On p-harmonic mappings in the plane

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Abstract

To every p-harmonic vector field in the plane with $p \geq 2$ there corresponds a quasilinear system of first order PDE's which couples the complex gradients of the coordinate functions of the field. The ellipticity of such system is proved. A relation between planar quasiregular mappings and p-harmonic fields is discussed. The p-harmonic conjugate problem is stated.

Keywords: p-harmonic mapping, complex gradient, quasilinear system, quasiregular mapping

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1 Introduction and preliminaries.

One of the most challenging and difficult problems in the theory of nonlinear PDEs is to understand the geometry of solutions. In that setting the profound position is held by the question on the topological structure of the set of critical points. Even for the simplest nonlinear equations pursuing the answer to this question is difficult and often requires using advanced methods. Such is the case of the p-harmonic equation - a model differential equation of nonlinear analysis and its applications (e.g. in nonlinear elasticity theory [6], quantum physics [7], fluid dynamics [11], glaciology [2]). The set of critical points for p-Laplacian in the plane is understood due to results by Bojarski and Iwaniec [8], Manfredi [20, 21], Aronsson [3] and Lindqvist [4]; see also Lewis [18]. These results were established largely by employing quasiregular mappings and their topological properties (see [15] for comprehensive account of quasiregular transformations and recent developments in that area; see [9],[14] for more on the interplay between topology and quasiregular mappings; see [17] and Chapter 12 in [13] for more on connections between quasiregular mappings and elliptic equations).

The structure of the set of critical points for p-harmonic functions in dimensions higher than 2 is an open problem and seems to go beyond the scope of existing methods.

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The classical result by Bojarski and Iwaniec [8] (see also [20]) asserts that the complex gradient of p-harmonic function in the planar domain for $p > 2$ is a quasiregular mapping. This in turn implies via properties of quasiregular mappings that the singular set of the p-harmonic function consists of isolated points. The essential part of their proof is that this complex gradient satisfies a first order quasilinear elliptic system of PDEs.

The purpose of this note is to establish the similar result in the vectorial setting, that is for p-harmonic mappings in the plane.

These are $W^{1,p}_{loc}(\Omega, \mathbb{R}^2)$ solutions to the so called p-harmonic system:

$$\operatorname{div}(|Du|^{p-2}Du) = 0, \quad u = (u^1, u^2) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad 1 < p < \infty, \tag{1}$$

where Du is the Jacobi matrix. Equivalently, this system can be written as follows.

$$\begin{cases} \operatorname{div}(|Du|^{p-2}\nabla u^1) = 0 \\ \\ \operatorname{div}(|Du|^{p-2}\nabla u^2) = 0 \end{cases} \tag{2}$$

As p-harmonic system is strongly coupled by the appearance of Du, many methods known in the scalar case fail; also complexity of technical details is much higher.

For such mapping $u = (u^1, u^2)$ we denote by f, g complex gradients of the first and second coordinate function of u, respectively.

$$f = \frac{1}{2}(u^1_x - i\, u^1_y), \quad g = \frac{1}{2}(u^2_x - i\, u^2_y). \tag{3}$$

We adopt a computation in [8], which requires assumption $p > 2$. The following lemma is a generalization of Proposition 2 in [8] (see also Theorem (1.10) [23] for the setting of elliptic complexes).

Lemma 1 *For $p > 2$ let $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^2)$ be a p-harmonic planar transformation. Define $F = |Du|^{\frac{p-2}{2}} Du \in L^2(\Omega)$. Then $F \in W^{1,2}_{loc}(\Omega, \mathbb{R}^{2\times 2})$. Moreover for any compact set $K \subset \Omega$*

$$\|DF\|_{L^2(K)} \leqslant \frac{C(p)}{\operatorname{dist}(K, \partial\Omega)} \|F\|_{L^2(\Omega)}.$$

The proof is similar to that of [Proposition 2, [8]] and, therefore, is omitted.

We are now in a position to formulate the keypoint result of this note. Consequences of this result will be exploited in further sections.

Theorem 1 *For $p > 2$ let $u = (u^1, u^2) \in W^{1,p}_{loc}(\Omega, \mathbb{R}^2)$ be a p-harmonic mapping. Consider complex gradients f, g of coordinate functions u^1, u^2, respectively (eqs. (3)). We have the following system of quasilinear equations.*

$$\begin{cases} \left(2p + \frac{4|g|^2}{|f|^2}\right) f_{\overline{z}} = (2-p)\left(\frac{\overline{f}}{f}\, f_z + \frac{f}{\overline{f}}\, \overline{f_z}\right) \\ \qquad\qquad + (2-p)\left[\frac{\overline{g}}{f}\, g_z + \frac{g}{\overline{f}} \overline{g_z} + \left(\frac{\overline{g}}{\overline{f}} + \frac{g}{f}\right) g_{\overline{z}}\right] \\ \left(2p + \frac{4|f|^2}{|g|^2}\right) g_{\overline{z}} = (2-p)\left(\frac{\overline{g}}{g}\, g_z + \frac{g}{\overline{g}}\, \overline{g_z}\right) \\ \qquad\qquad + (2-p)\left[\frac{\overline{f}}{g}\, f_z + \frac{f}{\overline{g}} \overline{f_z} + \left(\frac{\overline{f}}{\overline{g}} + \frac{f}{g}\right) f_{\overline{z}}\right] \end{cases} \tag{4}$$

at the points where $f \neq 0$ and $g \neq 0$.

Notice that for a scalar p-harmonic equation $u^2 \equiv 0$ and the above system reduces to equation (5) in [8]:

$$f_{\overline{z}} = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\overline{f}}{f}\, f_z + \frac{f}{\overline{f}}\,\overline{f_z}\right).$$

This immediately implies that f is a quasiregular mapping with

$$|\mu| = \left|\frac{f_{\overline{z}}}{f_z}\right| \leq 1 - \frac{2}{p},$$

where μ is the Beltrami coefficient of f (thorough discussion of this topic can be found in Chapter 11, [15]; see also [1]).

System (4) can be solved for $f_{\overline{z}}$ and $g_{\overline{z}}$. Tedious but straightforward computations allow us to express $f_{\overline{z}}, g_{\overline{z}}$ in terms of $f_z, g_z, \overline{f_z}, \overline{g_z}$ as follows.

$$\begin{bmatrix} f \\ g \end{bmatrix}_{\overline{z}} = A(f,g)\begin{bmatrix} f \\ g \end{bmatrix}_z + \overline{A(f,g)}\,\overline{\begin{bmatrix} f \\ g \end{bmatrix}_z}, \tag{5}$$

where

$$A(f,g) =$$

$$\frac{2-p}{\Phi}\begin{bmatrix} B\frac{\overline{f}}{f} + (2-p)D & \frac{\overline{g}}{f}\big(B + (2-p)D\big) \\ \frac{\overline{f}}{Bg}\big(\Phi + (2-p)^2C\big) + (2-p)\frac{\overline{f}}{g}D & \frac{\overline{g}}{Bg}\big(\Phi + (2-p)^2C\big) + (2-p)D \end{bmatrix}$$

for

$$\begin{aligned}
\Phi &:= \Phi(f,g) = \left(2p + \frac{4|g|^2}{|f|^2}\right)\left(2p + \frac{4|f|^2}{|g|^2}\right) - (2-p)^2\left(2 + \frac{\overline{g}}{g}\frac{f}{\overline{f}} + \frac{g}{\overline{g}}\frac{\overline{f}}{f}\right), \\
B &:= B(f,g) = 2p + 4\frac{|f|^2}{|g|^2}, \\
C &:= C(f,g) = 2 + \frac{\overline{g}}{g}\frac{f}{\overline{f}} + \frac{g}{\overline{g}}\frac{\overline{f}}{f}, \\
D &:= D(f,g) = \frac{\overline{g}}{g} + \frac{\overline{f}}{f}.
\end{aligned}$$

Notice that Φ is real and $\Phi > 32p$.

The ellipticity of such quasilinear system requires some discussion. We will postpone it to Section 2. There we also prove the following result.

Theorem 2 *Let $\beta = \overline{\alpha} = -\overline{A(f,g)}$. If $\|\alpha\| < \frac{1}{2}$, then the system (5) is elliptic for p sufficiently close to 2; exact value of p will be specified in the proof. Here we use the Hilbert-Schmidt norm of a matrix, i.e. $\|\alpha\|^2 = \mathrm{tr}(\alpha\overline{\alpha}^T)$.*

In Section 3 we use the system (4) to establish conditions for f to be quasiregular. For that reason we show that (see (16))

$$\frac{|f_{\overline{z}}|}{|f_z|} \leq 1 - \frac{2}{p} + \frac{(p-2)(p-1)}{2p}\frac{|g_z|}{|f_z|}.$$

Similar inequality holds for g. We illustrate discussion by formulating this condition for radial mappings. We also point out that seldom f and g are quasiregular simultaneously. In the same section we make an observation for radial mappings that taking conjugate complex gradients instead of complex gradients leads to simpler conditions.

Following this idea, in Section 4 we discover and discuss differential expression which depends on all entries of the Jacobi matrix of u and which turns out to be quasiregular. This is obtained in the class of radial transformations, but we conjecture that the same expression remains quasiregular for the general solutions of system (4).

The last section is devoted to demonstrate other possible application of our results. That is, p-harmonic conjugate problem formulated in the analogy to the planar harmonic conjugate problem.

2 Proofs of Theorems 1 and 2

Proof of Theorem 1.
Denote by Du the Jacobi matrix of u. Then

$$|Du|^2 \;=\; (u^1_x)^2 + (u^1_y)^2 + (u^2_x)^2 + (u^2_y)^2 = 4|f|^2 + 4|g|^2,$$

where f, g are complex gradients as in (3). The following holds in the sense of distributions,

$$\begin{cases} \frac{\partial}{\partial x}\left[(|f|^2+|g|^2)^{\frac{p-2}{2}}(f+\overline{f})\right] + i\frac{\partial}{\partial y}\left[(|f|^2+|g|^2)^{\frac{p-2}{2}}(f-\overline{f})\right] \;=\; 0 \\[2ex] \frac{\partial}{\partial x}\left[(|f|^2+|g|^2)^{\frac{p-2}{2}}(g\,+\,\overline{g})\right] + i\frac{\partial}{\partial y}\left[(|f|^2+|g|^2)^{\frac{p-2}{2}}(g-\overline{g})\right] \;=\; 0 \end{cases}$$

because u is a solution of p-harmonic system of equations. This system is equivalent to the following

$$\begin{cases} \mathrm{Re}\frac{\partial}{\partial \overline{z}}\left[(|f|^2+|g|^2)^{\frac{p-2}{2}} f\right] \;=\; 0 \\[2ex] \mathrm{Re}\frac{\partial}{\partial \overline{z}}\left[(|f|^2+|g|^2)^{\frac{p-2}{2}} g\right] \;=\; 0 \end{cases}$$

due to Lemma 1. Now, derivation of system (4) is straightforward. ■

2.1 The ellipticity of system (4)

The purpose of this section is to discuss the ellipticity of system (4) and to prove Theorem 2.

The following definition of ellipticity comes naturally from that of linear systems (see Chapter 6, [5] for expository discussion of this topic).

Definition 1 *Consider general quasilinear operator of the first order in two variables*

$$\mathfrak{L} \;=\; A(f(z), g(z))\frac{\partial}{\partial x} \;+\; B(f(z), g(z))\frac{\partial}{\partial y}, \tag{6}$$

where $A(f(z),g(z)), B(f(z),g(z))$ *are* $2n\times 2n$ *matrices depending on a vector function* $(f(z),g(z))$ *defined on the domain* $\Omega\subset\mathbb{R}^2$.
We say that $\mathfrak{L}$ *is elliptic at* $f(z)$ *and* $g(z)$ *if*

$$\det\big(aA(f(z),g(z))+bB(f(z),g(z))\big)\neq 0,\ \textit{for } a^2+b^2\neq 0 \textit{ and for a.e. } z\in\Omega \quad (7)$$

By the standard complexification we arrive at the following form of this operator:

$$\mathfrak{L}=\alpha(f(z),g(z))\frac{\partial}{\partial z}+\beta(f(z),g(z))\overline{\frac{\partial}{\partial z}}+\gamma(f(z),g(z))\frac{\partial}{\partial \overline{z}}+\delta(f(z),g(z))\overline{\frac{\partial}{\partial \overline{z}}} \quad (8)$$

Here α,β,γ and δ are complex matrices $n\times n$. The following lemmas are quasilinear generalizations of these of constant coefficient versions discussed in [Chapter 6, [5]].

Lemma 2 *The operator* $\mathfrak{L}$ *of type (5) is elliptic if and only if the linear map*

$$H\to\big[\alpha(f(z),g(z))\overline{\xi}+\gamma(f(z),g(z))\xi\big]H+\big[\delta(f(z),g(z))\overline{\xi}+\beta(f(z),g(z))\xi\big]\overline{H}$$

is nonsingular for all $\xi\in\mathbb{C}\setminus\{0\}$ *and a.e.* $z\in\Omega\subset\mathbb{R}^2$.

Lemma 3 *Let* $P(z)$ *and* $Q(z)$ *be complex* $n\times n$ *matrices depending on the point* $z\in\Omega$. *The linear map*

$$H\to P(z)H+\overline{Q(z)}\,\overline{H},\quad H\in\mathbb{C}^n$$

is nonsingular if and only if

$$\det\begin{bmatrix} P(z) & Q(z)\\ \overline{Q(z)} & \overline{P(z)}\end{bmatrix}\neq 0,\qquad \textit{for a.e. } z\in\Omega.$$

Using the above definition and lemmas we can derive a sufficient condition for the ellipticity of operators of type (5).

Theorem 3 *Let* $\alpha=\alpha(f(z),g(z)),\beta=\beta(f(z),g(z))$ *be complex functions defined on the set* $\Omega\subset\mathbb{R}^2$ *with values in* $\mathbb{C}^{2\times 2}$ *such that*

$$\|\alpha(f(z),g(z))\|+\|\beta(f(z),g(z))\|<1,\qquad \textit{for a.e. } z\in\Omega.$$

Then the quasilinear differential operator

$$\mathfrak{L}=\frac{\partial}{\partial\overline{z}}+\alpha\frac{\partial}{\partial z}+\beta\overline{\frac{\partial}{\partial z}}$$

is elliptic. Here we use the Hilbert-Schmidt norm, i.e. $\|\alpha\|^2=\mathrm{tr}(\alpha\overline{\alpha}^T)$.

We remark that this theorem provides us with only sufficient condition and, therefore, does not describe all elliptic operators.

Proof of Theorem 2.
We need to estimate norm of matrix $A(f,g)$. In order to do this we estimate first its square, i.e.

$$\|A(f,g)\|^2 = |A(f,g)_{11}|^2 + |A(f,g)_{12}|^2 + |A(f,g)_{21}|^2 + |A(f,g)_{22}|^2,$$

where the double subindices indicate the entries of $A(f,g)$. The key point of our considerations is that we can bound coefficients of $A(f,g)$ only in terms of number p. To avoid tedious calculations we will give argument only for $|A(f,g)_{11}|$ as the remaining estimates follow the same line.
First notice that

$$\begin{aligned}
|\Phi| &= \left|\left(2p+\frac{4|g|^2}{|f|^2}\right)\left(2p+\frac{4|f|^2}{|g|^2}\right)-(2-p)^2\left(2+\frac{\overline{g}}{g}\frac{f}{\overline{f}}+\frac{g}{\overline{g}}\frac{\overline{f}}{f}\right)\right| \\
&\geq \left|\left(2p+\frac{4|g|^2}{|f|^2}\right)\left(2p+\frac{4|f|^2}{|g|^2}\right)\right| - \left|(2-p)^2\left(2+\frac{\overline{g}}{g}\frac{f}{\overline{f}}+\frac{g}{\overline{g}}\frac{\overline{f}}{f}\right)\right| \\
&\geq 4p^2+16+8p\left(\frac{|f|^2}{|g|^2}+\frac{|g|^2}{|f|^2}\right)-4(2-p)^2 \\
&= 16p+8p\left(\frac{|f|^2}{|g|^2}+\frac{|g|^2}{|f|^2}\right). \qquad (9)
\end{aligned}$$

Then

$$\begin{aligned}
|A(f,g)_{11}| &= \left|\frac{(2-p)\left[(2p+\frac{4|f|^2}{|g|^2})\frac{\overline{f}}{f}+(2-p)(\frac{\overline{g}}{g}+\frac{\overline{f}}{f})\right]}{\left(2p+\frac{4|g|^2}{|f|^2}\right)\left(2p+\frac{4|f|^2}{|g|^2}\right)-(2-p)^2\left(2+\frac{\overline{g}}{g}\frac{f}{\overline{f}}+\frac{g}{\overline{g}}\frac{\overline{f}}{f}\right)}\right| \\
&\leq \frac{|2-p|\left[2p+\frac{4|f|^2}{|g|^2}+2|2-p|\right]}{16p+8p\left(\frac{|f|^2}{|g|^2}+\frac{|g|^2}{|f|^2}\right)} \\
&\leq \frac{2|2-p|(2p+|2-p|)|g|^2|f|^2+4|2-p||f|^4}{8p(|f|^2+|g|^2)^2} \\
&\leq (|f|^2+|g|^2)^2\frac{\max\{4|2-p|,(2p+|2-p|)|2-p|\}}{8p(|f|^2+|g|^2)^2} \\
&= \frac{(p-2)(3p-2)}{8p}. \qquad (10)
\end{aligned}$$

where the last equality is the result of the assumption that $p>2$.
Similarly we find that:

$$\begin{aligned}
|A(f,g)_{12}| &\leq \frac{(p-2)(p-1)}{4p}, \\
|A(f,g)_{21}| &\leq \frac{(p-2)(p^2+4p-4)}{32p}, \\
|A(f,g)_{22}| &\leq \frac{(p-2)(p^2+p+2)}{16p}. \qquad (11)
\end{aligned}$$

Henceforth, after adding the squares of the above coefficients, we obtain the inequality.

$$\|A(f,g)\|^2 \leq \frac{(p-2)^2(5p^4+16p^3+236p^2-346p+160)}{1024\,p^2} \qquad (12)$$

The numerator is increasing for $p \geq 2$ faster than the denominator. Numerical estimate gives us that for $p = 3$ $\|A(f,g)\| \lesssim 0.47$. However, already for $p = 4$, $\|A(f,g)\| > 1$. Thus the ellipticity condition $\|A(f,g)\| \leq \frac{1}{2}$ fails for $p \geq 4$. Nevertheless, for $2 < p < 4$ the ellipticity is satisfied for p in the range

$$p \in (2, p_*), \quad \text{where } p_* \in [3, 4). \tag{13}$$

Exact value of the critical parameter p_* for which our proof holds is, according to formula (12), given by the condition:

$$(p_* - 2)\sqrt{5p_*^4 + 16p_*^3 + 236p_*^2 - 346p_* + 160} = 16p_*.$$

■

***Remark* 1** Let us point out that the restriction imposed on the admissible range of parameter p in the above theorem is only due to the method we have used, namely Theorem 3.

3 p-Harmonic transformations and quasiregular mappings

For the complex gradient $\tilde{f}$ of p-harmonic function in the plane one can establish that

$$|\tilde{f}_{\bar{z}}| \leq (1 - \frac{2}{p})|\tilde{f}_z|, \tag{14}$$

provided that $p > 2$. Therefore, the complex gradient $\tilde{f}$ is a quasiregular mapping (see [8]).

The goal of this section is to present a counterpart of the above result in the category of p-harmonic systems. One of the main characteristic features of vectorial setting is that technical complexity of computations is considerably higher. This in turn enforces us, in many examples below, to restrict our considerations to the class of radial mappings. Nevertheless, even for that class we detect interesting phenomena. It will appear for instance that instead of complex gradient it is more convenient to consider its conjugate.

According to the operator form (5) of the system at (4) we find that

$$f_{\bar{z}} = A(f,g)_{11} f_z + A(f,g)_{12} g_z + \overline{A(f,g)_{11}}\,\overline{f_z} + \overline{A(f,g)_{12}}\,\overline{g_z}.$$

This results in the following estimation.

$$\begin{aligned} |f_{\bar{z}}| &\leq |A(f,g)_{11}||f_z| + |A(f,g)_{12}||g_z| + |\overline{A(f,g)_{11}}||\overline{f_z}| + |\overline{A(f,g)_{12}}||\overline{g_z}| \\ &\leq \frac{p-2}{p}|f_z| + \frac{(p-2)(p-1)}{2p}|g_z|. \end{aligned} \tag{15}$$

Here we appealed to inequalities (10) and (11).
Therefore

$$\frac{|f_{\bar{z}}|}{|f_z|} \leq 1 - \frac{2}{p} + \frac{(p-2)(p-1)}{2p}\frac{|g_z|}{|f_z|}. \tag{16}$$

Such form of inequality reveals difficulty in showing that f and g are quasiregular mappings. Without additional assumption on relation between g_z and f_z it seems to be impossible to estimate the Beltrami coefficient of f exclusively in terms of p as in (14).
To overcome this difficulty let us assume that there exists function $\gamma(z) < \frac{4}{(p-2)(p-1)}$ for which:

$$\frac{|g_z|}{|f_z|} < \gamma(z). \tag{17}$$

This assumption together with condition (16) imply

$$\frac{|f_{\bar{z}}|}{|f_z|} < 1$$

and so f is a quasiregular mapping. Notice that by choosing $p \approx 2$ we increase the level of freedom for boundedness of γ. We recall from the discussion of ellipticity condition in Section 2.1 that this is the case of system (4). To illustrate we consider the class of radial solutions for which the above estimate can be described more explicitly.

Example 1 Let u be a radial p-harmonic mapping, i.e. solution of system (1) in the form $u(z) = H(|z|)\, z$. Then Jacobi matrix Du takes the form

$$Du = \begin{bmatrix} \dot{H}\,\frac{x^2}{|z|} + H & \dot{H}\,\frac{xy}{|z|} \\ & \\ \dot{H}\,\frac{xy}{|z|} & \dot{H}\,\frac{y^2}{|z|} + H \end{bmatrix} \qquad z = x + i\, y.$$

Direct calculations imply that

$$\begin{aligned} f_z &= \frac{1}{4}\Bigg[\ \ddot{H}\, x\frac{x^2-y^2}{|z|^2} + \dot{H}\, x\frac{x^2+3y^2}{|z|^3} - 2i\,\frac{y}{|z|^2}(x^2\,\ddot{H} + \frac{y^2}{|z|}\,\dot{H})\Bigg], \\ g_z &= \frac{1}{4}\Bigg[\ \ddot{H}\, y\frac{x^2-y^2}{|z|^2} + \dot{H}\, y\frac{y^2+3x^2}{|z|^3} - 2i\,\frac{x}{|z|^2}(y^2\,\ddot{H} + \frac{x^2}{|z|}\,\dot{H})\Bigg]. \end{aligned} \tag{18}$$

Then condition (17) reads as:

$$\begin{aligned} &\frac{|g_z|^2}{|f_z|^2} < \gamma(z)^2 \iff \\ \ddot{H}\left(\ddot{H} + \frac{\dot{H}}{|z|}\right)&\left(x^2\gamma(z)^2 - y^2\right) + \dot{H}^2\left(1 + \frac{3x^2y^2}{|z|^4}\right)\left(\gamma(z)^2 - 1\right) + \\ &x^2y^2\frac{\dot{H}}{|z|^5}\left(\frac{3\,\dot{H}}{|z|} - 4\,\ddot{H}\right)\left(y^2\gamma(z)^2 - x^2\right) > 0. \end{aligned} \tag{19}$$

From this we immediately observe, that for instance if $\ddot{H}, \dot{H} > 0$ and $\frac{3\dot{H}}{|z|} - 4\,\ddot{H} > 0$ then on the domain described by the set of inequalities below, f is quasiregular.

$$\begin{cases} |y| < |\gamma(z)||x| \\ |x| < |\gamma(z)||y| \\ |\gamma(z)| > 1 \end{cases}$$

The above considerations give us condition only for f to be quasiregular. The similar chain of inequalities as in (15) results in analogous condition for g (we use the set of inequalities (11)):

$$\frac{|g_{\overline{z}}|}{|g_z|} \leq \frac{(p-2)(p^2+p+2)}{8p} + \frac{(p-2)(p^2+4p-4)}{16p}\frac{|f_z|}{|g_z|}. \tag{20}$$

As in the case of f, we assume that there exists function $\delta(z) < \frac{2(p+2)(-p^2+3p+2)}{(p-2)(p^2+4p-4)}$ such that

$$\frac{|f_z|}{|g_z|} \leq \delta(z). \tag{21}$$

Then $\frac{|g_{\overline{z}}|}{|g_z|} < 1$ and therefore g is quasiregular.

Separate question that requires answer is when both f and g are quasiregular. In such a case conditions (17) and (21) have to be satisfied simultaneously. This results in strong pointwise estimate

$$\frac{(p-2)(p^2+4p-4)}{(p+1)^2(4-p)} \leq \frac{|g_z|}{|f_z|} \leq \frac{4}{(p-2)(p-1)}. \tag{22}$$

This condition is satisfied when the left hand side is nonnegative and does not exceed the right hand side of the inequality. This holds for $p \in (2, 3^*)$, where $3^* > 3$ but close to 3. Henceforth, by taking the minimum of p_* in (13) and 3^* we attain the range of admissible p's.

The above discussion illustrates how seldom are the complex gradients f or g quasiregular. This suggests that we should look for other differential expressions involving u to be quasiregular. It turns out that if we consider

$$f = \frac{1}{2}(u^1_x + i\, u^1_y), \;\; g = \frac{1}{2}(u^2_x + i\, u^2_y). \tag{23}$$

instead of complex gradient, then for radial p-harmonics we obtain much simpler condition than (19) (the similar phenomenon has been recently observed in the scalar setting [19]). In the next example we discuss this remarkable fact. In that setting Beltrami coefficients for f and g can be computed explicitly without additional assumptions on relation between them (compare with (16)).

Example 2 Let

$$u(z) = H(|z|)\, z \tag{24}$$

be a radial p-harmonic mapping as in the previous example. Then, the complex conjugate of gradient of the first component equals

$$f = \frac{1}{2}(u^1_x + i\, u^1_y) \; = \; \frac{1}{2}\left(\dot{H}\,\frac{x^2}{|z|} + H + i\;\dot{H}\,\frac{xy}{|z|}\right). \tag{25}$$

Computations similar to that in (18) lead us to formulas

$$f_z \;=\; \frac{1}{4}(f_x - i\, f_y) = \frac{1}{4}\left(\ddot{H}\, x + 3\,\dot{H}\,\frac{x}{|z|}\right), \tag{26}$$

$$f_{\overline{z}} \;=\; \frac{1}{4}(f_x + i\, f_y) = \frac{1}{4}\left(\ddot{H}\, x\frac{x^2-y^2}{|z|^2} + \dot{H}\, x\frac{x^2+3y^2}{|z|^3} + 2iy\big(\;\ddot{H}\,\frac{x^2}{|z|^2} + \dot{H}\,\frac{y^2}{|z|^3}\big)\right).$$

By the same reasoning we obtain equations for g_z and $g_{\overline{z}}$.

$$g_z \;=\; \frac{1}{4}(g_x - i\,g_y) = \frac{1}{4}\left(\ddot{H}\,y + 3\,\dot{H}\,\frac{y}{|z|}\right), \tag{27}$$

$$g_{\overline{z}} \;=\; \frac{1}{4}(g_x + i\,g_y) = \frac{1}{4}\left(\ddot{H}\,y\frac{x^2-y^2}{|z|^2} + \dot{H}\,y\frac{y^2+3x^2}{|z|^3} + 2ix\big(\;\ddot{H}\;\frac{y^2}{|z|^2} + \;\dot{H}\;\frac{x^2}{|z|^3}\big)\right).$$

We will first give conditions for f and g to be both sense preserving or to be both sense reversing mappings as we will need this assumption in further investigations of Beltrami coefficients of f and g.

Observation 1 *Let u be a radial p-harmonic mapping as in (24). Let $J(f)$ $(J(g))$ denote the Jacobian of mapping f, (g) respectively.*
Then we have.

(a) If $\dot{H}\;\left[(\ddot{H}\;|z| + 3\;\dot{H})x^2 - \;\dot{H}\;|z|^2\right] \geq 0\;(\leq 0)$, then $J(f) \geq 0\;(\leq 0)$ and therefore f is sense preserving (reversing, respectively).

(b) If $\dot{H}\;\left[(\ddot{H}\;|z| + 3\;\dot{H})y^2 - \;\dot{H}\;|z|^2\right] \geq 0\;(\leq 0)$, then $J(g) \geq 0\;(\leq 0)$ and therefore g is sense preserving (reversing, respectively).

Proof.
We will give an argument only for part a). The part b) follwos the same lines. In part a) we only demonstrate the case of nonnegative Jacobian, the other being similar.
Recall that Jacobian of f in complex variables is defined via the following formula:

$$J(f) \;=\; |f_z|^2 - |f_{\overline{z}}|^2.$$

Using equations (18) and (26) we get,

$$16J(f) \;=\; x^2(\ddot{H} + \frac{3\,\dot{H}}{|z|})^2 - x^2(\ddot{H} + \frac{\dot{H}}{|z|})^2 - 4y^2\frac{\dot{H}^2}{|z|^2} \geq 0,$$

provided

$$\dot{H}\ddot{H}\;x^2|z| + 3\;\dot{H}^2\;x^2 - \;\dot{H}^2\;|z|^2 \geq 0. \qquad (y^2 = |z|^2 - x^2)$$

This immediately implies the assertion of point a). ■

We are now in a position to study the behavior of μ and ν, the Beltrami coefficients of f and g, respectively. The collection of identities (26, 27) permits us to find the square of $|\mu|$ for orientation preserving f.

$$|\mu|^2 = \frac{|f_{\overline{z}}|^2}{|f_z|^2} = \frac{x^2(\ddot{H} + \frac{\dot{H}}{|z|})^2 + 4y^2\frac{\dot{H}^2}{|z|^2}}{x^2(\ddot{H} + \frac{3\dot{H}}{|z|})^2}. \tag{28}$$

Similarly we obtain that

$$|\nu|^2 = \frac{|g_{\overline{z}}|^2}{|g_z|^2} = \frac{y^2(\ddot{H} + \frac{\dot{H}}{|z|})^2 + 4x^2\frac{\dot{H}^2}{|z|^2}}{y^2(\ddot{H} + \frac{3\dot{H}}{|z|})^2}. \tag{29}$$

Straightforward computations based on the last proof imply the following theorem.

Theorem 4 *Let u be nontrivial p-harmonic radial mapping as above. Assume that f is orientation preserving.*

(a) The Beltrami coefficient $|\mu| < 1$ if and only if

$$\dot{H}\ \left[(\ddot{H}\ |z| + 3\ \dot{H})x^2 - \dot{H}\ |z|^2\right] > 0.$$

(b) The Beltrami coefficient $|\nu| < 1$ if and only if

$$\dot{H}\ \left[(\ddot{H}\ |z| + 3\ \dot{H})y^2 - \dot{H}\ |z|^2\right] > 0.$$

Therefore, f and g are quasiregular mappings.

***Remark* 2** Analogous theorem holds for f and g orientation reversing. In such a case all inequalities in the above formulation are reversed.

4 $f + i\,g$ is a quasiregular mapping for radial p-harmonics

In previous sections we derived conditions for complex gradients f and g of the p-harmonic mapping to be quasiregular. To obtain full analogy with the scalar case we need to find a differential expression Φ which depends on all entries of Jacobi matrix Du such that if one of the coordinate functions of u vanishes identically we obtain the classical (scalar) result. Below we make a first step to obtain such Φ.

Recall that in Section 3 (23) we pointed out that for the radial mappings, complex conjugates of f and g result in more transparent conditions for quasiregularity than these for complex gradients.

The above suggests to focus on mapping $f + i\,g$, where by f and g we mean complex conjugate gradients of components of planar p-harmonic radial mapping u. That is,

$$h = \frac{1}{2}(u^1_x - u^2_y) + \frac{i}{2}(u^1_y + u^2_x). \tag{30}$$

Recall formulas (26, 27) above. Elementary calculations lead us to the following identities.

$$\begin{aligned} |f_z + i\,g_z|^2 &= \frac{1}{4}|z|^2\big(\ \ddot{H} + \dot{H}\ \frac{3}{|z|}\big)^2, \\ |f_{\overline{z}} + i\,g_{\overline{z}}|^2 &= \frac{1}{4}x^2\big(\ \ddot{H} + \dot{H}\ \frac{3}{|z|}\big)^2 + \frac{1}{4}y^2\big(\ \ddot{H} - \frac{\dot{H}}{|z|}\big)^2. \end{aligned} \tag{31}$$

The main result of this section reads as

Theorem 5 *Let $u(z) = H(|z|)z$ be a radial p-harmonic mapping such that*

$$\dot{H}\,(\ddot{H}\,|z| + \dot{H}) > 0. \tag{32}$$

Then ν, the Beltrami coefficient of h satisfies

$$|\nu| < 1,$$

and hence h is a quasiregular transformation.

Proof: If $\dot{H}\,(\ddot{H}\,|z| + \dot{H}) > 0$, then

$$(\ddot{H} + \dot{H}\,\frac{3}{|z|})^2 > (\,\ddot{H} - \frac{\dot{H}}{|z|})^2.$$

Therefore

$$\frac{1}{4}x^2(\ddot{H} + \dot{H}\,\frac{3}{|z|})^2 + \frac{1}{4}y^2(\ddot{H} + \dot{H}\,\frac{3}{|z|})^2 > \frac{1}{4}x^2(\ddot{H} + \dot{H}\,\frac{3}{|z|})^2 + \frac{1}{4}y^2(\,\ddot{H} - \frac{\dot{H}}{|z|})^2$$

and henceforth

$$|\nu|^2 = \frac{|f_{\overline{z}} + i\,g_{\overline{z}}|^2}{|f_z + i\,g_z|^2} = \frac{\frac{1}{4}x^2\big(\,\ddot{H} + \dot{H}\,\frac{3}{|z|}\big)^2 + \frac{1}{4}y^2\big(\,\ddot{H} - \frac{\dot{H}}{|z|}\big)^2}{\frac{1}{4}|z|^2\big(\,\ddot{H} + \dot{H}\,\frac{3}{|z|}\big)^2} < 1. \tag{33}$$

■

***Remark* 3** Similar condition can be derived for sense reversing mapping h. In such a case inequality (32) has to be reversed.

***Remark* 4** We would like to mention that the condition (32) can be expressed only in terms of $\dot{H}$. This is due to the fact that the p-harmonic system of equations in the class of radial mappings reduces to one ODE

$$\begin{aligned}(p-1)\,\ddot{H}\dot{H}^2\,|z|^3 + (2p-1)\,\dot{H}^3\,|z|^2 + 2(p-1)H\,\dot{H}\ddot{H}\,|z|^2 + (5p-4)H\,\dot{H}^2|z| \\ +H^2\,\ddot{H}\,r + 3pH^2\,\dot{H} = 0,\end{aligned} \tag{34}$$

from which $\ddot{H}$ can be computed explicitly. For the sake of convenience we will skip computations and state the resulting condition. Identity (34) reduces condition (32) to the following.

$$\begin{aligned}\frac{\dot{H}}{H}|z| \quad &\in \quad (r_-,\, r_+),\ \text{where} \\ r_{-,+} \quad &= \quad \frac{2-3p \mp \sqrt{p^2-12p+4}}{2p},\end{aligned}$$

provided that $p > 6 + 4\sqrt{2}$. For $p \in (1, 6 + 4\sqrt{2})$ this condition fails to be satisfied.

***Remark* 5** Let us compare the above condition with its counterpart for radial solution of scalar p-harmonic equation. Let $u(z) = H(|z|)$ be such solution. Then $f = \frac{1}{2}(u_x + \mathrm{i}u_y) = \dot{H}\ \frac{z}{2|z|}$. Denote by ϱ Beltrami coefficient of f. It is not difficult to find that inequality $|\varrho| = \frac{|\ddot{H} - \frac{\dot{H}}{|z|}|}{|\ddot{H} + \frac{\dot{H}}{|z|}|} < 1$, holds provided that $\ddot{H}\dot{H} > 0$. Henceforth, fact that H is monotone increasing convex or monotone decreasing concave, implies that f is quasiregular. However, our condition for mappings is more involved.

***Remark* 6** Unfortunately the above proof cannot be repeated for f and g complex gradients of the general solution u. In this case we have to deal with the coefficients of matrix $A(f, g)$(see expression (5)). Resulting expressions are much more involved and therefore difficult to handle in as easy way as in Theorem 5. In addition, explicit equations obtained for radial mappings, become inequalities in the general case.

The above considerations justify the following problem.

PROBLEM 1 *Let u be a solution of the p-harmonic system of equations (2). Define expressions f, $\overline{f}$, g and $\overline{g}$ as follows.*

$$\begin{aligned} f &= \frac{1}{2}(u^1_x + i\,u^1_y), \quad g = \frac{1}{2}(u^2_x + i\,u^2_y), \\ \overline{f} &= \frac{1}{2}(u^1_x - i\,u^1_y), \quad \overline{g} = \frac{1}{2}(u^2_x - i\,u^2_y) \end{aligned}$$

Find conditions on u, p and the domain of u in which the **generalized complex gradients**:

$$\begin{aligned} h &= \frac{1}{2}(f + i\,g) = \frac{1}{2}(u^1_x - u^2_y) + \frac{i}{2}(u^1_y + u^2_x), \\ \tilde{h} &= \frac{1}{2}(\overline{f} + i\,\overline{g}) = \frac{1}{2}(u^1_x + u^2_y) - \frac{i}{2}(u^2_x - u^1_y) \end{aligned}$$

are **quasiregular** *mappings.*

5 p-Harmonic conjugate

The purpose of this short section is to point out one of the possible applications of the systems (4) and (5).

If u is harmonic on a planar domain, then function v is called harmonic conjugate if $u + i\,v$ is holomorphic (see e.g. [12]). Although question of finding harmonic conjugate is nowadays a mathematical folklore, it continuous to inspire research (see [22] for the n-dimensional setting; see [10],[16] for relations between conjugate harmonic and Clifford algebras).
The following is motivated by the harmonic conjugate problem.

PROBLEM 2 *For a given p-harmonic function $f : \Omega \to \mathbb{R}$ find a function $g : \Omega \to \mathbb{R}$, for which the mapping $h = (f, g)$ is p-harmonic.*

***Remark* 7** One can state the similar problem without requiring f to be p-harmonic.

***Remark* 8** The resulting function g need not be p-harmonic (and seldom it does, as p-harmonic system is strongly coupled by appearance of $|Dh|$ in both equations).

It is a subject of our future investigations to use systems (4) and (5) to solve Problem 2.

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