Combinatorics in Banach space theory

Lecture 1

1 Almost disjoint families and complementability of c_0

There are two basic constructions in the combinatorial set theory which are important in the structural theory of Banach spaces: almost disjoint families and independent families. We shall start our considerations by showing how the existence of uncountable almost disjoint families of subsets of $\mathbb{N}$ may be used to derive the uncomplementability of c_0 inside ℓ_∞. Later, we will see the usefulness of this construction in the definition of the Johnson–Lindenstrauss space which yields a classical and ground-breaking example in the theory of the 'three-space problem'.

Definition 1.1. Let Γ be any set. A family $\mathcal{F} \subset \mathcal{P}\Gamma$ is called *almost disjoint* whenever $A \cap B$ is finite for every $A, B \in \mathcal{F}$, $A \neq B$.

The following 'folklore' result has many different proofs. We present three of them; each is equally elegant and ingenious.

Proposition 1.2. *There exists an almost disjoint family consisting of infinite subsets of* $\mathbb{N}$, *which has the cardinality* $\mathfrak{c}$.

First proof. We may identify the set $\mathbb{N}$ with the countable set $\mathbb{Q}$. For each real number x pick any sequence $(q_n(x))_{n=1}^{\infty} \subset \mathbb{Q}$ converging to x and let $A_x = \{q_n(x) \colon n \in \mathbb{N}\}$. Then, $\mathcal{F} = \{A_x \colon x \in \mathbb{R}\} \subset \mathcal{P}\mathbb{Q}$ is almost disjoint and has the cardinality $\mathfrak{c}$. $\square$

Second proof. Now, we identify the set $\mathbb{N}$ with the set $\mathbb{Z}^2$ of all lattice points on the plane. Take any number $d > 2$ and for each $\vartheta \in [0, \pi)$ let A_ϑ be the set of all lattice points lying inside the strip bounded by two parallel lines at the distance $d/2$ from the origin and making the angle ϑ with the horizontal axis. Then, $\mathcal{F} = \{A_\vartheta \colon 0 \leqslant \vartheta < \pi\} \subset \mathcal{P}\mathbb{Z}^2$ is almost disjoint and has the cardinality $\mathfrak{c}$. $\square$

Third proof. In this proof we identify the set $\mathbb{N}$ with the set $\mathcal{N}$ of all nodes in the complete binary tree $\mathcal{N} = \bigcup_{\alpha<\omega}\{0,1\}^\alpha$. Let $\mathcal{F} \subset \mathcal{P}\mathcal{N}$ be the family of all branches. It has the cardinality $\mathfrak{c}$ and is almost disjoint, as every two distinct branches meet only at some finite initial segment. $\square$

Recall that a subspace Y of a Banach space X is called *complemented* in X provided that there exists a closed subspace Z of X such that $X = Y + Z$ and $Y \cap Z = \{0\}$. In this case we write $X = Y \oplus Z$. It is equivalent to the fact that there is a bounded projection from X onto Y, that is, a surjective operator $P \colon X \to Y$ satisfying $P^2 = P$ (see, e.g., [Rud91, Theorem 5.16]).

By the Hahn–Banach extension theorem, every finite-dimensional subspace of a Banach space X is complemented (see, e.g., [Rud91, Lemma 4.21]). Let us stress that the assumption of local convexity of X in this statement is essential. Without assuming local convexity one may still state that every subspace of finite co-dimension is complemented.

As c_0 is contained in ℓ_∞, and is equipped with the same supremum norm, one of the first questions concerning the notion of complementability was whether there exists a bounded projection from ℓ_∞ onto c_0. The negative answer was given by Phillips [Phi40] and Sobczyk [Sob41]. At that time it was certainly quite a surprising application of large almost disjoint families.

Theorem 1.3 (Phillips & Sobczyk, 1940-41)**.** *c_0 is not complemented in ℓ_∞.*

We shall derive this result from the following fact which will turn out to give somewhat more than expected. For any set $A \subset \mathbb{N}$ let us denote $\ell_\infty(A)$ the space of all bounded sequences which are supported on A, that is,

$$\ell_\infty(A) = \big\{(\xi_n)_{n=1}^\infty \in \ell_\infty \colon \xi_n = 0 \text{ for each } n \notin A\big\}.$$

Proposition 1.4. *Suppose $T\colon \ell_\infty \to \ell_\infty$ is an operator which vanishes on c_0. Then there exists an infinite set $A \subset \mathbb{N}$ such that T vanishes on $\ell_\infty(A)$.*

Proof. Let $\{A_i\}_{i\in\mathcal{I}}$ be an uncountable almost disjoint family of infinite subsets of $\mathbb{N}$. Suppose, in search of a contradiction, that T does not vanish on any of the subspaces $\ell_\infty(A_i)$, for $i \in \mathcal{I}$. Then, for every $i \in \mathcal{I}$ one may find $\xi^{(i)} \in \ell_\infty(A_i)$ with $\|\xi^{(i)}\| = 1$ and natural numbers k_i, n_i such that $|e_{k_i}^* T\xi^{(i)}| > n_i^{-1}$ (where e_k^* stands for the kth coordinate functional on ℓ_∞). This implies that for some particular choice of $(k,n) \in \mathbb{N}\times\mathbb{N}$ the set

$$\mathcal{I}_{k,n} := \big\{i \in \mathcal{I} \colon |e_k^* T\xi^{(i)}| > n^{-1}\big\}$$

is uncountable.

For each $i \in \mathcal{I}_{k,n}$ choose a scalar α_i with $|\alpha_i| = 1$ and $\alpha_i e_k^* T\xi^{(i)} = |e_k^* T\xi^{(i)}|$. Now, for any finite subset F of $\mathcal{I}_{k,n}$ put

$$y = \sum_{i\in F} \alpha_i \xi^{(i)} \in \ell_\infty.$$

Since the intersection of any two members of $\{A_i\}_{i\in F}$ is finite, we may write $y = u + v$, where $\|u\|_\infty \leqslant 1$ and v is finitely supported. In particular, $v \in c_0$, hence $Tv = 0$ by the assumption, so $\|Ty\| = \|Tu\| \leqslant \|T\|$. On the other hand,

$$e_k^* Ty = \sum_{i\in F} \alpha_i e_k^* T\xi^{(i)} > |F| n^{-1}$$

and, consequently, $|F| < n\|T\|$. However, this would mean that the set $\mathcal{I}_{k,n}$ is finite which is not the case. □

Proof of Theorem 1.3. Suppose, towards a contradiction, that there is a bounded projection $P\colon \ell_\infty \to c_0$. Since P is the identity on c_0, the operator $T = I_{\ell_\infty} - P$ vanishes on c_0 and it maps ℓ_∞ into ℓ_∞. According to Proposition 1.4, there is an infinite set $A \subset \mathbb{N}$ such that $P(\xi) = \xi$ for every sequence $\xi \in \ell_\infty$ which is supported on A. But this would mean that the range of P is not contained in c_0; a contradiction. □

We have thus shown that c_0 is not complemented in its bidual $c_0^{**} = \ell_\infty$. Moreover, since every dual Banach space is complemented in its bidual and being complemented in its bidual is an isomorphic invariant (see Problems 1.1 and 1.2), we may formulate the following corollary.

Corollary 1.5. *c_0 is not isomorphic to any dual space.*

Notice that if Y is a complemented subspace of a Banach space X then the quotient space X/Y is isomorphic to a subspace of X which is complementary to Y. This follows directly from the algebraic 'isomorphism theorem' and the Open Mapping Theorem applied to the canonical projection from X onto a subspace complementary to Y (Y is the kernel of this projection). So, in this case there is an embedding operator from X/Y into X. Proposition 1.4 implies that this is impossible for $X = \ell_\infty$ and $Y = c_0$, even if the desired embedding is not required to have a closed range.

Corollary 1.6. *There is no injective operator from ℓ_∞/c_0 into ℓ_∞.*

Proof. Suppose $S\colon \ell_\infty/c_0 \to \ell_\infty$ is such an operator and let $P\colon \ell_\infty \to \ell_\infty/c_0$ be the canonical quotient map. Then $P|_{c_0} = 0$, hence the operator $SP\colon \ell_\infty \to \ell_\infty$ also vanishes on c_0. However, since S is one-to-one, we would have $SP\xi \neq 0$ for every $\xi \notin c_0$, which contradicts the assertion of Proposition 1.4. □

2 Rosenthal's lemma

If Σ is a σ-algebra, $(E_n)_{n=1}^\infty$ is a sequence of pairwise disjoint members of Σ, and $\mu\colon \Sigma \to [0,\infty)$ is a bounded, finitely additive measure, then for every $\varepsilon > 0$ we may certainly extract a subsequence $(E_{n_j})_{j=1}^\infty$ such that $\mu(E_{n_j}) < \varepsilon$. Of course, generally this cannot be done simultaneously for infinitely many measures $(\mu_n)_{n=1}^\infty$, even if we require that they are uniformly bounded. However, it is possible to find subsequences $(E_{n_j})_{j=1}^\infty$ and $(\mu_{n_j})_{j=1}^\infty$ such that the values of all the measures μ_{n_j} at E_{n_j}'s are arbitrarily small, with spikes only on the diagonal, that is, only $\mu_{n_j}(E_{n_j})$ may be possibly large.

This is a classical 'sliding-hump' type result, called Rosenthal's lemma. It was proved in [Ros70] in order to characterise non-weakly compact operators acting on injective Banach spaces (we will discuss this topic in Section 4). The statement may look a bit innocent but in fact it is a powerful tool. Among various of its consequences we will derive Nikodým's boundedness principle and Phillips' lemma which concern sequences of finitely additive scalar measures defined on a discrete set. In the monograph [DU77], by Diestel and Uhl, Rosenthal's lemma is used to derive many of the major results in the theory of vector measures.

The original proof by Rosenthal is quite complicated and uses an uncoutable analogue of Proposition 1.2 (see [Ros70, Proposition]). We shall present a much shorter proof, due to Kupka [Kup74]. For transparency, we will first focus on the countable case.

Lemma 2.1 (Rosenthal, 1970)**.** *Let Σ be a σ-algebra of subsets of some set Ω and $(\mu_n)_{n=1}^\infty$ be a uniformly bounded sequence of finitely additive, non-negative measures defined on Σ. Then, for every pairwise disjoint sequence $(E_n)_{n=1}^\infty \subset \Sigma$ and every $\varepsilon > 0$ there exists a strictly increasing sequence $(n_k)_{k=1}^\infty \subset \mathbb{N}$ such that*

$$\mu_{n_k}\Bigg(\bigcup_{j\neq k} E_{n_j}\Bigg) < \varepsilon \quad \textit{for each } k \in \mathbb{N}.$$

Proof. Let us suppose, with no loss of generality, that $\mu_n(\Omega) \leqslant 1$ for each $n \in \mathbb{N}$. Consider any sequence $(M_p)_{p=1}^\infty$ of pairwise disjoint infinite subsets of $\mathbb{N}$ such that $\mathbb{N} = \bigcup_{p=1}^\infty M_p$. We shall distinguish two cases.

Case 1. First, if there is some $p \in \mathbb{N}$ for which

$$\mu_k\Bigg(\bigcup_{\substack{j\in M_p\\ j\neq k}} E_j\Bigg) < \varepsilon \quad \text{for each } k \in M_p,$$

then we get the assertion by simply enumerating $M_p = \{n_1 < n_2 < \ldots\}$.

Case 2. Now, suppose that for every $p \in \mathbb{N}$ there is $k_p \in M_p$ such that

$$\mu_{k_p}\Bigg(\bigcup_{\substack{j\in M_p\\ j\neq k_p}} E_j\Bigg) \geqslant \varepsilon. \tag{2.1}$$

Fix, for a moment, any $p \in \mathbb{N}$. Since $\bigcup_q E_{k_q}$ is disjoint from the set $\bigcup_{j\in M_p, j\neq k_p} E_j$, we have

$$\bigcup_{j\in M_p, j\neq k_p} E_j \subset \bigcup_{n\in\mathbb{N}} E_n \setminus \bigcup_{q\in\mathbb{N}} E_{k_q}. \tag{2.2}$$

Obviously, we have

$$\mu_{k_p}\Bigg(\bigcup_{q\in\mathbb{N}} E_{k_q}\Bigg) + \mu_{k_p}\Bigg(\bigcup_{n\in\mathbb{N}} E_n \setminus \bigcup_{q\in\mathbb{N}} E_{k_q}\Bigg) \leqslant 1,$$

so (2.2) and (2.1) imply

$$\mu_{k_p}\Bigg(\bigcup_{q\in\mathbb{N}} E_{k_q}\Bigg) \leqslant 1 - \varepsilon,$$

and this inequality is valid for every $p \in \mathbb{N}$.

Consequently, we could repeat the same argument replacing the sequences $(\mu_n)_{n=1}^\infty$ and $(E_n)_{n=1}^\infty$ by $(\mu_{k_p})_{p=1}^\infty$ and $(E_{k_p})_{p=1}^\infty$, respectively. By continuing this we would get subsequent upper bounds $1 - 2\varepsilon, 1 - 3\varepsilon, \ldots$ for some of the measures μ_n. Since this process has to terminate, we will end up with *Case 1* in which the assertion has been proved. $\square$

Now, let us see how this type of argument goes through in the uncountable case. For simplicity, the pairwise disjoint sets given in the assumption may be identified with points of a discrete set Γ, and all the given measures may be assumed to act on the σ-algebra of all subsets of Γ.

Lemma 2.2 (Rosenthal, 1970)**.** *Let Γ be an infinite set and let $\{\mu_\gamma\colon \gamma \in \Gamma\}$ be a uniformly bounded family of finitely additive, non-negative measures defined on $\mathcal{P}\Gamma$. Then, for every $\varepsilon > 0$ there exists a set $\Delta \subset \Gamma$ with $|\Delta| = |\Gamma|$ and such that*

$$\mu_\delta\big(\Delta \setminus \{\delta\}\big) < \varepsilon \quad \textit{for each } \delta \in \Delta.$$

Proof. Assuming the Axiom of Choice (no choice of not assuming it!), we have $|\Gamma| = |\Gamma\times\Gamma|$ for every infinite cardinal number Γ (see, e.g., [Jec00, Theorem 3.5]). Hence, we may write $\Gamma = \bigcup_{\gamma\in\Gamma} \Delta_\gamma$, where Δ_γ's are pairwise disjoint and $|\Delta_\gamma| = |\Gamma|$ for each $\gamma \in \Gamma$. Again, we distinguish two cases.

Case 1. First, if for every $\gamma \in \Gamma$ there is some $x_\gamma \in \Delta_\gamma$ such that $\mu_{x_\gamma}\big(\Gamma \setminus \Delta_\gamma\big) < \varepsilon$, then the set $\Delta := \{x_\gamma \colon \gamma \in \Gamma\}$ does the job.

Case 2. So, suppose that there is some $\gamma_0 \in \Gamma$ such that

$$\mu_x\big(\Gamma \setminus \Delta_{\gamma_0}\big) \geqslant \varepsilon \quad \text{for every } x \in \Delta_{\gamma_0}.$$

Then, repeating this argument with Δ_{γ_0} in the place of Γ we must arrive at *Case 1* after finitely many steps, since otherwise the uniform boundedness of the measures μ_γ would be violated. □

Regarding Lemma 2.1 one may ask whether a similar conclusion holds true for measures defined on a set algebra $\mathscr{F}$ which is not necessarily a σ-algebra. The answer is positive and it follows quite directly from the σ-algebra case and the Hahn–Banach theorem. This will not be used in the sequel, so we omit the proof. The interested reader should consult [DU77, p. 19].

Corollary 2.3. *Let $\mathscr{F}$ be a set algebra and $(\mu_n)_{n=1}^\infty$ be a uniformly bounded sequence of finitely additive, non-negative measures defined on $\mathscr{F}$. Then, for every pairwise disjoint sequence $(E_n)_{n=1}^\infty \subset \mathscr{F}$ and every $\varepsilon > 0$ there exists a strictly increasing sequence $(n_k)_{k=1}^\infty \subset \mathbb{N}$ such that*

$$\mu_{n_k}\Bigg(\bigcup_{j\neq k,\, j\in\Delta} E_{n_j}\Bigg) < \varepsilon$$

for every $k \in \mathbb{N}$ and every finite set $\Delta \subset \mathbb{N}$.