

## Mappings of finite distortion and PDE with nonstandard growth

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Quasiregular mappings with distortion  $K$  and solutions of the  $p$ -Laplace equation have both been recently extended to the case where the parameter  $K$  or  $p$  is a function depending on the space variable. For the constant parameter case, results by Bojarski–Iwaniec and Manfredi show that the gradient of a  $p$ -harmonic function in the plane is quasiregular or constant. We generalize the result, showing that a planar  $p(\cdot)$ -harmonic-type function, modeled on the strong equation, is a mapping of finite distortion under appropriate assumptions.

### 1 Introduction

If  $u \in C^2(\Omega)$ ,  $\Omega \subset \mathbb{C}$ , satisfies the Laplace equation  $\Delta u = 0$ , then the complex gradient  $\partial_z u = u_x - i u_y$  is an analytic function. This is a standard result in undergraduate Complex Analysis courses. The corresponding result for the  $p$ -Laplace equation was derived in the late 1980s: Bojarski and Iwaniec [9] ( $p > 2$ ) and Manfredi [47] ( $1 < p < \infty$ ) showed that that if  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$  and  $u \in W_{loc}^{1,p}$  is non-constant, then  $\partial_z u = \frac{1}{2}(u_x - i u_y)$  is  $K_p$ -quasiregular, with

$$K_p = \frac{1}{2} \left( p - 1 + \frac{1}{p-1} \right).$$

In other words, the gradient of every planar  $p$ -harmonic function is  $K_p$ -quasiregular. Recall that a mapping  $F \in W_{loc}^{1,1}(\Omega; \mathbb{R}^n)$  is  $K$ -quasiregular if

$$\|DF(x)\|^n \leq K J_F(x) \quad \text{a.e.,}$$

where  $DF$  is the derivative and the Jacobian  $J_F \in L_{loc}^1(\Omega)$ .

In recent years both  $p$ -harmonic functions and  $K$ -quasiregular mappings have been extended to include the case where the parameter  $p$  or  $K$  depends on the space variable, leading to  $p(\cdot)$ -harmonic functions and mappings of finite distortion  $K(x)$  (also called below  $K(\cdot)$ -quasiregular); see e.g. [3, 17, 20, 27, 30, 35] and [8, 25, 36, 39, 53] for some recent advances and [2, 7, 10, 40, 55] for applications. In this paper we study whether the Bojarski–Iwaniec–Manfredi result can be extended to this setting (for more on relations between PDE and quasiregular mappings see e.g. [4, 5, 6, 50]). We refer to Section 2 for definitions, notation and references and plunge here straight into the results.

An obvious conjecture would be that the gradient of a  $p(\cdot)$ -harmonic function is  $K_p(\cdot)$ -finite distortion with

$$K_p(x) = \frac{1}{2} \left( p(x) - 1 + \frac{1}{p(x)-1} \right).$$

Unfortunately, the relationship in the variable exponent case is not quite as simple as this; in fact, for arbitrarily regular  $p$ , say  $p \in C^\infty(\bar{\Omega})$ , we show in Example 3.1 that the gradient of a  $p(\cdot)$ -harmonic function need not be of finite distortion at all. In the rest of Section 3 we obtain an understanding of this phenomenon through more sophisticated examples.

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It turns out that the central question is how we generalize the  $p$ -Laplace equation to the variable exponent, non-standard growth setting. For background on the topic, see the surveys [30, 51]. Thus far, two approaches have been used with the starting point being either the minimization problem

$$\inf_u \int_{\Omega} |\nabla u|^p dx, \quad u \in u_0 + W_0^{1,p}(\Omega),$$

where the minimum is taken over all Sobolev functions with the same trace  $u_0$ , or the weak form of the differential equation  $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ . In the variable exponent case these lead to non-equivalent, although closely related, problems, namely, the minimization of

$$\int_{\Omega} |\nabla u(x)|^{p(x)} dx \quad \text{with Euler-Lagrange equation} \quad \operatorname{div}(p(x)|\nabla u(x)|^{p(x)-2}\nabla u) = 0,$$

or the minimization of

$$\int_{\Omega} \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx \quad \text{with Euler-Lagrange equation} \quad \operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u) = 0.$$

We denote both equations by  $\Delta_{p(\cdot)} u(x) = 0$ . In what follows it will be clear from the context which version we use. Neither of these equations gives solutions with the desirable geometric properties that we are interested in, such as being of finite distortion.

The correct starting point for this problem seems to be the strong form of the  $p$ -Laplace equation  $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ , i.e.

$$\Delta_p u = |\nabla u|^{p-4} \left[ (p-2) \sum_{i,j} u_{x_i x_j} u_{x_i} u_{x_j} + |\nabla u|^2 \Delta u \right] = 0,$$

where  $u_{x_i}$  denotes the partial derivative. If  $p$  is replaced by  $p(x)$ , we arrive at yet another generalization of the  $p$ -Laplace equation. We denote this operator by  $\tilde{\Delta}_{p(\cdot)}$ . In order to state our problem in a weak form, we use the following expression of this operator:

$$\tilde{\Delta}_{p(\cdot)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - |\nabla u|^{p(x)-2} \log(|\nabla u|) \nabla u \cdot \nabla p.$$

defined for  $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ . The corresponding weak formulation then requires that

$$-\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi dx = \int_{\Omega} |\nabla u|^{p(x)-2} \log(|\nabla u|) \nabla u \cdot \nabla p \varphi dx \quad (\star)$$

for all  $\varphi \in W_0^{1,p(\cdot)}(\Omega)$ . Obviously, this equation reduces to the ordinary  $p$ -Laplace equation when  $p$  is constant. Another point worth mentioning is that we must require that  $\nabla p$  exists in some suitable sense, e.g. that  $p$  is Lipschitz continuous.

The following simple result suggests that there may be some interest in looking at this equation.

**Proposition 1.1.** *Let  $p$  be Lipschitz with  $1 < p^- \leq p^+ < \infty$ . Then*

$$\tilde{\Delta}_{p(\cdot)}(\lambda u) = \lambda^{p(\cdot)-1} \tilde{\Delta}_{p(\cdot)} u$$

*in the sense of distributions for  $u \in W^{1,p(\cdot)}(\Omega)$  and  $\lambda \in [0, \infty)$ . In particular, if  $u$  is a solution, then so is  $\lambda u$ .  $\square$*

Recall that the equation  $\Delta_{p(\cdot)} u = 0$  is never homogeneous for variable  $p$ , which leads to problems with various proof techniques. A reflection of this phenomenon is that the constant  $C$  in the Harnack inequality

$$\operatorname{ess\,sup}_{x \in B} |u(x)| \leq C \operatorname{ess\,inf}_{x \in B} |u(x)|$$

cannot be chosen independent of the positive solution  $u$  of the  $\Delta_{p(\cdot)}$ -equation [34]. The example used in [34] to show that  $C$  depends on  $u$  does not work in our case, but it remains for future investigations to study whether it is possible to derive a Harnack inequality with absolute constant for solutions of  $(\star)$ .

Since our starting point is the strong form of the  $p$ -Laplace equation, it is natural to use this as leverage in order to prove existence and regularity results for solutions (see also Remark 1.6 on limitations in the current theory of weak solutions in this context). This is the approach adopted in this article. Our main result in the case  $1 < p^- \leq p^+ < \infty$  is the following generalization of the Bojarski-Iwaniec-Manfredi result:

**Theorem 1.2.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded  $C^2$  domain and let  $g \in C^{1,\gamma}(\partial\Omega)$ . Suppose that  $p$  is Lipschitz continuous and that  $1 < p^- \leq p^+ < \infty$ . Then there exists a weak solution  $u \in C^{1,\gamma}(\Omega)$  of*

$$\begin{cases} \tilde{\Delta}_{p(\cdot)} u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

which satisfies the strong maximum principle

$$\sup_D |u| \leq \sup_{\partial D} |u|$$

for every  $D \subset \Omega$ . Moreover, the complex gradient  $\frac{1}{2}(u_x - iu_y)$  of  $u$  is  $K_p(\cdot)$ -quasiregular with

$$K_p(x) = \frac{1}{2} \left( p(x) - 1 + \frac{1}{p(x) - 1} \right).$$

□

This result gives us good control locally of the mapping properties of  $\partial_z u$ ; for instance, if  $p = 2$  in some open set, then  $\partial_z u$  is conformal in this set. However, the condition  $1 < p^- \leq p^+ < \infty$  implies that  $K_p(\cdot) \in L^\infty$ , so in fact our mapping is properly quasiregular. In order to arrive at mappings of finite distortion, we must allow  $K_p(\cdot)$  to be unbounded, which happens if either  $p^- = 1$  or  $p^+ = \infty$ . Interest in such situation has arisen recently for the equation  $-\Delta_{p(\cdot)} u = 0$ , see [31, 32, 33, 44, 48, 49]. In this paper we pursue the lower limit,  $p^- = 1$ .

**Theorem 1.3.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded  $C^2$  domain and let  $g \in C^{1,\gamma}(\partial\Omega)$ . Suppose that  $p \in C(\bar{\Omega})$  such that*

- the set  $Y := \{p = 1\}$  has vanishing Hausdorff 1-measure, and
- for every  $\Omega' \subset\subset \Omega \setminus Y$ ,  $p|_{\Omega'}$  is Lipschitz continuous.

Then there exists a weak solution  $u$  of  $(\star)$  satisfying the strong maximum principle. If  $K_p(\cdot) \in \exp L(\Omega)$ , with  $K_p(\cdot)$  as before, then the complex gradient  $\frac{1}{2}(u_x - iu_y)$  of  $u$  is  $K_p(\cdot)$ -quasiregular. □

*Remark 1.4.* In order to deal with mappings of finite distortion, it is natural to assume that  $K_p(\cdot) \in \exp L(\Omega)$  (or some similar condition, see [36, 37] and Section 2.1). If we now assume that  $p$  is Lipschitz, then we can easily conclude that  $p^- > 1$ . Therefore it is crucial that the assumption on  $p$  be relaxed from Theorem 1.2, even if it leads to a less transparent condition. For instance the exponent

$$p(x) = 1 + \frac{c}{\log(e + 1/|x|)}$$

satisfies the conditions of the theorem when  $c \in (0, 1)$ . □

*Remark 1.5.* If  $Y \subset\subset \Omega$  in the theorem, then the solution  $u$  has boundary values given by  $g$ . See the end of Section 6 for a justification. □

*Remark 1.6.* The right hand side of  $(\star)$  involves a function of the gradient of the solution. The recent survey [30] found that existence for such equations has hardly been studied at all, and regularity only to a quite limited extent. (Mingione (private communication) pointed out that perturbation techniques [13] are likely to work here.)

Such equations have been studied largely under the assumption that the differential expression on the right hand side is in divergence form, e.g.  $-\operatorname{div}(|F|^{q-2}F)$  (see [3, 51] and references therein). To place our results in the wider context, we note that there is a plethora of literature on perturbed  $p$ -harmonic equations (especially for the one dimensional case) and one of the most discussed cases is

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x, u),$$

under variety of growth conditions imposed on  $f$  and different boundary data (see [12, 18, 22, 24] and references therein). □

The structure of the rest of this paper is as follows. In the next section we present background on mappings of finite distortion and variable exponent spaces that are needed later on. In Section 3 we present the three examples which were alluded to in the introduction. These examples show that solutions of the usual variable exponent  $p(\cdot)$ -Laplace equation are very far from being mappings of finite distortion. In Section 4, we prove Theorem 1.2, which can be done in a rather straightforward way using tools from [29]. Section 5 contains a variety of lemmas, including a Caccioppoli estimate which is used in the proof of the second main theorem, and a proof of Proposition 1.1. Finally, in Section 6 we prove the second main result, Theorem 1.3, via a combination of a limiting argument, a refinement of the method from Section 4, and a result from [33]. Although our main results only apply to the planar case, some auxiliary results are given in the Euclidean space  $\mathbb{R}^n$  for the benefit of future investigations.

## 2 Preliminaries

By  $\Omega \subset \mathbb{R}^n$  we denote a bounded open set. By  $c$  we denote a generic constant, whose value may change between appearances even within a single line. By  $f_A$  we denote the integral average of  $f$  over  $A$ . We identify the complex plane  $\mathbb{C}$  with  $\mathbb{R}^2$ .

### 2.1 Mappings of finite distortion

A mapping  $F \in W_{loc}^{1,1}(\Omega; \mathbb{R}^n)$  on a domain  $\Omega \subset \mathbb{R}^n$  with Jacobian  $J_F \in L_{loc}^1(\Omega)$  is said to be of *finite distortion*  $K(x)$  if there exists measurable function  $K: \mathbb{R}^n \rightarrow \mathbb{R}$ , finite almost everywhere and

$$\|DF(x)\|^n \leq K(x) J_F(x) \quad \text{for a.e. } x \in \Omega.$$

If  $K \in L^\infty(\Omega)$ , then the mapping is called *quasiregular*. Such mappings in the planar case have been studied since the late 1920s (under various names), the higher dimensional version since the late 1950s. Mappings of finite distortion (MFD) have been studied for about 10 years. We refer to the monograph [39] by Iwaniec and Martin for more background on these mappings. Also the research webpage of the Mathematics Department of Jyväskylä University is the excellent source for the past and current developments in the area.

In what follows we will be largely concerned with compactness properties of mappings of finite distortion. For the sequence  $(F_k)$  of  $K$ -quasiregular mappings, its uniform limit on compact subsets is a  $K$ -quasiregular by [54, Chapter 9] or [45, Chapter 5]. The situation becomes more complicated if distortion function can be unbounded. In such a case the integrability properties of  $DF$  and  $J_F$  play a fundamental role (see e.g. [37] or [39, Chapter 8]). Let us briefly discuss the relevant results.

Let  $P: [0, \infty] \rightarrow \mathbb{R}_+$  be an Orlicz function such that:

1.  $\int_1^\infty P(t) t^{-n-1} dt = \infty$ ; and
2. the function  $t \mapsto P(t^{\frac{2n+1}{2n^2}})$  is convex.

Let  $(F_k)$  be a sequence of mappings of finite distortion, bounded in the Orlicz-Sobolev spaces  $W^{1,P}(\Omega, \mathbb{R}^n)$  with distortions bounded by  $M$ :

$$K_{F_k}(x) \leq M(x) < \infty \quad \text{a.e. } \Omega, \quad k = 1, 2, \dots$$

Then [39, Theorem 8.10.2] implies, in particular, that one can extract the subsequence converging locally uniformly to a mapping of finite distortion  $F \in W^{1,P}(\Omega, \mathbb{R}^n)$  with  $K_F \leq M$  a.e. For  $P(t) = t^n$  and  $M(x) = \text{const}$  we regain the bounded distortion case.

In order to carry various results known for mappings of bounded distortion to the setting of MFD, one often requires the distortions to be *exponentially integrable*, see e.g. [41, 50, 52]. This works also for compactness properties. For a given bounded family  $\mathcal{F}$  of MFD, assume that there exist  $\lambda > 0$  and  $M > 0$  such that

$$\int_{\Omega} e^{\lambda K_F(x)} dx \leq M, \quad \text{for all } F \in \mathcal{F}. \quad (2.1)$$

Then [39, Theorem 8.14.1] implies, among other things, that  $\mathcal{F}$  is equicontinuous on all compact subsets of  $\Omega$  and that  $\mathcal{F}$  is the closed normal family of MFD.

### 2.2 Variable exponent function spaces

For background on variable exponent function spaces we refer to the surveys [16, 56] or the (upcoming) monograph [15]. Most of the results in this subsection were proved in [42]. The variable exponent Lebesgue space is a special case of an Orlicz-Musielak space. For a constant function  $p$ , it coincides with the standard Lebesgue space. Often it is assumed that  $p$  is bounded, since this condition is known to imply many desirable features for  $L^{p(\cdot)}(\Omega)$ .

A measurable function  $p: \Omega \rightarrow [1, \infty]$  is called a *variable exponent*, and we denote

$$p_A^+ := \operatorname{ess\,sup}_{x \in A} p(x), \quad p_A^- := \operatorname{ess\,inf}_{x \in A} p(x), \quad p^+ := p_\Omega^+ \quad \text{and} \quad p^- := p_\Omega^-$$

for  $A \subset \Omega$ . We define a *(semi)modular* on the set of measurable functions by setting

$$\varrho_{L^{p(\cdot)}(\Omega)}(u) := \int_{\Omega} |u(x)|^{p(x)} dx;$$

here we use the convention  $t^\infty = \infty \chi_{(1, \infty]}(t)$  in order to get a left-continuous modular, see [15, Chapter 2] for details. The *variable exponent Lebesgue space*  $L^{p(\cdot)}(\Omega)$  consists of all measurable functions  $u: \Omega \rightarrow \mathbb{R}$  for which the modular  $\varrho_{L^{p(\cdot)}(\Omega)}(u/\mu)$  is finite for some  $\mu > 0$ . The Luxemburg norm on this space is defined as

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \mu > 0 : \varrho_{L^{p(\cdot)}(\Omega)}\left(\frac{u}{\mu}\right) \leq 1 \right\}.$$

Equipped with this norm,  $L^{p(\cdot)}(\Omega)$  is a Banach space.

If  $E$  is a measurable set of finite measure, and  $p$  and  $q$  are variable exponents satisfying  $q \leq p$ , then  $L^{p(\cdot)}(E)$  embeds continuously into  $L^{q(\cdot)}(E)$ . In particular, every function  $u \in L^{p(\cdot)}(\Omega)$  also belongs to  $L^{p(\cdot)}(\Omega)$ . The variable exponent Hölder inequality takes the form

$$\int_{\Omega} fg \, dx \leq 2 \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)},$$

where  $p'$  is the point-wise conjugate exponent,  $1/p(x) + 1/p'(x) = 1$ .

The function  $\alpha$  is said to be *log-Hölder continuous* if there is constant  $L > 0$  such that

$$|\alpha(x) - \alpha(y)| \leq \frac{L}{\log(e + 1/|x - y|)}$$

for all  $x, y \in \Omega$ . We denote  $p \in \mathcal{P}^{\log}(\Omega)$  if  $1/p$  is log-Hölder continuous. (Note that this definition is only appropriate in the bounded domains we consider.)

The *variable exponent Sobolev space*  $W^{1,p(\cdot)}(\Omega)$  consists of functions  $u \in L^{p(\cdot)}(\Omega)$  whose distributional gradient  $\nabla u$  belongs to  $L^{p(\cdot)}(\Omega)$ . The variable exponent Sobolev space  $W^{1,p(\cdot)}(\Omega)$  is a Banach space with the norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

In general, smooth functions are not dense in the variable exponent Sobolev space [57], but the log-Hölder condition suffices to guarantee that they are [15, Section 8.1]. In this case, we define *the Sobolev space with zero boundary values*,  $W_0^{1,p(\cdot)}(\Omega)$ , as the closure of  $C_0^\infty(\Omega)$ .

The Sobolev conjugate exponent is also defined point-wise,  $p^*(x) := \frac{np(x)}{n-p(x)}$  for  $p^+ < n$ . If  $p$  is log-Hölder continuous, the Sobolev–Poincaré inequality

$$\|u - u_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \leq c \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds when  $\Omega$  is a nice domain, for instance convex or John [15, Section 7.2].

### 3 Examples

Our first example shows that the direct generalization of the Bojarski–Iwaniec–Manfredi result to the usual variable exponent Laplacian  $\Delta_{p(\cdot)}$  fails.

**Example 3.1.** Let  $\Omega = [0, 1]^2$  and  $p(x, y) = p(x)$  depending on  $x$  only and bounded away from 1. In this situation we easily check that

$$u_c(x, y) = \int_0^x \left( \frac{c}{p(\xi)} \right)^{\frac{1}{p(\xi)-1}} d\xi$$

is a  $p(\cdot)$ -harmonic function for every  $c > 0$ . For this function we have

$$\partial_z u_c(x, y) = \left( \frac{c}{p(x)} \right)^{\frac{1}{p(x)-1}}.$$

Note that  $\text{Im } \partial_z u_c = 0$ . If  $p$  is constant, then the right hand side is a real constant, which is excluded from consideration by assumption. When  $p$  is variable, however,  $\partial_z u$  maps the square  $\Omega$  onto a line segment in the real axis, hence it has vanishing Jacobian, and is certainly not quasiregular or even of finite distortion.  $\square$

For constant  $p$ , one excludes the trivial case of affine solutions in the Bojarski–Iwaniec–Manfredi result by assuming that the complex gradient is non-constant. In the variable exponent case, the previous example showed that the trivial case is not quite as trivial, but one might nevertheless hope to exclude this and have the result apply to other cases. The next example shows that this approach will not be successful, as we obtain non-trivial quasiregular mappings with distortion not controlled by a function of  $p$ .

**Example 3.2.** Let  $\Omega$  be an annulus centered at the origin and let  $p$  be spherically symmetric. Suppose that  $u \in C^2(\Omega)$  and  $u(z) = \varphi(|z|)$  with  $\varphi' > 0$ . Then

$$\Delta_{p(\cdot)} u(z) = |\varphi'|^{p(z)-2} [(p(z) - 1)\varphi'' + \frac{n-1}{|z|}\varphi' + p'(z)\varphi' \log \varphi'].$$

If  $\Delta_{p(\cdot)} u = 0$ , then  $(p(z) - 1)\varphi'' + \frac{n-1}{|z|}\varphi' + p'(z)\varphi' \log \varphi' = 0$ . Hence

$$(r\varphi'' - \varphi')(p(z) - 1) = -(\varphi' + rp'(z)\varphi' \log \varphi') - (p(z) - 1)\varphi' = [-rp'(z) \log \varphi' - p(z)]\varphi'$$

and

$$(r\varphi'' + \varphi')(p(z) - 1) = [-rp'(z) \log \varphi' + p(z) - 2]\varphi'.$$

Let the dimension  $n = 2$  and set  $F := u_x - iu_y$  (here  $z = x + iy$ ). Then

$$2\partial_z F = u_{xx} - u_{yy} - 2iu_{xy} \quad \text{and} \quad 2\partial_{\bar{z}} F = u_{xx} + u_{yy} = \Delta u.$$

Denoting  $r = \sqrt{x^2 + y^2}$ , we find that  $2\partial_{\bar{z}} F = \varphi'' + \frac{1}{r}\varphi'$  and

$$2\partial_z F = \varphi'' \frac{x^2}{r^2} + \varphi' \frac{r^2 - x^2}{r^3} - \varphi'' \frac{y^2}{r^2} - \varphi' \frac{r^2 - y^2}{r^3} - 2i[\varphi'' \frac{xy}{r^2} - \varphi' \frac{xy}{r^3}] = [\varphi'' - \frac{1}{r}\varphi'] \frac{x^2 - y^2 - 2ixy}{r^2}.$$

Note that the last factor on the right hand side is unimodular. Thus we obtain

$$\frac{|\partial_z F| + |\partial_{\bar{z}} F|}{|\partial_z F| - |\partial_{\bar{z}} F|} = \frac{|r\varphi'' - \varphi'| + |r\varphi'' + \varphi'|}{|r\varphi'' - \varphi'| - |r\varphi'' + \varphi'|} = \frac{|rp'(z) \log \varphi' + p(z)| + |p(z) - 2 - rp'(z) \log \varphi'|}{|rp'(z) \log \varphi' + p(z)| - |p(z) - 2 - rp'(z) \log \varphi'|}.$$

From this expression we can see that the value of  $K_p(\cdot)$  depends not only on  $p(z)$ , but also on the derivative of  $p$ .  $\square$

The previous example showed that  $K_p(\cdot)$  for solutions of  $\Delta_{p(\cdot)} u = 0$  depends point-wise on  $\nabla p$  in addition to  $p$ . The next example shows that this is not the case for  $\tilde{\Delta}_{p(\cdot)} u = 0$ .

**Example 3.3.** Let  $u$  be as in the previous example. Then

$$\tilde{\Delta}_{p(\cdot)} u(z) = |\varphi'|^{p(z)-2} [(p(z) - 2)\varphi'' + \varphi'' + \frac{n-1}{|z|}\varphi'].$$

Note that we do not need to assume that  $p$  is radial to obtain this formula. Thus  $\tilde{\Delta}_{p(\cdot)} u = 0$  is equivalent to  $(p(z) - 1)\varphi'' + \frac{1}{|z|}\varphi' = 0$  for every  $z \in \Omega$ .

Let  $F$  be as before. If  $\tilde{\Delta}_{p(\cdot)} u = 0$ , then  $\varphi' = -(p(z) - 1)r\varphi''$ , and

$$\frac{|\partial_z F| + |\partial_{\bar{z}} F|}{|\partial_z F| - |\partial_{\bar{z}} F|} = \frac{p(z) + |2 - p(z)|}{p(z) - |2 - p(z)|} = \max \left\{ p(z) - 1, \frac{1}{p(z) - 1} \right\}.$$

Thus we see that  $F$  is MFD. Moreover, the bound for  $K_p(\cdot)$  from the main theorems is seen to be of the right order as  $p \rightarrow 1$ .  $\square$

## 4 Proof of Theorem 1.2

Our strategy for existence and generalizing the Bojarski–Iwaniec–Manfredi result is based on approximation by strong solutions. Of course, as in the constant exponent case, the solution of the  $p(\cdot)$ -Laplace equation will not in general be strong. Therefore, we look at the regularized versions of the problem:

$$\begin{cases} \operatorname{div} A_\varepsilon(x, \nabla u) = B_\varepsilon(x, \nabla u) & \text{in } \Omega \subset \mathbb{R}^n \\ u = g & \text{on } \partial\Omega \end{cases} \quad (4.1)$$

where the functions  $A: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $B: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  are given by:

$$\begin{aligned} A_\varepsilon(x, \xi) &= (\varepsilon + |\xi|^2)^{\frac{p(x)-2}{2}} \xi \\ B_\varepsilon(x, \xi) &= (\varepsilon + |\xi|^2)^{\frac{p(x)-2}{2}} \log \sqrt{\varepsilon + |\xi|^2} \xi \cdot \nabla p. \end{aligned}$$

The weak formulation, a non-degenerate version of  $(\star)$ , is

$$-\int_{\Omega} (\varepsilon + |\nabla u|^2)^{\frac{p(x)-2}{2}} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} (\varepsilon + |\nabla u|^2)^{\frac{p(x)-2}{2}} \log \sqrt{\varepsilon + |\nabla u|^2} \nabla p \cdot \nabla u \varphi \, dx \quad (4.2)$$

for  $\varphi \in C_0^\infty(\Omega)$ . If  $u \in C^2$ , we can express this in a strong form as used in Chapter 10 of [29]:

$$Qu := \sum_{i,j} a_{i,j}(x, \nabla u) u_{x_i, x_j} + b(x, \nabla u) = 0, \quad (4.3)$$

where

$$a_{i,j}(x, \xi) = (\varepsilon + |\xi|^2)^{\frac{p(x)-4}{2}} [(p(x) - 2)\xi_i \xi_j + \delta_{ij}(\varepsilon + |\xi|^2)]$$

and (in our case)  $b = 0$ . The ellipticity bounds for the operator  $Q$  are

$$\lambda(x, \xi) := \min\{p(x) - 1, 1\} (\varepsilon + |\xi|^2)^{\frac{p(x)-2}{2}} \quad \text{and} \quad \Lambda(x, \xi) := \max\{p(x) - 1, 1\} (\varepsilon + |\xi|^2)^{\frac{p(x)-2}{2}}.$$

To deal with problem (4.3) we need some variable exponent regularity results. Mingione and collaborators (cf. [11, 51]) have developed the regularity theory greatly, but their results are stated mostly for the variational case, although apparently extending by standard means to equations as well. We use a result by Xianling Fan, who, following [1], has extended some of these results to the non-variational cases in [20].

**Theorem 4.4** (Theorem 1.2 in [20]). *Let  $\partial\Omega$  be of class  $C^{1,\gamma}$ ,  $g \in C^{1,\gamma}(\partial\Omega)$  and suppose that  $p$  is Lipschitz continuous on  $\Omega$ . If  $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  is a solution of (4.1), then  $u \in C^{1,\alpha}(\bar{\Omega})$ , where  $\alpha$  and  $|u|_{C^{1,\alpha}(\bar{\Omega})}$  depend only on  $p^-$ ,  $p^+$ ,  $\|\nabla p\|_\infty$ ,  $n$ ,  $\|u\|_\infty$ ,  $\gamma$ ,  $|g|_{C^{1,\alpha}(\partial\Omega)}$  and  $\Omega$ .  $\square$*

In order to put this result to use, we need to establish the boundedness of our solutions, for which we use a maximum principle:

**Theorem 4.5** (Theorem 10.3 in [29]). *Let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  be a solution of (4.3). Suppose that*

$$|b(x, \xi)| |\xi|^2 \leq (\mu_1 |\xi| + \mu_2) \sum_{i,j} a_{i,j}(x, \xi) \xi_i \xi_j$$

for all  $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$  and  $x \in \Omega$ . Then  $u$  satisfies the strong maximum-type principle

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C(\mu_1, \text{diam } \Omega) \mu_2.$$

$\square$

With these tools the proof of the first main theorem is quite straightforward.

**Proof of Theorem 1.2.** We first prove the existence part. For  $\varepsilon > 0$  we consider the auxiliary Dirichlet boundary problem given by (4.1). This problem is uniformly elliptic for every  $\varepsilon > 0$  (since  $p^- > 1$ ). Because the boundary data  $g$  and the boundary  $\partial\Omega$  are regular, the equation has a strong solution by [29, Theorem 12.5]. We denote this solution by  $u^\varepsilon$ . Since  $b = 0$  in (4.3) in our case, the condition in Theorem 4.5 is satisfied with  $\mu_2 = 0$ ; hence  $u^\varepsilon$  satisfies the strong maximum principle, and

$$\sup_{\Omega} |u^\varepsilon| \leq \sup_{\partial\Omega} |u^\varepsilon| = \sup_{\partial\Omega} |g|.$$

Since  $u^\varepsilon$  is a bounded weak solution, Theorem 4.4 implies that  $u^\varepsilon \in C^{1,\gamma}(\Omega)$  with  $|u^\varepsilon|_{C^{1,\gamma}(\Omega)}$  independent of  $\varepsilon$ . In particular, there exists  $M > 0$  such that  $|\nabla u^\varepsilon| < M$  for all  $\varepsilon$ . We easily see that

$$d_\varepsilon := \sup_{x \in \Omega} \sup_{|\xi| < M} |A_\varepsilon(x, \xi) - A(x, \xi)| + |B_\varepsilon(x, \xi) - B(x, \xi)| \rightarrow 0$$

as  $\varepsilon \rightarrow 0$  since  $1 < p^- \leq p^+ < \infty$ . Fix  $\varphi \in W_0^{1,p(\cdot)}(\Omega)$ . Then

$$\begin{aligned} & \left| \int_{\Omega} -A(x, \nabla u^\varepsilon) \cdot \nabla \varphi - B(x, \nabla u^\varepsilon) \varphi \, dx \right| \\ & \leq \left| \int_{\Omega} -A_\varepsilon(x, \nabla u^\varepsilon) \cdot \nabla \varphi - B_\varepsilon(x, \nabla u^\varepsilon) \varphi \, dx \right| + d_\varepsilon \|\varphi\|_{W^{1,1}(\Omega)}. \end{aligned}$$

Since  $u^\varepsilon$  is a solution of (4.1), the first term on the right hand side equals zero. Taking into account that  $d_\varepsilon \rightarrow 0$ , we obtain that

$$\int_{\Omega} -A(x, \nabla u^\varepsilon) \cdot \nabla \varphi \, dx - \int_{\Omega} B(x, \nabla u^\varepsilon) \varphi \, dx \rightarrow 0.$$

Since  $|u^\varepsilon|_{C^{1,\gamma}(\Omega)}$  is bounded, we can choose a subsequence and  $u \in W^{1,p(\cdot)}(\Omega)$  such that  $u^\varepsilon \rightarrow u$  a.e. and  $\nabla u^\varepsilon \rightarrow \nabla u$  a.e. in  $\Omega$ . Then the uniform estimate of  $|u^\varepsilon|_{C^{1,\gamma}}$  implies that  $u^\varepsilon \rightarrow u$  everywhere in  $\bar{\Omega}$ , so that  $u|_{\partial\Omega} = u^\varepsilon|_{\partial\Omega} = g$ . Furthermore, since  $|\nabla u^\varepsilon|$  is uniformly bounded, it follows by dominated convergence that

$$\int_{\Omega} -A(x, \nabla u) \cdot \nabla \varphi - B(x, \nabla u) \varphi \, dx = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} -A(x, \nabla u^\varepsilon) \cdot \nabla \varphi - B(x, \nabla u^\varepsilon) \varphi \, dx = 0,$$

so  $u$  is a solution of  $(\star)$  with boundary values  $g$ . This completes the proof of the existence part. Since each  $u^\varepsilon$  satisfies the strong maximum principle, so does the point-wise limit.

We now prove that  $u$  is a mapping of finite distortion. We follow Manfredi's proof in [47, Theorem 1]. For the sake of completeness we repeat some steps. Let  $\Omega' \subset\subset \Omega$  have smooth boundary. Let  $u^\varepsilon$  be a solution of the auxiliary problem as before and recall that

$$u^\varepsilon \rightarrow u \quad \text{in } W^{1,p(\cdot)}(\Omega') \quad \text{as } \varepsilon \rightarrow 0.$$

Since  $u^\varepsilon \in C^2(\Omega')$ , we can use the strong form of (4.1), which in two dimensions reads

$$a u_{xx}^\varepsilon + 2b u_{xy}^\varepsilon + c u_{yy}^\varepsilon = 0,$$

where

$$\begin{aligned} a &= (\varepsilon + |\nabla u^\varepsilon|^2) + (p(x) - 2)(u_x^\varepsilon)^2, \\ b &= (p(x) - 2)u_x^\varepsilon u_y^\varepsilon, \\ c &= (\varepsilon + |\nabla u^\varepsilon|^2) + (p(x) - 2)(u_y^\varepsilon)^2. \end{aligned}$$

We define the complex gradient  $F^\varepsilon := (u_x^\varepsilon, -u_y^\varepsilon)$ . Note that  $a$ ,  $b$  and  $c$  are exactly the same as in [47] when  $p$  is replaced by  $p(x)$ . Therefore the same point-wise computations around formula (5) in Theorem 1 of [47] imply that

$$\frac{\|DF^\varepsilon\|^2}{J_{F^\varepsilon}} \leq \frac{1}{2} \left( p(x) - 1 + \frac{1}{p(x) - 1} \right) = K_p(x).$$

and hence the distortion inequality holds for  $F^\varepsilon$ . That  $F^\varepsilon$  belong to  $W_{\text{loc}}^{1,1}(\Omega')$  and  $J_{F^\varepsilon} \in L_{\text{loc}}^1(\Omega')$  are immediate consequences of the  $C^2$  regularity of  $u^\varepsilon$  and the definition of  $F^\varepsilon$ . Therefore, the  $F^\varepsilon$  are mappings of finite distortion for all  $\varepsilon$ .

From the construction of  $u$ , we know  $F^{\varepsilon_i} \rightarrow F$  a.e. The uniform Hölder estimate for  $|u^\varepsilon|_{C^{1,\gamma}(\Omega)}$  implies the equicontinuity of the family  $(F^{\varepsilon_i})$ . From the Arzela-Ascoli Theorem we infer existence of a subsequence  $F^{\varepsilon_i}$  which converges to  $F$  uniformly on compact subsets of  $\Omega'$ , for  $\varepsilon_i \rightarrow 0$ .

Since  $1 < p^- \leq p^+ < \infty$ , we see that  $K_p(\cdot) \in L^\infty$ . In particular, the distortion  $K_p(\cdot)$  for  $F^\varepsilon$  is exponentially integrable. Theorem 8.14.1 in [39] implies that  $F$  as the limit of the uniformly convergent sequence  $F^{\varepsilon_i}$  is the mapping of finite distortion (see condition (2.1) and the discussion in Section 2.1). This completes the proof of theorem.  $\blacksquare$

## 5 Auxilliary results

In this section we collect some miscellaneous results. We start with the homogeneity result as an example of how the operator  $\tilde{\Delta}_{p(\cdot)}$  is better behaved than  $\Delta_{p(\cdot)}$ .

**Proof of Proposition 1.1.** Fix  $\varphi \in W_0^{1,\infty}(\Omega)$  and define

$$J_\varphi(u) := \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi + |\nabla u|^{p(x)-2} \log |\nabla u| \nabla u \cdot \nabla p \, dx.$$



Since  $p$  is Lipschitz we see that  $J_\varphi : W^{1,p(\cdot)}(\Omega) \rightarrow \mathbb{R}$  is continuous; for the same reason, smooth functions are dense [15, Section 8.1]. Thus we can choose functions  $u_i \in W^{1,p(\cdot)}(\Omega) \cap C^\infty(\Omega)$  such that  $u_i \rightarrow u$  in  $W^{1,p(\cdot)}(\Omega)$ . Since  $u_i$  is smooth, integration by parts yields

$$\begin{aligned} J_\varphi(u_i) &= \int_{\Omega} \left[ -\operatorname{div}(|\nabla u_i|^{p(x)-2} \nabla u_i) + |\nabla u_i|^{p(x)-2} \log |\nabla u_i| \nabla u_i \cdot \nabla p \right] \varphi \, dx \\ &= \int_{\Omega} |\nabla u_i|^{p(x)-3} \left[ |\nabla u_i| \Delta u + (p(x) - 2) \nabla(|\nabla u_i|) \cdot \nabla u_i \right] \varphi \, dx. \end{aligned}$$

Therefore  $J_\varphi(\lambda u_i) = J_{\lambda^{p(\cdot)-1}\varphi}(u_i)$  for any  $\lambda > 0$  and so it follows by the continuity of  $J_{\lambda^{p(\cdot)-1}\varphi}$  that

$$J_\varphi(\lambda u) = \lim_{i \rightarrow \infty} J_\varphi(\lambda u_i) = \lim_{i \rightarrow \infty} J_{\lambda^{p(\cdot)-1}\varphi}(u_i) = J_{\lambda^{p(\cdot)-1}\varphi}(u),$$

which is the weak formulation of the claim of the proposition. Since  $\varphi$  can be any test function, we conclude that  $\lambda u$  is also a weak solution if  $u$  is.  $\blacksquare$

Next, we derive a Caccioppoli type inequality for  $(\star)$ . In the theory of elliptic PDE, establishing the Caccioppoli inequality is one of the first steps in analyzing the equation. It is then used for example to investigate higher integrability properties of solutions as well as to show Harnack type estimates.

**Theorem 5.1** (Caccioppoli inequality). *Let  $u$  be a solution of equation  $(\star)$  and let the exponent  $p$  be Lipschitz with  $1 < p^- \leq p^+ < \infty$ . For  $\delta \in (0, 1)$  there exists a constant  $C = C(p^-, p^+, \|\nabla p\|_\infty, \delta)$  such that*

$$\int_{\Omega} |\nabla u|^{p(x)} \eta^{p^+} \, dx \leq C \int_{\Omega} |u|^{p(x)} |\nabla \eta|^{p(x)} + |u|^{\frac{p(x)}{1-\delta}} \eta^{p^+} \, dx$$

for all test functions  $\eta \in C_0^\infty(\Omega)$ .  $\square$

**Proof.** Let  $\eta \in C_0^\infty(\Omega)$  be a test function with  $0 \leq \eta \leq 1$  and define  $\varphi \in W_0^{1,p(\cdot)}(\Omega)$  by

$$\varphi := u \eta^{p^+} \quad \text{so that} \quad \nabla \varphi = \eta^{p^+} \nabla u + p^+ \eta^{p^+-1} u \nabla \eta.$$

For such  $\varphi$ , equation  $(\star)$  yields

$$\begin{aligned} \int_{\Omega} |\nabla u|^{p(x)-2} \eta^{p^+} |\nabla u|^2 \, dx &= - \int_{\Omega} |\nabla u|^{p(x)-2} \log |\nabla u| \nabla u \cdot \nabla p u \eta^{p^+} \, dx \\ &\quad - p^+ \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \eta u \eta^{p^+-1} \, dx. \end{aligned}$$

From this we obtain

$$\int_{\Omega} |\nabla u|^{p(x)} \eta^{p^+} \, dx \leq c \int_{\Omega} |\nabla u|^{p(x)-1} |u| \left[ |\log |\nabla u|| \eta^{p^+} + |\nabla \eta| \eta^{p^+-1} \right] \, dx. \quad (5.2)$$

As usual, we want to apply Young inequalities, and absorb all terms with  $|\nabla u|$  into the left hand side. For the second term on the right hand side we have

$$|\nabla u|^{p(x)-1} |\nabla \eta| |u| \eta^{(p^+ - \frac{p^+}{p'(x)} - 1) + \frac{p^+}{p'(x)}} \leq \sigma |\nabla u|^{p(x)} \eta^{p^+} + c_\sigma \eta^{p^+ - p(x)} |u|^{p(x)} |\nabla \eta|^{p(x)},$$

where  $\sigma \in (0, 1)$ . For the other term, we use that  $|\nabla u|^{p(x)-1} |\log |\nabla u|| \leq c |\nabla u|^{p(x)-1+\delta}$  for  $\delta > 0$  since  $p^- > 1$ . Thus Young's inequality gives

$$|\nabla u|^{p(x)-1} |\log |\nabla u|| |u| \leq \sigma |\nabla u|^{p(x)} + c_\sigma |u|^{\frac{p(x)}{1-\delta}}.$$

Next we apply these two Young-type inequalities in (5.2):

$$\int_{\Omega} |\nabla u|^{p(x)} \eta^{p^+} \, dx \leq c \int_{\Omega} \sigma |\nabla u|^{p(x)} \eta^{p^+} + c_\sigma (|u|^{p(x)} |\nabla \eta|^{p(x)} + |u|^{\frac{p(x)}{1-\delta}} \eta^{p^+}) \, dx$$

With the choice  $\sigma := c/2$  we can absorb the first term on the right hand side into the left hand side, which gives the claim.  $\blacksquare$

Using the Caccioppoli inequality from the previous theorem we can show a compactness result similar to the one used in Corollary 5.5 of [31]. For the proof we recall the following well known result which is valid when  $p^+ < \infty$  [15, Chapter 2]:

$$\int_{\Omega} |u_j|^{p(x)} dx \rightarrow \infty \quad \text{if and only if} \quad \|u_j\|_{L^{p(\cdot)}(\Omega)} \rightarrow \infty. \quad (5.3)$$

**Proposition 5.4.** *Let the exponent  $p$  be Lipschitz with  $1 < p^- \leq p^+ < n$ . Let  $(u_j)$  be a sequence of solutions of  $(\star)$ , such that  $u_j \rightarrow u$  in  $L_{\text{loc}}^{p(\cdot)}(\Omega)$  for some  $u \in W_{\text{loc}}^{1,p(\cdot)}(\Omega)$ . Then there exists constant  $C$  such that*

$$\limsup_{j \rightarrow \infty} \|\nabla u_j\|_{L^{p(\cdot)}(B)} \leq C \|\nabla u\|_{L^{p(\cdot)}(2B)} \quad (5.5)$$

for all balls  $2B \subset \subset \Omega$ . □

**Proof.** We first note that  $u - c$  is a solution if and only  $u$  is. Therefore, we may assume that  $u_{2B} = 0$  by subtracting a constant from all functions. By homogeneity,  $u'_j := \frac{u_j}{\lambda}$  is also a solution for every  $\lambda > 0$ . Let us choose  $\lambda := \|\nabla u\|_{L^{p(\cdot)}(2B)}$ . (If  $\lambda = 0$ , then we apply the proof to  $\lambda' > 0$  and let  $\lambda' \rightarrow 0$  in the end.)

Choose  $\eta \in C_0^\infty(2B)$  with  $\eta|_B = 1$  and  $|\nabla \eta| \leq 2/\text{diam } B$ . Then Theorem 5.1 yields

$$\int_B |\nabla u'_j|^{p(x)} dx \leq c \int_{\Omega} \left( \frac{|u'_j|}{\text{diam}(B)} \right)^{p(x)} + |u'_j|^{p^*(x)} dx.$$

By assumption,  $u_j \rightarrow u$  in  $L_{\text{loc}}^{p(\cdot)}(\Omega)$ . Hence, by dominated convergence,

$$\limsup_{j \rightarrow \infty} \int_B |\nabla u'_j|^{p(x)} dx \leq c \int_{\Omega} \left( \frac{|u'|}{\text{diam}(B)} \right)^{p(x)} + |u'|^{p^*(x)} dx. \quad (5.6)$$

Since  $(u')_{2B} = 0$ , it follows from the Poincaré inequality [15, Section 7.2] that

$$\left\| \frac{u'}{\text{diam}(B)} \right\|_{L^{p(\cdot)}(2B)} \leq c \text{diam}(B) \left\| \nabla \frac{u'}{\text{diam}(B)} \right\|_{L^{p(\cdot)}(2B)} = c \|\nabla u'\|_{L^{p(\cdot)}(2B)};$$

and from the Sobolev–Poincaré inequality [15, Section 7.2] that

$$\|u'\|_{L^{p^*(\cdot)}(2B)} \leq c \|\nabla u'\|_{L^{p(\cdot)}(2B)}.$$

The choice of  $\lambda$  implies that  $\|\nabla u'\|_{L^{p(\cdot)}(2B)} = 1$ . Therefore, it follows from these inequalities and (5.3) that right hand side of (5.6) is bounded by a constant. Using (5.3) again, we see that

$$c \geq \limsup_{j \rightarrow \infty} \|\nabla u'_j\|_{L^{p(\cdot)}(\Omega)} = \frac{1}{\lambda} \limsup_{j \rightarrow \infty} \|\nabla u_j\|_{L^{p(\cdot)}(\Omega)}.$$

We obtain the claim by multiplying this by  $\lambda = \|\nabla u\|_{L^{p(\cdot)}(2B)}$ . ■

## 6 Proof of Theorem 1.3

Recall that  $Y = \{x \in \Omega : p(x) = 1\}$  and define  $p_\lambda = \max\{p, \lambda\}$  for  $\lambda > 1$ . Let  $p$  be locally Lipschitz in  $\bar{\Omega} \setminus Y$  and let the domain and boundary data be as in Theorem 1.3. Then  $p_\lambda$  is Lipschitz for every  $\lambda > 1$ , so there exists a solution  $u_\lambda$  of  $(\star)$  with exponent  $p_\lambda$  by Theorem 1.2. The following proposition shows us how we can construct a solution for the case  $p^- = 1$  using these auxiliary solutions of the  $p_\lambda$ -equation. This approach is similar to [31, Proposition 6.1] for the equation  $\Delta_{p(\cdot)} u = 0$ .

**Proposition 6.1.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded  $C^2$  domain and let  $g \in C^{1,\gamma}(\partial\Omega)$ . Let  $p$  be a bounded exponent such that  $p_{\bar{\Omega}'} > 1$  and  $p|_{\Omega'}$  is Lipschitz for every  $\Omega' \subset \subset \bar{\Omega} \setminus Y$ , and the set  $Y$  has Lebesgue measure zero.*

*Then there exists a sequence  $(\lambda_j)$  decreasing to 1,  $p_{\lambda_j}(\cdot)$ -solutions  $(u_{\lambda_j})$  of  $(\star)$  with boundary values  $g$ , and  $u \in L^\infty(\Omega)$  satisfying the strong maximum principle such that*

1.  $u_{\lambda_j} \rightarrow u$  in  $L^{p(\cdot)}(\Omega)$ ;
2.  $u_{\lambda_j} \rightarrow u$  in  $W_{\text{loc}}^{1,p(\cdot)}(\Omega \setminus Y)$ ; and
3.  $u$  is a  $p(\cdot)$ -solution in  $\Omega \setminus Y$ .

□

**Proof.** The existence of the sequence  $(u_{\lambda_j})$  of solutions follows by Theorem 1.2. Let us denote  $u_j := u_{\lambda_j}$  and  $p_j := p_{\lambda_j}$ . By the strong maximal principle,  $\sup_{\Omega} |u_j| \leq \sup_{\partial\Omega} |g| =: M < \infty$ .

Fix  $\varepsilon > 0$  and choose open sets  $\Omega'' \subset\subset \Omega' \subset\subset \Omega \setminus Y$  such that  $|\Omega \setminus \Omega''| < \varepsilon/M$ . Fix also a function  $\eta \in C_0^\infty(\Omega')$  with  $\chi_{\Omega''} \leq \eta \leq 1$ . Since  $p_{\Omega'}^- > 1$ , it follows by the Caccioppoli inequality, Theorem 5.1, that  $\|\nabla u_j\|_{L^{p(\cdot)}(\Omega'')} \leq \|\nabla u_j\|_{L^{p_j(\cdot)}(\Omega'')} \leq C$ . The constant does not depend on  $j$ . By the point-wise inequality  $t^{p(x)} \leq t^{p_j(x)} + 1$  we obtain

$$\int_{\Omega''} |\nabla u_j(x)|^{p(x)} dx \leq \int_{\Omega''} |\nabla u_j(x)|^{p_j(x)} dx + |\Omega''|;$$

hence we conclude that the sequence  $(u_j)$  is bounded in  $W^{1,p(\cdot)}(\Omega'')$ . The compact embedding

$$W^{1,p(\cdot)}(\Omega'') \hookrightarrow L^{p(\cdot)}(\Omega'')$$

holds since  $p \in \mathcal{P}^{\log}(\Omega'')$  [15, Section 7.4]. Combined with the boundedness of  $(u_j)$  in  $W_{\text{loc}}^{1,p(\cdot)}(\Omega'')$ , this implies that we may choose a subsequence (relabelled  $(u_j)$ ) which converges in  $L^{p(\cdot)}(\Omega'')$ . Then

$$\int_{\Omega} |u_j - u_k|^{p(x)} dx \leq 2M |\Omega \setminus \Omega''| + \int_{\Omega''} |u_j - u_k|^{p(x)} dx < 4\varepsilon$$

provided  $j$  and  $k$  are large enough. Moving to a diagonal subsequence, we obtain a Cauchy sequence in  $L^{p(\cdot)}(\Omega)$ , hence we may define  $u$  such that Claim (1) holds. Since  $|u_j| \leq M$  for every  $j$ , the same holds for  $u$ .

Since  $1 < p_{\Omega'}^- \leq p_{\Omega'}^+ < \infty$ , the space  $W^{1,p(\cdot)}(\Omega')$  is reflexive [15, Section 7.1]. Since  $(u_j)$  is bounded in  $W^{1,p(\cdot)}(\Omega'')$  we find that there exists a weakly converging subsequence. Thus (2) is proved.

It follows that  $u \in W_{\text{loc}}^{1,p(\cdot)}(\Omega \setminus Y)$ , and hence to prove (3) we need to check that  $(\star)$  is satisfied for every test function  $\varphi \in W^{1,p(\cdot)}(\Omega \setminus Y)$  with compact support in  $\Omega \setminus Y$ . Assume  $\lambda_j \in (1, p_{\Omega'}^-)$  so that  $p_j = p$  in  $\Omega'$ . Let  $(u_j)$  be a subsequence that converges weakly to  $u \in W^{1,p(\cdot)}(\Omega')$ . We use  $\eta(u - u_j)$  as a test function for the solution  $u_j$ . Recalling that  $p_j = p$  in  $\Omega'$ , we obtain

$$\begin{aligned} & \int_{\Omega'} |\nabla u_j|^{p(x)-2} (u - u_j) \nabla u_j \cdot \nabla \eta dx + \int_{\Omega'} |\nabla u_j|^{p(x)-2} \eta \nabla u_j \cdot (\nabla u - \nabla u_j) dx \\ &= - \int_{\Omega'} |\nabla u_j|^{p(x)-2} \log |\nabla u_j| \nabla u_j \cdot \nabla p \eta (u - u_j) dx. \end{aligned}$$

From this we conclude, using Hölder's inequality, that

$$\begin{aligned} & \int_{\Omega'} |\nabla u_j|^{p(x)-2} \eta \nabla u_j \cdot (\nabla u - \nabla u_j) dx \\ & \leq \|\nabla \eta\|_{\infty} \int_{\Omega'} |\nabla u_j|^{p(x)-1} |u - u_j| dx + \|\nabla p\|_{\infty} \int_{\Omega'} |\nabla u_j|^{p(x)-1} |\log |\nabla u_j|| |u - u_j| dx \\ & \leq 2 \left\| |\nabla u_j|^{p(x)-1} \right\|_{L^{p'(\cdot)}(\Omega')} \|u - u_j\|_{L^{p(\cdot)}(\Omega')} + C \left\| |\nabla u_j|^{p(x)-1} \log |\nabla u_j| \right\|_{L^{\alpha(\cdot)}(\Omega')} \|u - u_j\|_{L^{2p(\cdot)}(\Omega')}, \end{aligned}$$

where  $\alpha(x) := p(x)/(p(x) - 1/2)$ . Since  $p_{\Omega'}^- > 1$ , there exists a constant  $C > 0$  such that

$$\left| |\nabla u_j|^{p(x)-1} \log |\nabla u_j| \right| \leq C |\nabla u_j|^{p(x)/\alpha(x)}.$$

Since  $\|\nabla u_j\|_{p(\cdot)} \leq C$ , we conclude by the previous inequality that  $\left\| |\nabla u_j|^{p(x)-1} \log |\nabla u_j| \right\|_{L^{\alpha(\cdot)}(\Omega')} \leq C$ . Also,  $(p^* - 2p)_{\Omega'}^- > 0$ , so the Rellich–Kondrachov theorem [15, Section 7.4] implies that  $\|u - u_j\|_{L^{2p(\cdot)}(\Omega')} \rightarrow 0$ . Similarly,

$$\left\| |\nabla u_j|^{p(x)-1} \right\|_{L^{p'(\cdot)}(\Omega')} \|u - u_j\|_{L^{p(\cdot)}(\Omega')} \rightarrow 0.$$

Thus we have proved that

$$\int_{\Omega'} |\nabla u_j|^{p(x)-2} \eta \nabla u_j \cdot (\nabla u - \nabla u_j) dx \rightarrow 0 \tag{6.2}$$

as  $j \rightarrow \infty$ .

Since  $|\nabla u|^{p(x)-2}\eta\nabla u \in L^{p'(\cdot)}(\Omega')$  and  $\nabla u_j \rightharpoonup \nabla u$  in  $L^{p(\cdot)}(\Omega')$ , we see that

$$\int_{\Omega'} |\nabla u|^{p(x)-2}\eta\nabla u \cdot (\nabla u - \nabla u_j) dx \rightarrow 0$$

Subtracting the right hand side of the previous line from (6.2) yields

$$\int_{\Omega'} \eta [|\nabla u_j|^{p(x)-2}\nabla u_j - |\nabla u|^{p(x)-2}\nabla u] \cdot (\nabla u_j - \nabla u) dx \rightarrow 0.$$

Since the integrand is non-negative and  $\eta = 1$  in  $\Omega''$ , we conclude that  $\nabla u_j \rightarrow \nabla u$  a.e. in  $\Omega''$ . Since  $(|\nabla u_j|^{p(x)-2}\nabla u_j)$  is a bounded sequence in  $L^{p'(\cdot)}(\Omega'')$ , it has (after moving to a subsequence) a weak limit, which necessarily coincides with the point-wise limit, hence

$$|\nabla u_j|^{p(x)-2}\nabla u_j \rightharpoonup |\nabla u|^{p(x)-2}\nabla u$$

weakly in  $L^{p'(\cdot)}(\Omega'')$ . Fix a test function  $\varphi \in W^{1,p(\cdot)}(\Omega \setminus Y)$  with  $\text{supp } \varphi \subset \Omega''$ . The weak convergence implies that

$$\begin{aligned} \int_{\Omega} |\nabla u_j|^{p(x)-2}\nabla u_j \cdot \nabla \varphi dx &= \int_{\Omega''} |\nabla u_j|^{p(x)-2}\nabla u_j \cdot \nabla \varphi dx \\ &\rightarrow \int_{\Omega''} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla \varphi dx = \int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla \varphi dx. \end{aligned} \quad (6.3)$$

Since  $(p^*)' \leq \alpha$ , the weak convergence in  $L^{p(\cdot)}(\Omega')$  implies that  $|\nabla u_j|^{p(\cdot)/\alpha(\cdot)} \rightharpoonup |\nabla u|^{p(\cdot)/\alpha(\cdot)}$  in  $L^{(p^*)'(\cdot)}(\Omega')$ . Because  $||\nabla u_j|^{p(x)-2} \log |\nabla u_j| \nabla u_j \cdot \nabla p| \leq C|\nabla u_j|^{p(x)/\alpha(x)}$ , we conclude that also

$$|\nabla u_j|^{p(x)-2} \log |\nabla u_j| \nabla u_j \cdot \nabla p \rightharpoonup |\nabla u|^{p(x)-2} \log |\nabla u| \nabla u \cdot \nabla p$$

in the same space. By the Sobolev inequality,  $\varphi \in L^{p^*(\cdot)}(\Omega'')$ . Thus we find that

$$\int_{\Omega''} |\nabla u_j|^{p(x)-2} \log |\nabla u_j| \nabla u_j \cdot \nabla p \varphi dx \rightarrow \int_{\Omega''} |\nabla u|^{p(x)-2} \log |\nabla u| \nabla u \cdot \nabla p \varphi dx.$$

Combining this with (6.3), we obtain that

$$\begin{aligned} &-\int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla \varphi - |\nabla u|^{p(x)-2} \log |\nabla u| \nabla u \cdot \nabla p \varphi dx \\ &= \lim_{j \rightarrow \infty} -\int_{\Omega} |\nabla u_j|^{p(x)-2}\nabla u_j \cdot \nabla \varphi - |\nabla u_j|^{p(x)-2} \log |\nabla u_j| \nabla u_j \cdot \nabla p \varphi dx = 0. \end{aligned}$$

Therefore  $u$  is a solution in  $\Omega \setminus Y$ . ■

We need the following result on local regularity of solutions. The comments regarding Theorem 4.4 apply also here.

**Theorem 6.4** (Theorem 1.1 in [20]). *Let  $p$  be Lipschitz continuous on  $\Omega$ . If  $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  is a solution of (4.1), then  $u \in C_{\text{loc}}^{1,\alpha}(\Omega')$  for  $\Omega' \subset\subset \Omega$ , where  $\alpha$  and  $|u|_{C^{1,\alpha}(\Omega')}$  depend only on  $p^-$ ,  $p^+$ ,  $\|\nabla p\|_\infty$ ,  $n$ ,  $\|u\|_\infty$ , and  $\text{dist}(\Omega', \mathbb{R}^n \setminus \Omega)$ . □*

We are now ready to prove the second main result by upgrading the solutions from the previous result to a solution in the whole domain.

**Proof of Theorem 1.3.** Let  $u$  be as in Proposition 6.1. Then  $u$  is a solution of  $(\star)$  in  $\Omega \setminus Y$ . Since  $Y$  has zero 1-Hausdorff measure, it is removable for Sobolev functions in  $W^{1,p(\cdot)}(\Omega)$  and  $u$  is a solution in  $\Omega$  by [33, Theorem 3.2].

Let next  $x \in \Omega$  be a point with  $p(x) > 1$ . Since  $p$  is continuous, we may choose  $B := B(x, r)$  such that  $p_{2B}^- > 1$ . By Theorem 6.4,  $|\nabla u_j|$  is bounded in  $B$ , uniformly in  $j$ . Since  $\nabla u_j|_B$  in  $K_p(\cdot)$ -quasiregular and uniformly bounded, it follows by the compactness result at the end of Section 2.1 that  $\nabla u|_B$  is also  $K_p(\cdot)$ -quasiregular. Therefore the distortion estimate holds in the set  $\Omega \setminus Y$ , i.e. almost everywhere in  $\Omega$ , which concludes the proof of the theorem. ■

Let us conclude by justifying Remark 1.5. If  $Y \subset\subset \Omega$ , then we can choose  $E \subset\subset \Omega$  closed with  $C^2$  boundary such that  $p_{\Omega \setminus E}^- > 1$ . By Theorem 6.4 the functions  $u_j$  from Proposition 6.1 are  $C^{1,\alpha}$  on  $\partial E$ , uniformly in  $j$ . Hence  $u_j$  satisfies the assumptions of Theorem 4.4 in the set  $\Omega \setminus E$ . Then we may conclude that the limit  $u$  has boundary values  $g$  in  $\Omega$  by the same method as in Theorem 1.2.

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