overline{u}(\zeta_i)$, and verify the Lipschitz condition (3.135).

For details see the discussion following Corollary 3.40 through the end of section 3.3.3.b.

4.5.4 Computational analysis of an example

This section contains a computational analysis of the approximations studied in sections 4.5.1–4.5.3 and consists of the construction of nearly optimal controls for a particular case of the model of Example 2 in section 4.4.2.

Let us first notice that, contrary to what we did in section 4.5.2 where we were motivated by the desire to present the most general approach, here we present an alternative possibility where we discretize the observation space R^d only towards the end.

Once a partition of the state space E is given, there is in fact no strict necessity to discretize the observations right from the beginning. So, if we discretize according to section 4.5.2.b only the state space, we obtain a partially observed ergodic control problem with the state $\overline{x}^m_i$ (see II in section 4.5.2.b) evolving in E_m, but the observations (y_i) are in R^d; the cost functional $J^m_{\overline{\mu}}(u)$ remains of the form (4.148), but $(\overline{\pi}^m_i)$ is now the filtering process corresponding to $(\overline{x}_i^m)$ and (y_i) that satisfies a recursion analogous to (3.57) where $\overline{r}_m(j, \overline{y}_i^m)$ is replaced by $r(j, y_i) = r(b_j, y_i)$ with $r(x, y)$ determined in (1.2); finally $J_\mu^m(u)$ is minimized over $u \in \mathcal{A}_m(L, n)$.

Equivalently, we may consider a completely observed control problem on the simplex $P(E_m)$ with transition kernel $\overline{\prod}_m^u$ as in (3.58), where $\overline{M}_m^u$ is as in (3.57) but with $\overline{r}_m(j, \overline{y}_i^m)$ replaced by $r(b_j, y_i)$, cost functional $J_\mu^m(u)$ given by (4.173), and control functions u from the class $\overline{\mathcal{A}}_m(L, n)$. A further discretization of the simplex $P(E_m)$ as in section 4.5.3 leads to a problem for which the computations can actually be carried out. However, to compute the transition kernel for $(\overline{\pi}_i^m)$, one has to compute probabilities of sets of the form $\{y \colon \overline{M}_m^u(y, e_k^q) \in G_p^q\}$. Since the transformation $\overline{M}_m^u(y, s)$ is highly nonlinear as a function of y, these probabilities are difficult to determine explicitly. Therefore, to make computations feasible, at this stage a discretization of the observations becomes now unavoidable.

Notice that the alternative possibility not to discretize the observations already from the beginning and that was just described for a model of the type of Example 2, can also be applied to models corresponding to Example 1.

Based on the above comments, we now present our computational example, dividing the rest of the section into three further subsections. In subsection 4.5.4.a, based on an approach that is described in detail in [30], we present a description of the computational method. Its implementation is given in subsection 4.5.4.b and finally numerical results are reported in subsection 4.5.4.c.

4.5.4.a The computational method

The first step consists in partitioning the state space E, by which we obtain a partially observed Markov chain (x_i) on $E_m = \{1, 2, \ldots, k\}$, with transition matrix $P^a(j, l)$, $j, l \in E_m$ and observations (y_i) of the form

$$y_i = h(x_i) + w_i \tag{4.181}$$

where w_i are i.i.d. d-dimensional standard Gaussian vectors, and $h(x) \in R^d$ for $x \in E_m$.

We choose $U = [0, K]$ and the class of admissible control functions of the form $\overline{\mathcal{A}}_m(L,n)$ i.e. for $\eta \in P(E_m)$

$$u(\eta) = \overline{u}\Big(\sum_{j=1}^{k} \varphi_1(j)\eta_j, \dots, \sum_{j=1}^{k} \varphi_n(j)\eta_j\Big) r\Big(\sum_{j=1}^{k} \psi(j)\eta_j\Big) \tag{4.182}$$

where, $\varphi_i(j)$, $\psi(j)$ are fixed nonnegative numbers, with (see (4.91)) $\psi(k) = 0$, $\psi(j) \leq 1$, and (see (4.90)) $r\colon [0,1] \to [0,1]$ satisfying for $0 < b < c < 1$

$$r(x) = \begin{cases} 0 & \text{for } x \leq b \\ c\dfrac{(x-b)}{c-b} & \text{for } b \leq x \leq c \\ 1 & \text{for } c \leq x \leq 1 \end{cases} \tag{4.183}$$

and where $\overline{u}\colon R^n \to U = [0,K]$ is a Lipschitz function with Lipschitz constant L.

Moreover, for the sake of our specific example, for $a \in U = [0,K]$, let

$$P^a(j,l) = \frac{p^1(j,l)a^2 + p^2(j,l)a + \lambda(l)}{a^2 + a + 1} \tag{4.184}$$

where $p^1(j,l)$, $p^2(j,l)$ are fixed transition matrices and $\lambda \in P(E_m)$. Finally, for fixed numbers $c(j)$, $d(j)$, $j \in E_m$, let

$$c(j,a) = c(j)a^2 + d(j) \tag{4.185}$$

The problem is now completely described by the constants $k, d, h(j)$ for $j \in E_m$, K, L, n, $\varphi_i(j)$ for $i = 1, 2, \dots, n$, $j \in E_m$, $\psi(j)$ for $j \in E_m$, b, c, $\lambda(j)$ for $j \in E_m$, $p^1(j,l)$, $p^2(j,l)$ for $j, l \in E_m$, $c(j)$ and $d(j)$ for $j \in E_m$.

To guarantee the existence of a unique invariant measure of the corresponding filtering process $\overline{\pi}_i^m$ for any Lipschitz function $\overline{u}\colon R^n \to U$, by analogy to (E 2.1), (E 2.3) we also assume that the choice of $p^1(i,j)$, $p^2(i,j)$, $\lambda(j)$ and $h(j)$ is such that

$$\inf_{a \in U} \inf_{i \in E_m} P^a(i,k) > 0 \tag{4.186}$$

and

for some $j \in \{1, 2, \ldots, d\}$ we have either

$$h^j(k) > h^j(i) \quad \text{for } i \neq k \tag{4.187}$$

or

$$h^j(k) < h^j(i) \quad \text{for } i \neq k$$

As we mentioned above, to make computations feasible, we now discretize also the observation space R^d. For this purpose we introduce two parameters β and $\gamma > 0$. For a given β a parameter $M(\beta)$ is determined in such way that

$$\sup_{j \in E_m} \int\limits_{R^d \setminus [-M(\beta), M(\beta)]^d} (2\pi)^{-\frac{d}{2}} \exp[-\frac{1}{2}(y - h(j), y - h(j)] dy < \beta \tag{4.188}$$

Then the rectangle $[-M(\beta), M(\beta)]^d$ is partitioned into $D_1, \ldots, D_{s(\beta,\gamma)}$ subrectangles with diameters less than γ. Let $D_0 = R^d \setminus [-M(\beta), M(\beta)]^d$ and define as its representative element d_0 the point in R^d that has its coordinates equal to $M(\beta) + 1$. In each D_s, $s = 1, 2, \ldots, s(\beta, \gamma)$ we choose the center d_s as a representative element.

Put

$$m^u_{\beta,\gamma}(\eta) := \sum_{s=1}^{s(\beta,\gamma)} (2\pi)^{-\frac{d}{2}} \sum_{j=1}^{k} \exp[-\frac{1}{2}(d_s - h(j), d_s - h(j))] P^{u(\eta)}(\eta, j) \int\limits_{D_s} dy \tag{4.189}$$

and for $B \in \mathcal{B}(P(E_m))$ define, with $\overline{M}^u_m$ as described at the beginning of this section,

$$\begin{aligned} \prod\nolimits^u_{\beta,\gamma}(\eta, B) := \quad & \sum_{s=1}^{s(\beta,\gamma)} (2\pi)^{-\frac{d}{2}} \sum_{j=1}^{k} \exp[-\frac{1}{2}(d_s - h(j), d_s - h(j))] \\ & P^{u(\eta)}(\eta, j) \chi_B(\overline{M}^u_m(d_s, \eta)) \int\limits_{D_s} dy + \chi_B(\overline{M}^u_m(d_0, \eta)) \Big(1 - \\ & \sum_{s=1}^{s(\beta,\gamma)} (2\pi)^{-\frac{d}{2}} \sum_{j=1}^{k} \exp[-\frac{1}{2}(d_s - h(j), d_s - h(j))] \Big) \\ & P^{u(\eta)}(\eta, j) \int\limits_{D_s} dy, \qquad \text{if } m^u_{\beta,\gamma}(\eta) \leq 1 \end{aligned} \tag{4.190}$$

and

$$\prod^u_{\beta,\gamma}(\eta, B) := \sum_{s=1}^{s(\beta,\gamma)} (2\pi)^{-\frac{d}{2}} \sum_{j=1}^{k} \exp[-\frac{1}{2}(d_s - h(j), d_s - h(j))] P^{u(\eta)}(\eta, j)$$
$$\chi_B(\overline{M}^u_m(d_s, \eta)) \int_{D_s} dy (m^u_{\beta,\gamma}(\eta))^{-1} \quad \text{when } m^u_{\beta,\gamma}(\eta) > 1$$

Using an argument similar to the proof of Theorem 4.60, it is clear that the control problem for a Markov process on $P(E_m)$ with the transition operator $\prod^u_{\beta,\gamma}$ approximates the problem corresponding to the transition operator $\overline{\prod}^u_m$ provided β and γ are sufficiently small.

In the next step, according to section 4.5.3, the simplex $P(E_m)$ is partitioned into sets $(G^q_k)_{k=1,2,\ldots,k_q}$. In order to determine this sets we choose a simplex coordinate, say l, and partition the range of l, i.e. the interval $[0,1]$ into subintervals $[0,t_1),[t_1,t_2),\ldots,[t_{k_q-2},t_{k_q-1}),[t_{k_q-1},1]$ having the following property: letting, for $p = 1,2,\ldots,k_q$, G^q_p be given by the subset of the simplex with the l-th coordinate belonging to $[t_{p-1},t_p)$, these sets G^q_p have their volumes close to one another. In each G^q_p consider as representative element the center point e^q_p with coordinates

$$e^q_p(l) = \frac{t_p + t_{p-1}}{2} \qquad e^q_p(j) = \frac{1 - \frac{t_p + t_{p-1}}{2}}{k-1} \text{ for } j \neq l \tag{4.191}$$

with $t_0 = 0$ and $t_{k_q} = 1$.

A transition matrix $\hat{\prod}^u_{\beta,\gamma,q}(e^q_p, e^q_s)$ for a process taking values in the set of representative elements is then defined (compare with section 4.5.3) by

$$\hat{\prod}^u_{\beta,\gamma,q}(e^q_p, e^q_s) := \prod^u_{\beta,\gamma}(e^p_q, G^q_s) \tag{4.192}$$

and an invariant vector $\hat{\Phi}^u_{\beta,\gamma,q}$ corresponding to $\hat{\prod}^u_{\beta,\gamma,q}$ is computed.

Finally, for a given u of the form of (4.182) consider the cost functional

$$\hat{J}^q_{\beta,\gamma}(u) := \sum_{p=1}^{k_q} \sum_{j=1}^{k} (c(j)(u(e^q_p))^2 + d(j)) e^q_p(j) \hat{\Phi}^u_{\beta,\gamma,q}(p) \tag{4.193}$$

As was pointed out at the end of section 4.5.3, to compute the matrix $\hat{\prod}^u_{\beta,\gamma,q}$ and the vector $\hat{\Phi}^u_{\beta,\gamma,q}$ we need only a finite number of values of the control function u, namely $u(e^q_p)$, for $p = 1,2,\ldots,k_q$.

Furthermore, by (4.182) it is clear that the control functions can differ only through the values of $\overline{u}(\zeta_i)$ where

$$\zeta_i := \Big(\sum_{j=1}^{k} \varphi_1(j) e_i^q(j), \ldots, \sum_{j=1}^{k} \varphi_n(j) e_i^q(j)\Big) \tag{4.194}$$

Given therefore k_q elements $a^1, \ldots, a^{k_q} \in U$ for which the Lipschitz condition (3.135), namely

$$\rho_U(a^j, a^p) \le L \max_{i=1,2,\ldots,n} \Big| \sum_{s=1}^{k} \varphi_i(s)(e_j^q(s) - e_p^q(s)) \Big| \tag{4.195}$$

is satisfied, they can be considered as values at ζ_p of a Lipschitz function $\overline{u}$, constructed from a^p, $p = 1, 2, \ldots, k_q$ by linear interpolation.

Therefore the problem of determining a nearly optimal control function is now reduced to determine $a^1, \ldots, a^{k_q} \in U$ that satisfy the Lipschitz condition (4.195) and such that, with

$$u(e_p^q) = a^p r\Big(\sum_{j=1}^{k} \psi(j) e_p^q(j)\Big) \tag{4.196}$$

the cost functional $\hat{J}^q_{\beta,\gamma}(u)$ defined in (4.193) is minimal. To stress the fact that $\hat{J}^q_{\beta,\gamma}$ depends only on the values $a^1, \ldots, a^{k_q}$, by an abuse of notation we shall also write $\hat{J}^q_{\beta,\gamma}((a^i))$.

For actual computations, analogously to what we did in section 3.3.3.b, we still need to discretize U choosing N representative elements of $U = [0, K]$, namely $(\alpha_n)_{n=1,2,\ldots,N}$, with $\alpha_n = \frac{(n-1)K}{N-1}$ for $n = 1, 2, \ldots, N$. In this way we have only a finite number of k_q-tuples $a^1, \ldots, a^{k_q} \in U_N = \Big\{0, \frac{1}{N-1}, \ldots, \frac{(N-2)K}{N-1}, K\Big\}$ satisfying (3.195) to choose from. The problem is therefore reduced to finding a global minimum of $\hat{J}^q_{\beta,\gamma}((a^i))$ over a finite but large number of k_q-tuples (a^i) with values in U_N under the condition that (4.195) holds. To this end one can apply any of the stochastic global optimization algorithms.

4.5.4.b The implementation

In our example we use a simulated annealing algorithm (see e.g. [33] and [41]) formulated as follows:

For $i = 1, 2, \ldots, k_q$, $n = 1, 2, \ldots$, let z_n^i be a sequence of i.i.d. random variables such that, for each given i and n, and with N denoting the number of representative elements of U,

$$P\{z_n^i = -\frac{1}{N}\} = P\{z_n^i = 0\} = P\{z_n^i = -\frac{1}{N}\} = \frac{1}{3},$$

and let R_n be a sequence of i.i.d. uniformly distributed random variables on the interval $(0, 1)$. Moreover let $T_n > 0$ be a sequence of so called "cooling parameters", fixed apriori. Take admissible control values (a^i) namely such that, for $i = 1, 2, \ldots, k_q$, $a^i \in U_N$ and (4.195) is satisfied, and compute the value of $\hat{J}^q_{\beta,\gamma}((a^i))$. Then proceed recursively along the following steps

1. put $(a_1^i) = (\hat{a}_0^i) := (a^i)$ for $i = 1, 2, \ldots, k_q$,

2. compute $\overline{a}_n^i := a_n^i + z_n^i$, for $i = 1, 2, \ldots, k_q$; if for some i', $a_n^{i'} + z_n^{i'} = K + \frac{1}{N-1}$ or $a_n^{i'} + z_n^{i'} = -\frac{1}{N-1}$, then in the first case assign $a_n^{i'} + z_n^{i'}$ the value $\frac{N-2}{N-1}K$ or K with probability $\frac{1}{2}$ each; analogously in the second case assign $a_n^{i'} + z_n^{i'}$ the value 0 or $\frac{1}{N-1}$ with probability $\frac{1}{2}$ each; then check whether $\overline{a}_n^i$ satisfy the condition (4.195); if not repeat step 2 with a new copy of (z_n^i) $i = 1, 2, \ldots, k_q$.

3. compute $\hat{J}^q_{\beta,\gamma}((\overline{a}_n^i))$; if $\hat{J}^q_{\beta,\gamma}((\overline{a}_n^i)) \leq \hat{J}^q_{\beta,\gamma}((a_n^i))$

   put

$$(\hat{a}_{n-1}^i) = \begin{cases} (\overline{a}_n^i) & \text{if } \hat{J}^q_{\beta,\gamma}((\overline{a}_n^i)) < \hat{J}^q_{\beta,\gamma}(\hat{a}_{n-1}^i) \\ (\hat{a}_{n-1}^i) & \text{otherwise} \end{cases}$$

   and $a_n^i = \overline{a}_n^i$, then set $n = n + 1$ and go to the step 2.

4. if $\hat{J}^q_{\beta,\gamma}((\overline{a}_n^i)) > \hat{J}^q_{\beta,\gamma}((a_n^i))$ then, in the case when

$$R_n < \exp\Big[- \frac{\hat{J}^q_{\beta,\gamma}((\overline{a}_n^i)) - \hat{J}^q_{\beta,\gamma}((a_n^i))}{T_n} \Big]$$

   put $a_n^i = \overline{a}_n^i$, $n = n + 1$ and go to the step 2; otherwise put $a_n^i = a_n^i$, $n = n + 1$ and go to the step 2.

From the general theory of simulated annealing it follows that for a sufficiently large number of iterations the values of $(\hat{a}_n^i)$, $i = 1, 2, \ldots, k_q$ form nearly optimal controls for $\hat{J}^q_{\beta,\gamma}$.

Following the algorithm described above, dr A. Zemła from IM PAN wrote a computer program called CONT, which for given β, γ, q computes the parameters (a^i), $i = 1, 2, \ldots, k_q$ of a nearly optimal control function. Since of a nearly optimal control function u we need only its values at the points $e^q_p \in P(E_m)$, these values are easily computed from the parameters (a^i) according to (4.196).

Notice that the method used in the program CONT is based on a discretization of the observations and of the simplex, where the filter takes its values, in addition to a discretization of the controls. Since a natural way to perform numerical calculations is by means of Monte Carlo simulations, dr Zemła also wrote an alternative program called CONTMC. The approach in this latter program involves only the discretization of the controls and allows to evaluate via simulations the cost functional $\hat{J}^q_{\beta,\gamma}(u)$ in (4.193) corresponding to a control function $u \in \overline{\mathcal{A}}_m(L, n)$ that, by analogy to the method described in the previous subsection 4.5.4.a, is parametrized by a finite number of parameters a^p $(p = 0, 1, \ldots, k_q + 1)$ that take values in U_N and satisfy the Lipschitz condition

$$\rho_U(a^j, a^p) \leq L \max_{i=1,2,\ldots,n} \Big| \sum_{s=1}^{k} \varphi_i(s)(e^q_j(s) - e^q_p(s)) \Big| \tag{4.197}$$

with $j,\ p = 0, 1, \ldots, k_q + 1$. The parametrization is again obtained in the following way: first choose a simplex coordinate l (the same as in CONT) and partition it into intervals $[t_{p-1}, t_p)$ with $p = 1, \ldots, k_q$ where $t_0 = 0$, $t_{k_q} = 1$. Then select points e^q_p $(p = 0, 1, \ldots, k_q + 1)$ according to the scheme

$$\begin{array}{lll} e^q_0(l) = 0 \qquad e^q_0(j) = \dfrac{1}{k-1} & \text{for } j \neq l \\ e^q_p \ \text{ as in (4.191)} & \text{for } p = 1, 2, \ldots, k_q \\ e^q_{k_q+1}(l) = 1 \quad e^q_{k_q+1}(j) = 0 & \text{for } j \neq l \end{array} \tag{4.198}$$

For a given $(k_q + 2)$-tuple of parameters a^p satisfying (4.197), the control function $u \in \overline{\mathcal{A}}_m(L,n)$ is then determined for $\eta \in P(E_m)$ by putting

$$u(\eta) = \Big(\frac{t_{p+1} - t_p - 2\eta_l}{t_{p+1} - t_{p-1}} a^p + \frac{2\eta_l - t_p - t_{p-1}}{t_{p+1} - t_{p-1}} a^{p+1}\Big) r\Big(\sum_{j=1}^{k} \psi(j)\eta_j\Big)$$

$$\text{for } \eta_l \in [e^q_p(l), e^q_{p+1}(l)] \quad \text{with } p = 1, 2, \ldots, k_q - 1, \text{ and}$$

$$u(\eta) = \Big(\frac{t_1 - 2\eta_l}{t_1} a^0 + \frac{2\eta_l}{t_1} a^1\Big) r\Big(\sum_{j=1}^{k} \psi(j)\eta_j\Big) \tag{4.199}$$

$$\text{for } \eta_l \in [0, e^q_1(l)], \text{ and}$$

$$u(\eta) = \Big(\frac{2(1 - \eta_l)}{1 - t_{k_q-1}} a^{k_q} + \frac{2\eta_l - 1 - t_{k_q-1}}{1 - t_{k_q-1}} a^{k_q+1}\Big) r\Big(\sum_{j=1}^{k} \psi(j)\eta_j\Big)$$

$$\text{for } \eta_l \in [e^q_{k_q}(l), 1].$$

Since the goal is the value of the cost functional $\hat{J}^q_{\beta,\gamma}(u) = \hat{J}^q_{\beta,\gamma}((a^p))$ corresponding to an optimal u among those parametrized as in (4.199), the program CONTMC proceeds as follows: start with any $(k_q + 2)$ tuple (a^p) satisfying (4.197) and, using the corresponding control function (4.199), simulate a sufficiently large number of independent trajectories of the controlled process (x_i) along with the observations (y_i); at each step i the value of the filter π_i is computed, which, according to (4.199), allows to obtain the control value at that step i.

Given a sufficiently large positive integer T, perform S_m simulation runs and for each run compute then the value of

$$T^{-1} \sum_{i=0}^{T-1} \sum_{j=1}^{k} [c(j)(u(\pi_i))^2 + d(j)]\pi_i(j) \tag{4.200}$$

taking the average of the various simulation runs as the value of $\hat{J}^q_{\beta,\gamma}(a^p)$.

At this point one needs the optimal among the values $\hat{J}^q_{\beta,\gamma}(a^p)$. For this purpose CONTMC uses an adaptation of the simulated annealing approach described previously that repeats the simulation procedure just described for a sequence of suitably determined parametrizations (a^p) until a global minimum of $\hat{J}^q_{\beta,\gamma}(a^p)$ is reached.

4.5.4.c Numerical example

In this subsection we present the results of testing the programs CONT and CONTMC for a particular set of data.

Assume $k = 10$, $d = 1$, $n = 5$, $K = 10$, $L = 100$, $b = 0.2$, $c = 1$ and the matrices $(p^1(j,l))$ and $(p^2(j,l))$ are given respectively by

$$(p^1(j,l)) = \begin{pmatrix} 0.10 & 0.20 & 0.00 & 0.10 & 0.10 & 0.10 & 0.00 & 0.01 & 0.02 & 0.37 \\ 0.10 & 0.20 & 0.10 & 0.10 & 0.10 & 0.20 & 0.10 & 0.00 & 0.00 & 0.10 \\ 0.00 & 0.00 & 0.10 & 0.10 & 0.20 & 0.30 & 0.20 & 0.10 & 0.00 & 0.00 \\ 0.10 & 0.00 & 0.00 & 0.00 & 0.00 & 0.10 & 0.20 & 0.10 & 0.50 & 0.00 \\ 0.10 & 0.20 & 0.10 & 0.00 & 0.00 & 0.00 & 0.30 & 0.30 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.10 & 0.20 & 0.10 & 0.10 & 0.10 & 0.00 & 0.00 & 0.40 \\ 0.10 & 0.10 & 0.20 & 0.10 & 0.30 & 0.20 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.20 & 0.30 & 0.50 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.50 & 0.50 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.10 & 0.10 & 0.20 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.00 & 0.10 \end{pmatrix}$$

$$(p^2(j,l)) = \begin{pmatrix} 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 \\ 0.20 & 0.00 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 \\ 0.10 & 0.10 & 0.30 & 0.00 & 0.00 & 0.00 & 0.10 & 0.10 & 0.10 & 0.20 \\ 0.20 & 0.20 & 0.00 & 0.00 & 0.00 & 0.00 & 0.10 & 0.10 & 0.00 & 0.40 \\ 0.10 & 0.10 & 0.00 & 0.30 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.00 \\ 0.10 & 0.20 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.00 \\ 0.10 & 0.20 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.00 \\ 0.10 & 0.20 & 0.10 & 0.00 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 \\ 0.10 & 0.10 & 0.10 & 0.20 & 0.30 & 0.00 & 0.00 & 0.00 & 0.00 & 0.20 \\ 0.00 & 0.00 & 0.10 & 0.20 & 0.30 & 0.10 & 0.10 & 0.10 & 0.10 & 0.00 \end{pmatrix}$$

Moreover

$$\lambda(1) = 0.80 \quad \lambda(2) = 0.00 \quad \lambda(3) = 0.01 \quad \lambda(4) = 0.01 \quad \lambda(5) = 0.01$$

$$\lambda(6) = 0.00 \quad \lambda(7) = 0.02 \quad \lambda(8) = 0.05 \quad \lambda(9) = 0.05 \quad \lambda(10) = 0.05$$

$$h(1) = 1.0 \quad h(2) = 1.0 \quad h(3) = 2.0 \quad h(4) = 3.0 \quad h(5) = 5.0 \quad h(6) = 7.0$$

$$h(7) = 5.0 \quad h(8) = 8.0 \quad h(9) = 4.0 \quad h(10) = 9.0$$

$c(1) = 0.5 \quad c(2) = 3.0 \quad c(3) = 5.0 \quad c(4) = 7.0 \quad c(5) = 4.0 \quad c(6) = 7.0$

$c(7) = 3.0 \quad c(8) = 5.0 \quad c(9) = 3.0 \quad c(10) = 1.0$

$d(1) = 100.00 \quad d(2) = 25.00 \quad d(3) = 40.00 \quad d(4) = 35.00 \quad d(5) = 50.00$

$d(6) = 45.00 \quad d(7) = 70.00 \quad d(8) = 15.00 \quad d(9) = 4.00 \quad d(10) = 30.00$

and finally

$\varphi_1(1) = 10.00 \quad \varphi_1(2) = 6.00 \quad \varphi_1(3) = 7.00 \quad \varphi_1(4) = 9.00 \quad \varphi_1(5) = 5.00$

$\varphi_1(6) = 2.00 \quad \varphi_1(7) = 1.50 \quad \varphi_1(8) = 2.00 \quad \varphi_1(9) = 2.00 \quad \varphi_1(10) = 2.50$

$\varphi_2(1) = 2.00 \quad \varphi_2(2) = 2.50 \quad \varphi_2(3) = 2.50 \quad \varphi_2(4) = 3.00 \quad \varphi_2(5) = 3.00$

$\varphi_2(6) = 3.00 \quad \varphi_2(7) = 3.00 \quad \varphi_2(8) = 3.00 \quad \varphi_2(9) = 2.00 \quad \varphi_2(10) = 2.00$

$\varphi_3(1) = 1.00 \quad \varphi_3(2) = 1.00 \quad \varphi_3(3) = 1.00 \quad \varphi_3(4) = 1.00 \quad \varphi_3(5) = 1.00$

$\varphi_3(6) = 1.50 \quad \varphi_3(7) = 1.50 \quad \varphi_3(8) = 2.00 \quad \varphi_3(9) = 2.00 \quad \varphi_3(10) = 2.00$

$\varphi_4(1) = 2.00 \quad \varphi_4(2) = 2.00 \quad \varphi_4(3) = 2.00 \quad \varphi_4(4) = 2.00 \quad \varphi_4(5) = 2.00$

$\varphi_4(6) = 2.00 \quad \varphi_4(7) = 2.00 \quad \varphi_4(8) = 2.00 \quad \varphi_4(9) = 2.00 \quad \varphi_4(10) = 2.00$

$\varphi_5(1) = 2.00 \quad \varphi_5(2) = 2.50 \quad \varphi_5(3) = 3.00 \quad \varphi_5(4) = 2.50 \quad \varphi_5(5) = 2.50$

$\varphi_5(6) = 2.00 \quad \varphi_5(7) = 2.00 \quad \varphi_5(8) = 2.00 \quad \varphi_5(9) = 2.00 \quad \varphi_5(10) = 2.00$

$\psi(1) = 1.0 \quad \psi(2) = 1.0 \quad \psi(3) = 1.0 \quad \psi(4) = 1.0 \quad \psi(5) = 1.0 \quad \psi(6) = 1.0$

$\psi(7) = 1.0 \quad \psi(8) = 0.7 \quad \psi(9) = 0.4 \quad \psi(10) = 0.0$

Let $\beta = 0.1$, $\gamma = 0.15$ and $k_q = 15$. Assume the first coordinate of the simplex $P(E_m)$ is partitioned in the following way

$$t_1 = 0.016 \quad t_2 = 0.080 \quad t_3 = 0.150 \quad t_4 = 0.220 \quad t_5 = 0.290 \quad t_6 = 0.370$$

$$t_7 = 0.460 \quad t_8 = 0.540 \quad t_9 = 0.630 \quad t_{10} = 0.720 \quad t_{11} = 0.810 \quad t_{12} = 0.900$$

$$t_{13} = 0.970 \quad t_{14} = 0.990$$

Moreover choose "the cooling parameter" $T_n = n^2$, and $N = 50$ as the number of representative elements in the set of control parameters $[0, 10]$.

For the Monte Carlo simulations we set $T = 500$ and $S_m = 5$. Notice that, due to the ergodicity of the process (π_i), having taken T rather large, the small number $S_m = 5$ of runs turned out to be sufficient.

We recall that we had chosen the partition of the simplex in R^{10} on the basis of a partition of the first coordinate into $k_q = 15$ subintervals. Accordingly, the program CONT generates at each iteration a vector of 15 control parameter values $a^i, (i = 1, ..., 15)$. On the other hand, since the program CONTMC uses control values obtained according to the linear interpolation (4.199), in this latter case $k_q + 2 = 17$ values of the control parameters $a_i (i = 0, 1, ..., 16)$ are needed which include two additional values at the end points of the partition.

The program CONT was tested with three sets of initial control parameters $a_1^1 = \ldots = a_1^{15} = 0$, $a_2^1 = \ldots = a_2^{15} = 5$, $a_3^1 = \ldots = a_3^{15} = 10$. The optimal control parameters $(\hat{a}_1^i)$, $(\hat{a}_2^i)$ and $(\hat{a}_3^i)$ for the above initial values, which we obtained using CONT, were the following: For the first set they were

$$\hat{a}_1^1 = 1.6 \quad \hat{a}_1^2 = 1.6 \quad \hat{a}_1^3 = 1.6 \quad \hat{a}_1^4 = 1.6 \quad \hat{a}_1^5 = 1.6 \quad \hat{a}_1^6 = 1.6$$

$$\hat{a}_1^7 = 1.6 \quad \hat{a}_1^8 = 1.6 \quad \hat{a}_1^9 = 1.8 \quad \hat{a}_1^{10} = 1.8 \quad \hat{a}_1^{11} = 2.0 \quad \hat{a}_1^{12} = 4.6$$

$$\hat{a}_1^{13} = 3.8 \quad \hat{a}_1^{14} = 4.4 \quad \hat{a}_1^{15} = 3.6$$

and were obtained in 1458 iterations with corresponding value of the cost functional given by $J((\hat{a}_1^i)) = 57.90264572267466$.

For the second set we obtained

$$\hat{a}_2^1 = 1.6 \quad \hat{a}_2^2 = 1.6 \quad \hat{a}_2^3 = 1.6 \quad \hat{a}_2^4 = 1.6 \quad \hat{a}_2^5 = 1.6 \quad \hat{a}_2^6 = 1.6$$

$$\hat{a}_2^7 = 1.6 \quad \hat{a}_2^8 = 1.6 \quad \hat{a}_2^9 = 1.8 \quad \hat{a}_2^{10} = 1.8 \quad \hat{a}_2^{11} = 2.0 \quad \hat{a}_2^{12} = 2.0$$

$$\hat{a}_2^{13} = 5.6 \quad \hat{a}_2^{14} = 7.4 \quad \hat{a}_2^{15} = 4.6$$

in 2908 iterations and with corresponding value of the cost functional $J((\hat{a}_2^i)) = 57.90264572267466$.
For the third set

$$\hat{a}_3^1 = 1.6 \quad \hat{a}_3^2 = 1.6 \quad \hat{a}_3^3 = 1.6 \quad \hat{a}_3^4 = 1.6 \quad \hat{a}_3^5 = 1.6 \quad \hat{a}_3^6 = 1.6$$

$$\hat{a}_3^7 = 1.6 \quad \hat{a}_3^8 = 1.6 \quad \hat{a}_3^9 = 1.8 \quad \hat{a}_3^{10} = 1.8 \quad \hat{a}_3^{11} = 2.0 \quad \hat{a}_3^{12} = 3.4$$

$$\hat{a}_3^{13} = 1.0 \quad \hat{a}_3^{14} = 3.6 \quad \hat{a}_3^{15} = 3.2$$

were found in 3957 iterations and with corresponding value of the cost functional $J((\hat{a}_3^i)) = 57.90264572267465$.

The program CONTMC was similarly tested with three sets of initial control parameters $a_1^0 = \ldots = a_1^{16} = 0$, $a_2^0 = \ldots = a_2^{16} = 5$, $a_3^0 = \ldots = a_3^{16} = 10$. The optimal control parameters $(\breve{a}_1^i)$, $(\breve{a}_2^i)$ and $(\breve{a}_3^i)$ for the above initial values, which we obtained using CONTMC, were the following:

$$\breve{a}_1^0 = 1.4 \quad \breve{a}_1^1 = 1.6 \quad \breve{a}_1^2 = 0.8 \quad \breve{a}_1^3 = 1.4 \quad \breve{a}_1^4 = 1.2 \quad \breve{a}_1^5 = 1.0$$

$$\breve{a}_1^6 = 1.6 \quad \breve{a}_1^7 = 1.4 \quad \breve{a}_1^8 = 0.6 \quad \breve{a}_1^9 = 1.0 \quad \breve{a}_1^{10} = 1.6 \quad \breve{a}_1^{11} = 1.0$$

$$\breve{a}_1^{12} = 2.6 \quad \breve{a}_1^{13} = 3.0 \quad \breve{a}_1^{14} = 1.8 \quad \breve{a}_1^{15} = 2.4 \quad \breve{a}_1^{16} = 5.0$$

found in 1594 iterations and with corresponding value of the cost functional $J((\breve{a}_1^i)) = 60.17891686754708$,

$$\breve{a}_2^0 = 1.2 \quad \breve{a}_2^1 = 2.8 \quad \breve{a}_2^2 = 0.4 \quad \breve{a}_2^3 = 1.2 \quad \breve{a}_2^4 = 1.8 \quad \breve{a}_2^5 = 1.8$$

$$\breve{a}_2^6 = 1.4 \quad \breve{a}_2^7 = 1.6 \quad \breve{a}_2^8 = 2.0 \quad \breve{a}_2^9 = 1.8 \quad \breve{a}_2^{10} = 1.6 \quad \breve{a}_2^{11} = 2.0$$

$$\breve{a}_2^{12} = 3.0 \quad \breve{a}_2^{13} = 3.2 \quad \breve{a}_2^{14} = 2.2 \quad \breve{a}_2^{15} = 1.6 \quad \breve{a}_2^{16} = 3.4$$

found in 2541 iterations and with corresponding value of the cost functional $J((\breve{a}_2^i)) = 60.50849097426033$,

$$\breve{a}_3^0 = 1.4 \quad \breve{a}_3^1 = 1.4 \quad \breve{a}_3^2 = 1.0 \quad \breve{a}_3^3 = 1.6 \quad \breve{a}_3^4 = 1.0 \quad \breve{a}_3^5 = 1.6$$

$$\breve{a}_3^6 = 2.4 \quad \breve{a}_3^7 = 0.8 \quad \breve{a}_3^8 = 1.8 \quad \breve{a}_3^9 = 1.2 \quad \breve{a}_3^{10} = 1.8 \quad \breve{a}_3^{11} = 1.6$$

$$\breve{a}_3^{12} = 1.8 \quad \breve{a}_3^{13} = 3.4 \quad \breve{a}_3^{14} = 1.6 \quad \breve{a}_3^{15} = 2.2 \quad \breve{a}_2^{16} = 2.6$$

found in 4511 iterations and with corresponding value of the cost functional $J((\breve{a}_3^i)) = 59.93256322024737$.

The successive updating of the control parameters and corresponding control values of the cost functional along the search iterations of the simulated annealing algorithm is illustrated in Figures 1-6 below (Fig. 1-3 corresponding to the program CONT, Fig. 4-6 to CONTMC). Although for each of the six cases corresponding to the six Figures, 5000 search iterations were performed, it was found that for almost all of them the results did not improve considerably beyond the 1000-th iteration; consequently only the first 1000 iterations are reported. In the upper half of each Figure the behaviour of the values of the cost functional is shown, in the lower half that of the control parameters. Since for each iteration one obtains a vector of control parameters, to make the representation more accessible, we have chosen instead of a standard three-dimensional diagram a two-dimensional one, where the changes in the magnitude of the values of the control parameters are represented by a grading of the color intensity: white corresponding to the lowest value equal to 0, intense black to the largest equal to 10.

For programming convenience at each iteration 17 control parameters were generated not only by CONTMC but also by CONT, whereby in this latter case the two extreme values are kept fixed.

In the graphs of the cost functional the base line corresponds to the minimal value of 57.91 obtained over all six cases during the first 1000 iterations. Since the value of the cost functional corresponding to initial control parameters equal to 10 turned out to be large compared to the other cases, we chose a nonuniform scaling of the graphs, more precisely: in Fig.s $1, 2$ and $4, 5$ the upper line corresponds to a value of 200, while in Fig.s 3 and 6 it is 321.4 (the maximal value in Fig. 6 is in fact 321.315). To supplement the information in the six graphs of the cost functional, in the following Table we show the minimal values obtained for each of the six cases during the first 1000 iterations together with the iteration count number where it occurred.

Fig. no.:	1	2	3	4	5	6
minimal value:	57.911	57.963	58.380	60.214	60.879	77.277
iteration no.:	923	995	965	837	987	979

We would like to point out that the iterations reported in the Figures above correspond to the search iterations of the simulated annealing algorithm which, before settling down on a local minimum, has to explore possible alternative local minima as candidates for the global minimum. This explains the large excursions in the values of the cost functional exactly when a minimal value seems to have been reached.

The above numerical results indicate that the method used in CONT has the following advantages over the simulation program CONTMC: the nearly optimal controls corresponding to CONT as well as the corresponding values of the cost functional are almost the same for all sets of starting control parameters; furthermore, the cost function is evaluated exactly and not via simulations as in CONTMC. The disadvantage of CONT is that the control is allowed to take only a finite number of values at the discretized points of the simplex of possible filter values. In CONTMC on other hand, the controls are obtained from the linear interpolation (4.199). This explains the about 5% discrepancy between the values of the cost functional computed according to the two programs. The numerical results also indicate that, for our specific example, the values of the control parameters at the upper end of the partition have little effect on the value of the cost functional.

We remark here also that, for simplicity, the discretization of the simplex is determined by a discretization of only the first coordinate. One can easily work out less crude discretizations that will lead to improved results. Notice furthermore that the peculiar form (4.184) of the transition probability matrix was chosen only to test our methods. It corresponds to the last step of the discretization of a given initial model; different initial models will lead to different transition kernels at this final discretization level and our programs have then to be adjusted accordingly.

4.5.5 Discounted cost approximation

In section 4.2 a discounted cost approximation of the ergodic cost functional (1.6) ((1.12)) was studied with the use of Bellman's equation. As a result it was shown (see Corollary 4.9), that ε-optimal control functions for discounted cost problems with β close to 1 are nearly optimal also for the ergodic cost functional. Below, for the mixed observation model and the models of Examples 1 and 2 from section 4.4.2, we obtain an analogous result in a direct way: we do not require the existence of solutions to the ergodic Bellman equation

(4.4), instead we exploit ergodic properties of an embedded Markov chain and consider control functions in the class $\mathcal{A} = C(P(E), E)$ for the cases of mixed observations and Example 1 and $\overline{\mathcal{A}}$ (see section 4.5.1.b) for the case of Example 2.

In the case of the mixed observations model, for the purpose of this section we shall need a strengthened version of assumption (A11), namely

(A 11′) (i)—(iii) the same as in (A 11)
(iv) there exist $0 < \alpha < 1$ and, for $u \in \mathcal{A}$, a measure $\eta^u \in P(\Gamma_1)$ such that for the embedded Markov chain x_{τ_n}, where $\tau_1 = \tau$, $\tau_{n+1} = \tau_n + \tau \circ \theta_{\tau_n}$, we have for $n = 1, 2, \ldots$

$$\sup_{B \in \mathcal{B}(\Gamma_1)} \sup_{x \in \Gamma_1} \sup_{u \in \mathcal{A}} |P_x^u\{x_{\tau_n} \in B\} - \eta^u(B)| < \alpha^n$$

By sections 5.5 and 5.6 of [13] it is clear that (A 11′) (iv) implies (A 11) (iv).

We have

Theorem 4.62 *Assume, in the case of mixed observations: (A1), (A2), (A5), (A8)–(A10), (A 11′), (A12), in the case of Example 1: (A1)–(A5), (E1.1)–(E1.5) and finally (A1), (A2), (A5), (E2.1)–(E2.4) in the case of Example 2.*

Then, for $L, n > 0$

$$\sup_{u \in \mathcal{A}(L,n)} |(1-\beta) J_\mu^\beta(u) - J_\mu(u)| \to 0 \tag{4.201}$$

as $\beta \to 1$, *uniformly in* $\mu \in P(\Gamma_1)$ *in the case of mixed observations, in* $\mu \in P(E)$ *in the case of Examples 1 and 2 and where, for Example 2 the class* $\mathcal{A}(L,n)$ *is replaced by* $\overline{\mathcal{A}}(L,n)$ *(defined in section 4.5.1.b).*

P r o o f. We show (4.201) for the case of mixed observations only. The proofs for Examples 1 and 2 proceed in an analogous way exploiting the fact that the corresponding filtering processes have embedded i.i.d. sequences independently of the control chosen.

Assume, that (4.201) does not hold. Then there exist $\beta_m \to 1$, $\mu_m \Rightarrow \mu$ and $u_m \to u$ uniformly in $P(E)$ as $m \to \infty$, such that $\mu_m \in P(\Gamma_1)$, $u_m \in \mathcal{A}(L,n)$ and

$$|(1-\beta_m) J_{\mu_m}^{\beta_m}(u_m) - J_{\mu_m}(u_m)| \geq \delta > 0 \tag{4.202}$$

for $m = 1, 2, \ldots$.

Clearly, putting $\tau_0 = 0$,

$$J_{\mu_m}^{\beta_m}(u_m) = E_{\mu_m}^{u_m}\Big\{\sum_{i=0}^{\infty} \beta_m^{\tau_i} F_m^{\beta_m}(x_{\tau_i})\Big\} \tag{4.203}$$

with

$$F_m^{\beta_m}(x) := E_x^{u_m}\Big\{\sum_{i=0}^{\tau-1} \beta_m^i c(x_i, u_m(\pi_i))\Big\}$$

By considerations similar to those of the proof of Corollary 4.40 we obtain

$$F_m^{\beta_m}(x) \to F(x) := E_x^{u}\Big\{\sum_{i=0}^{\tau-1} c(x_i, u(\pi_i))\Big\} \tag{4.204}$$

as $m \to \infty$, uniformly in $x \in \Gamma_1$.

Hence

$$(1-\beta_m)\Big|J_{\mu_m}^{\beta_m}(u_m) - E_{\mu_m}^{u_m}\Big\{\sum_{i=0}^{\infty} \beta_m^{\tau_i} F(x_{\tau_i})\Big\}\Big| \to 0 \tag{4.205}$$

as $m \to \infty$.

By Proposition 4.32, and considerations similar to those of the proof of Proposition 4.50 (see also Remark 4.51), we have that

$$\begin{aligned} J_{\mu_m}(u_m) &= \int_{P(E)} \int_E c(x, u_m(\nu))\nu(dx)\Phi^{u_m}(d\nu) \to \\ &\int_{P(E)} \int_E c(x, u(\nu))\nu(dx)\Phi^{u}(d\nu) = J_\mu(u) \end{aligned} \tag{4.206}$$

as $m \to \infty$.

Now

$$\begin{aligned} &(1-\beta_m)E_{\mu_m}^{u_m}\Big\{\sum_{i=0}^{\infty} \beta_m^{\tau_i} F(x_{\tau_i})\Big\} - J_\mu(u) \\ &= (1-\beta_m)E_{\mu_m}^{u_m}\Big\{\sum_{i=0}^{\infty} \beta_m^{\tau_i}(F(x_{\tau_i}) - J_\mu(u)G_m^{\beta_m}(x_{\tau_i}))\Big\} \end{aligned} \tag{4.207}$$

where

$$G_m^{\beta_m}(x) := E_x^{u_m}\Big\{\sum_{i=0}^{\tau-1} \beta_m^i\Big\}$$

and as in (4.204)

$$G_m^{\beta_m}(x) \to G(x) := E_x^u\{\tau\} \tag{4.208}$$

as $m \to \infty$, uniformly in $x \in \Gamma_1$.

Therefore, using first (4.207) with (4.208) and then (4.206) and (4.205), we have that

$$\begin{aligned} &|(1-\beta_m)J_{\mu_m}^{\beta_m}(u_m) - J_{\mu_m}(u_m) \\ &-(1-\beta_m)E_{\mu_m}^{u_m}\Big\{\sum_{i=0}^{\infty}\beta_m^{\tau_i}[F(x_{\tau_i}) - J_\mu(u)G(x_{\tau_i})]\Big\}| \to 0 \\ &\text{as } m \to \infty \end{aligned} \tag{4.209}$$

Fix a positive integer k and let

$$H_{m,k}^{\beta_m}(x) := E_x^{u_m}\{\beta_m^{\tau_k}(F(x_{\tau_k}) - J_\mu(u)G(x_{\tau_k})\}$$

putting $H_{m,k}^1(x)$ equal to $H_{m,k}^{\beta_m}(x)$ for $\beta_m = 1$.
Using (A11) (iii), (A11′) (iv) and noticing that, by (4.98), $J_\mu(u) = \frac{\eta^u(F)}{\eta^u(G)}$, we obtain

$$\begin{aligned} &(1-\beta_m)\Big|E_{\mu_m}^{u_m}\Big\{\sum_{i=0}^{\infty}\beta_m^{\tau_i}[F(x_{\tau_i}) - J_\mu(u)G(x_{\tau_i})]\Big\}\Big| \\ &\le (1-\beta_m)k(\|F\| + J_\mu(u)\|G\|) \\ &+(1-\beta_m)\Big|E_{\mu_m}^{u_m}\Big\{\sum_{i=k}^{\infty}\beta_m^{\tau_i}[F(x_{\tau_i}) - J_\mu(u)G(x_{\tau_i})]\Big\}\Big| \\ &= (1-\beta_m)k(\|F\| + J_\mu(u)\|G\|) \\ &+(1-\beta_m)\Big|E_{\mu_m}^{u_m}\Big\{\sum_{i=0}^{\infty}\beta_m^{\tau_i}E_{x_{\tau_i}}^{u_m}\{\beta_m^{\tau_k}(F(x_{\tau_k}) - J_\mu(u)G(x_{\tau_k}))\}\Big\}\Big| \\ &+(1-\beta_m)E_{\mu_m}^{u_m}\Big\{\sum_{i=0}^{\infty}\beta_m^{\tau_i}\Big\}\sup_{x\in\Gamma_1}|H_{m,k}^{\beta_m}(x) - H_{m,k}^1(x)| \\ &\le (1-\beta_m)k(\|F\| + J_\mu(u)\|G\|) \\ &+(1-\beta_m)\Big|E_{\mu_m}^{u_m}\Big\{\sum_{i=0}^{\infty}\beta_m^{\tau_i}(\eta^u(F) - J_\mu(u)\eta^u(G))\Big\}\Big| \\ &+(1-\beta_m)E_{\mu_m}^{u}\Big\{\sum_{i=0}^{\infty}\beta_m^{\tau_i}\Big\}\alpha^k(\|F\| + J_\mu(u)\|G\|) \end{aligned}$$

$$
\begin{aligned}
&+ \sup_{x\in\Gamma_1} |H^{\beta_m}_{m,k}(x) - H^1_{m,k}(x)| \\
&\le (1-\beta_m)k(\|F\| + J_\mu(u)\|G\|) + \alpha^k(\|F\| + J_\mu(u)\|G\|) \\
&+ \sup_{x\in\Gamma_1} |H^{\beta_m}_{m,k}(x) - H^1_{m,k}(x)|
\end{aligned}
$$

By considerations analogous to (4.204) and previous results it is easy to see that

$$\lim_{m\to\infty} \sup_{x\in\Gamma_1} |H^{\beta_m}_{m,k}(x) - H^1_{m,k}(x)| = 0$$

Letting now $m \to \infty$ and then $k \to \infty$ we finally have that

$$\lim_{m\to\infty}(1-\beta_m)\Big|E^{u_m}_{\mu_m}\Big\{\sum_{i=0}^{\infty}\beta_m^{\tau_i}[F(x_{\tau_i}) - J_\mu(u)G(x_{\tau_i})]\Big\}\Big| = 0$$

which together with (4.209) leads to a contradiction of (4.202).

The proof of (4.201) for the case of mixed observation is therefore completed.

■

Similarly as in Corollary 4.56 (compare also with Corollary 4.9) from Theorem 4.62 we immediately have

Corollary 4.63 *Under the assumptions of Theorem 4.62 we have*

(i)

$$\inf_{u\in\mathcal{A}(L,n)}(1-\beta)J^\beta_\mu(u) \to \inf_{u\in\mathcal{A}(L,n)} J_\mu(u)$$

as $\beta \to 1$, uniformly in $\mu \in P(\Gamma_1)$ in the case of mixed observations and in $\mu \in P(E)$ in the case of Examples 1 and 2, replacing for Example 2 the class $\mathcal{A}(L,n)$ by $\overline{\mathcal{A}}(L,n)$

(ii) *if for $\beta \le 1$ we have*

$$\sup_{u\in\mathcal{A}(L,n)} |(1-\beta)J^\beta_\mu(u) - J_\mu(u)| < \varepsilon \tag{4.210}$$

then an ε-optimal control function u^β for $J^\beta(u)$ over $u \in \mathcal{A}(L,n)$ is 3ε-optimal for the cost functional $J_\mu(u)$ over $u \in \mathcal{A}(L,n)$.

■

Combining finally Corollary 4.63 with Proposition 4.50 (or Proposition 4.52 and 4.53, for Examples 1 and 2 respectively) we have that, for β sufficiently close to 1 and L, n sufficiently large, an ε-optimal control function u^β for $J_\mu^\beta(u)$ is nearly optimal for $J_\mu(u)$ also over the class $\mathcal{A}$ ($\overline{\mathcal{A}}$ in the case of Example 2).

4.6 Filter approximations and near optimal control values

For a given control u the filtering process (π_n^u) takes its values in the infinite dimensional space $P(E)$ and therefore $u(\pi_n^u)$ can practically not be computed. Therefore it appears reasonable to approximate (π_n^u) be a process $(\overline{\pi}_n^m)$ with values in the simplex $P(E_m)$, which is a finite dimensional space.

Notice now that, corresponding to section 3.3.2, in section 4.5.2 we already considered a filtering process $(\overline{\pi}_i^m)$ that takes values in the simplex $P(E_m)$ and evolves essentially according to (3.57). This process $(\overline{\pi}_i^m)$ was however introduced only to make the computation of nearly optimal controls feasible and is not a real filtering process since it corresponds to fictitions D_m-valued observations generated by the approximating observation "densities" $r_m(x, y)$.

Recall now that in (2.48) we introduced a projection operator $W_m \colon R^d \to D_m$; applying this operator to the real observations (y_i) generated according to (1.1), we get "real" D_m valued observations. By analogy to what we did in section 3.5.1 in the context of infinite horizon problems with discounting, these real observations will now be used to construct a computable approximation $(\overline{\pi}_i^{m(\overline{\mu})})$ of the true filter according to a formula of the type (3.57) and taking values in $P(E_m)$. As was already pointed out in section 3.5, the approximating filter process itself is not Markov and so here too we shall consider pairs of "filter" processes that are Markov.

Given a control function u, that in what follows we shall assume to belong to the class $\mathcal{A} = C(P(E), U)$ (or $\overline{\mathcal{A}}$ defined in section 4.5.1.b) for $\nu,\ \mu \in P(E)$, $\eta \in P(E_m)$ define therefore according to (3.202)–(3.204) pairs of processes $(\pi_n^{(\nu)}, \pi_n^{(\mu,\nu)})$, $(\pi_n^{m(\nu)}, \pi_n^{m(\mu,\nu)})$ and $(\overline{\pi}_n^{m(\eta)}, \tilde{\overline{\pi}}_n^{m(\mu,\eta)})$ where: in the case of mixed observations, $M^a(y, \pi), M_m^a(y, \pi)$ and $\overline{M}_m^a(y, \overline{\pi})$ are given by (4.49), (4.144) and (4.145) respectively; in the case of Examples 1 and 2 by

(1.8), (3.14) and (3.57) requiring additionally for Example 2 that r and r_m are a d-dimensional version of (1.2) as considered in 4.5.2.b$_2$.

In the case of Examples 1 and 2, using Lemma 3.55 we have that the pairs $(\pi_n^{(\nu)}, \pi_n^{(\mu,\nu)})$, $(\pi_n^{m(\nu)}, \pi_n^{m(\mu,\nu)})$ and $(\overline{\pi}_n^{m(\eta)}, \widetilde{\pi}_n^{m(\mu,\eta)})$ form Markov processes with transition operators T, T_m and $\overline{T}_m$ defined in (3.205)–(3.207) respectively.

For the case of mixed observations we have by analogy to Lemma 3.55

Lemma 4.64 *In the case of mixed observations the pairs* $(\pi_n^{(\nu)}, \pi_n^{(\mu,\nu)})$, $(\pi_n^{m(\nu)}, \pi_n^{m(\mu,\nu)})$ *and* $(\overline{\pi}_n^{m(\eta)}, \widetilde{\pi}_n^{m(\mu,\eta)})$ *form Markov process with transition operators* T, T_m *and* $\overline{T}_m$ *defined as follows*

$$T^u F(\nu,\mu) := \int_\Gamma F(\delta_z,\delta_z) P^{u(\nu)}(\mu,dz) + \int_{\Gamma^c}\int_{\Gamma^c} F(M^{u(\nu)}(y,\nu), Q(y,\zeta(\mu,u(\nu))) r(z,y) dy P^{u(\nu)}(\mu,dz) \tag{4.211}$$

$$T_m^u F(\nu,\mu) := \int_\Gamma F(\delta_z,\delta_z) P^{\overline{\mathcal{L}}_m u(\nu)}(\mu,dz) + \int_{\Gamma^c}\int_{\Gamma^c} F(M_m^{\overline{\mathcal{L}}_m u(\nu)}(y,\nu), Q(y,\zeta(\mu,\overline{\mathcal{L}}_m u(\nu)))) r(z,y) dy P^{\overline{\mathcal{L}}_m u(\nu)}(\mu,dz) \tag{4.212}$$

$$\overline{T}_m^u f(\eta,\mu) := \int_\Gamma f(\delta_{W_m z},\delta_z) P^{\widetilde{\mathcal{L}}_m u(\eta)}(\mu,dz) + \int_{\Gamma^c}\int_{\Gamma^c} f(\overline{M}_m^{\widetilde{\mathcal{L}}_m u(\eta)}(W_m y,\eta), Q(y,\zeta(\mu,\widetilde{\mathcal{L}}_m u(\eta)))) r(z,y) dy P^{\widetilde{\mathcal{L}}_m u(\eta)}(\mu,dz) \tag{4.213}$$

for $F \in b\mathcal{B}(P(E)\times P(E))$, $f \in b\mathcal{B}(P(E_m)\times P(E)$, μ, $\nu \in P(E)$ *and* $\eta \in P(E_m)$, *where the operator* $Q(y,\nu)$ *has here the form*

$$Q(y,\nu)(A) = \frac{\int_{A\cap\Gamma^c} r(z,y)\nu(dz)}{\int_{\Gamma^c} r(z,y)\nu(dz)} \tag{4.214}$$

for $y \in \Gamma^c$, $\nu \in P(E)$, $A \in \mathcal{B}(E)$, ζ *is as in (3.201) and where (see (2.48) and section 4.5.2.b$_1$)* $W_m y = k$ *for* $y \in B_k^m$, *with* $k = 1,2,\ldots,k_m$.

Moreover, under (A1), (A2), (A8)–(A10) the operator T *is Feller.*

P r o o f. The proof of the Markov properties and of the forms of the transition operators is similar to that of Lemma 1.3 (see also Lemma 4.14 and Lemma 3.55). The Feller property of T can be shown as in Proposition 4.15. ■

In the case of mixed observations for the purpose of this section we need the following additional strengthening of assumption (A11) (ii)–(iv)

(B11) for any $u \in \mathcal{A}$ and the Markov times T_{Γ_1}, T_{Γ^c} with respect to (x_n) controlled by $a_n = \overline{\mathcal{L}}_m u(\pi_n^{m(\nu)}) = \widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\overline{\nu})})$ we have

- (i) $E_{\nu,\mu}^{\overline{\mathcal{L}}_m u} T_{\Gamma_1} < \infty$ for $\nu,\ \mu \in P(\Gamma^c)$, $m = 1, 2, \ldots$,
- (ii) $\sup_{x\in\Gamma_1} \sup_m E_{x,x}^{\overline{\mathcal{L}}_m u} \tau^2 < \infty$ with $\tau = T_{\Gamma^c} + T_{\Gamma^1} \circ \theta_{T_{\Gamma^c}}$
- (iii) there is a unique invariant measure η_m^u for (x_{τ_n}) where $\tau_1 = \tau$, $\tau_{n+1} = \tau_n + \tau \circ \theta_\tau$, and moreover the strong law of large numbers for (x_{τ_n}) holds.

Analogously to Corollary 3.57 we have with (see section 3.5.1) $\overline{\mu} = (\mu(B_1^m), \ldots, \mu(B_{k_m}^m))$

Theorem 4.65 *Under (A1), (A2), (A5), (A8)–(A12), (B11) in the case of mixed observation, (A1)–(A5), (E 1.1)–(E 1.5) in the case of Example 1, and (A1), (A2), (A5), (E 2.1)–(E 2.4) in the case of Example 2, for* $u \in \mathcal{A}$ *and* $u \in \overline{\mathcal{A}}$ *in the case of Example 2, we have for* $\mu \in P(E)$

$$J_\mu((\widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\overline{\mu})}))) \to J_\mu(u(\pi_n)) \tag{4.215}$$

as $m \to \infty$*.*

P r o o f. The proof is rather long and technical and follows the steps of Theorems 4.55, 4.58 and 4.59 which in turn are based on Theorems 4.35, 4.45 and 4.47 respectively. Therefore we sketch only some preliminary steps. Consider first the case with mixed observations. Notice that the measures

$$\begin{aligned}\overline{\Psi}_m^u(F_1) \colon = \int_{\Gamma_1} E_{W_m x,x}^{\widetilde{\mathcal{L}}_m u}\Big\{\sum_{i=0}^{\tau-1} F_1(\overline{\pi}_i^{m(W_m x)}, \widetilde{\pi}_i^{m(x,W_m x)})\Big\}\eta_m^u(dx)\\ \Big(\int_{\Gamma_1} E_{W_m x,x}^{\widetilde{\mathcal{L}}_m u}\{\tau\}\eta_m^u(dx)\Big)^{-1}\end{aligned} \tag{4.216}$$

and

$$\Psi^u_m(F_2) := \int_{\Gamma_1} E^{\overline{\mathcal{L}}_m u}_{x,x}\Big\{\sum_{i=0}^{\tau-1} F_2(\pi_i^{m(x)}, \pi_i^{m(x,x)})\Big\}\eta^u_m(dx) \Big(\int_{\Gamma_1} E^{\overline{\mathcal{L}}_m u}_{x,x}\{\tau\}\eta^u_m(dx)\Big)^{-1} \tag{4.217}$$

defined for $F_1 \in b\mathcal{B}(P(E_m) \times P(E))$ and $F_2 \in b\mathcal{B}(P(E) \times P(E))$, where to simplify notations we identify Dirac measures δ_x with x, are invariant for the operators $\overline{T}^u_m$, T^u_m respectively.

By (B11) and the proof of Proposition 4.32 (see also Remark 4.34)

$$J_\mu((\widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\overline{\nu})}))) = \int_{P(E_m)\times P(E)} \int_E c(x, \widetilde{\mathcal{L}}_m u(\eta))\nu(dx)\overline{\Psi}^u_m(d\eta, d\nu) \tag{4.218}$$

and from (4.216), (4.217) by direct substitutions and recalling the definition of the operators $\overline{\mathcal{L}}_m$ and $\widetilde{\mathcal{L}}_m$ we have

$$\int_{P(E_m)\times P(E)} \int_E c(x, \widetilde{\mathcal{L}}_m u(\eta))\nu(dx)\overline{\Psi}^u_m(d\eta, d\nu) = \\ = \int_{P(E)\times P(E)} \int_E c(x, \overline{\mathcal{L}}_m u(\nu_1))\nu_2(dx)\Psi^u_m(d\nu_1, d\nu_2) \tag{4.219}$$

In view of (A5), Lemma 3.21 (i) and Remark 4.34 it remains to show that

$$\Psi^u_m \Rightarrow \Psi^u \qquad \text{as } m \to \infty \tag{4.220}$$

where for $F \in b\mathcal{B}(P(E) \times P(E))$

$$\Psi^u(F) = \int_{P(E)\times P(E)} F(\nu, \nu)\Phi^u(d\nu) \tag{4.221}$$

and Φ^u is as in (4.98).

By (4.217) and (4.98), to obtain (4.220), it suffices to prove that

$$\eta^u_m \Rightarrow \eta^u \tag{4.222}$$

and that for $F \in C(P(E) \times P(E))$

$$
\begin{aligned}
&E_{x,x}^{\overline{\mathcal{L}}_m u}\Big\{ \sum_{i=0}^{\tau-1} F(\pi_i^{m(x)}, \pi_i^{m(x,x)}) \Big\} \\
&\to E_x^u \Big\{ \sum_{i=0}^{\tau-1} F(\pi_i^{\delta_x}, \pi_i^{\delta_x}) \Big\} \quad \text{as } m \to \infty
\end{aligned}
\tag{4.223}
$$

uniformly in $x \in \Gamma_1$.

The proofs of (4.222) and (4.223) are similar to those of (4.112) and (4.120) respectively. For this purpose one has first to show an analog of Lemma 4.38 which says that if $F_m \in b\mathcal{B}(P(E) \times P(E))$ are uniformly bounded, and converge uniformly on compact subsets of $P(E) \times P(E)$ to $F \in C(P(E) \times P(E))$, then for $A = \Gamma$ or $A = \Gamma_1$

$$
T_m^u(\nu, \mu, F_m \chi_{\widetilde{A} \times \widetilde{A}}) \to T^u(\nu, \mu, F \chi_{\widetilde{A} \times \widetilde{A}})
$$

as $m \to \infty$, uniformly on compact subsets of $P(E) \times P(E)$. Then by induction one establishes an analog of Lemma 4.39, from which by (B11) the convergence (4.223) follows as in Corollary 4.40. To prove (4.222) one has to show an analog of (4.111) and then repeat the considerations of the proof of Lemma 4.42. The details are left to the reader.

The proofs for the models of Examples 1 and 2 are similar. Notice in fact that, by Lemma 4.26 and (E 1.3), (E 1.4) in the case of Example 1, a suitable version of the assumption (B11) is clearly satisfied. In the case of Example 2, for $u \in \overline{\mathcal{A}}$ and m so large that h_m inherits the growth property (E 2.1) of h, by (E 2.3) and considerations similar to those of Lemma 4.28 (see also Corollary 4.29), we also obtain an analog of (B11).

■

Remark 4.66 *From the proof of Theorem 4.65 it is easily seen that, more generally, for any μ, $\nu \in P(E)$ we have*

$$
J_\mu(\widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\overline{\nu})})) \to J_\mu(u(\pi_n))
$$

Theorem 4.65 completes our approach for the construction of nearly optimal controls for infinite horizon problems with an ergodic cost functional: First determine a nearly optimal control function according to the methods described in section 4.5 (or section 4.2.2 provided the control function is continuous). Applying an extension procedure as in Chapter 3, this control function can be considered as an element of the class $\mathcal{A}$ (or $\overline{\mathcal{A}}$ for Example 2). For an initial measure μ and corresponding $\overline{\mu}$ compute the filter $\overline{\pi}_i^{m(\overline{\mu})}$ according to (3.204) with the changes as mentioned at the beginning of this section and m sufficiently large. A nearly optimal control a_n in the generic period n is then obtained by putting $a_n = \widetilde{\mathcal{L}}_m u(\overline{\pi}_n^{m(\overline{\mu})})$.

Appendix

(Bibliographical notes)

Chapter 1

Section 1.1: The formulation given here of a control problem is kept as general as possible and includes the models most commonly used in Control (see e.g. [4]) and Operation Research (see e.g. [24], [40]).

Section 1.2: The results presented here extend the validity of partly known results to the general discrete time setting described in section 1.1. In particular, the Markov and Feller properties of the filter process (Lemma 1.3 and Corollary 1.5) as well as the fact that the filter transition kernel preserves concavity (Proposition 1.7) are the discrete time counterparts of results that can be found in [19].

Section 1.3: The measure transformation results presented here correspond to what in continuous time usually called a Girsanov-type transformation [16]. To the best of our knowledge these results are the most general ones that have so far appeared concerning discrete time models. Earlier results can be found in [8], [9], [35] and references therein (for a recent account see also [14]).

Chapter 2

Section 2.1: The idea of the approximation approach follows the lines of [10].

Section 2.2: The convergence results of this section are essentially new. The technique, adopted recurrently in the convergence proofs of subsection 2.2.2, namely the use of a weighted L^1-norm, is taken from [11] and goes back to a suggestion made by Prof. A. Bensoussan.

Section 2.3: The idea of approximationg general transition kernels by kernels corresponding to finite state Markov chains is the most natural one and has been used by various authors; the approach followed here in subsection 2.3.1 is an extension of that in [10]. The idea, exploited in subsection 2.3.2, namely to obtain finite-dimensional approximations by means of transition kernals separated in the variables goes back to [11] where it was used for filter approximations; it was later extended in [32] to the context of adaptive control.

Chapter 3

Section 3.2: The Bellman equation for infinite horizon problems with discounting is well known and is e.g. studied in [5] and [17]; Remark 3.3 is essentially taken from [5].

Section 3.3: Subsection 3.3.1 is an adaptation of results in [30] and the further subsection 3.3.1.a is new. Subsection 3.3.2 deals with approximations along the lines of subsection 2.3.1, that are here extended to the case when a normalized filter process is used. The results obtained in 3.3.2.a are, as the ones in 3.3.1.a, essentially new while those of 3.3.2.b are again adaptations of similar results in [30]. Subsection 3.3.3 deals with further approximations that, for the part described in 3.3.3.a and particularly 3.3.3.a_1, are analogous to those in section 2.3.1. and go back to ideas in [10]; subsection 3.3.3.a_2 is instead a detailed elaboration of work by Sondik in [34] (in this context see also [39]). Finally, subsection 3.3.3 is again based on the metodology used by the authors in [30].

Section 3.4: Analogously to section 2.3.2, this section extends to infinite horizon problems with discounting the approach originated in a filtering context in [11] and used for approximations in adaptive control in [32]. The entire section is new; in particular, the results of subsection 3.4.1 were derived to be able to deal with the specific kind of approximations considered in this section. Measurable selection theorems can be found in various references, here we restrict ourselves to mention [26] and [27].

Section 3.5: This section is entirely new, even if subsection 3.5.1 is based on results in [30].

Chapter 4

Section 4.1: Only very few results are so far available from the literature concerning ergodic control of stochatic partially observed systems. This problem is in fact listed as open problem in [7] and its difficulty appears also from [1].

Section 4.2: A first approach to the discounted cost approximation of the Bellman equation for partially observed ergodic Markov control problems appears, under rather restrictive assumptions, in [20] and [38]. The approach presented here in subsection 4.2.2, is taken from [36]. The generalization to the case considered in subsection 4.2.3 is new, and so is also the discounted cost approximation discussed in 4.5.5 for the models of subsection 4.5.1.

Section 4.3: The definition of a "mixed observation model" and the basic results of this section are taken from [37]. Subsection 4.3.2 concerning the solution of the Bellman equation is an adaptation from [36].

Section 4.4: The counterexample of a well-known ergodic result for uncontrolled filtering processes (see [19] for continuous time and [35] for its extension to discrete time and locally compact state spaces) has been worked out for this monograph to show the kind of difficulties that arise when studying existence and especially uniqueness of invariant measures for controlled filtering processes. Except for the results obtained by the authors in [30], [31] and [37], uniqueness results appear to be available from the literature only for very particular models as in [1] and [12]. The results obtained here for the embedded i.i.d. case are new and generalize related ones in [30], [31] (Example 2 is taken from [30]). Section 4.4.3 follows from [37].

Section 4.5: Since the basic approach here parallels that of section 3.3, much of what was said for the latter section holds here as well. An exception is section 4.5.1 that is more specific to the ergodic case.

Section 4.6: Similarly to section 3.5, the underlying technique comes again from [30]; here however the approach is borrowed from [37].

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