Combinatorics in Banach space theory

Lecture 11

Proposition 11.9. *Let X be a finite-dimensional normed space. Then, the range of every σ-additive vector measure $\mu\colon\Sigma\to X$, defined on a σ-algebra Σ, is a closed subset of X.*

Proof. Let Σ be a σ-algebra of subsets of Ω and $\mathcal{A}\subset\Sigma$ be any maximal family of pairwise disjoint atoms of a given measure $\mu\colon\Sigma\to X$. In other words, $\mathcal{A}$ is a subset of Σ which consists of pairwise disjoint atoms and such that the restriction of μ to the σ-algebra $\Sigma|_{\Omega\setminus\bigcup\mathcal{A}}=\{A\cap(\Omega\setminus\bigcup\mathcal{A})\colon A\in\Sigma\}$ is non-atomic. Since μ is bounded, $\mathcal{A}$ is at most countable. Let Y and Z be the ranges of the measure μ restricted to the σ-algebras $\Sigma|_{\bigcup\mathcal{A}}$ and $\Sigma|_{\Omega\setminus\bigcup\mathcal{A}}$, respectively. Then, $\mu(\Sigma)=Y+Z$ and, by the Lyapunov theorem, Z is a compact (and convex) subset of X. If we prove that Y is also compact, then we will be done because the algebraic sum of any two compact sets is still compact.

The compactness of Y is obvious whenever $\mathcal{A}$ is finite (or empty), so suppose that $\mathcal{A}=\{A_1,A_2,\ldots\}$ consists of infinitely many atoms of μ and let $x_n=\mu(A_n)$. Obviously, we have $Y=\{\sum_{n\in M}x_n\colon M\in\mathcal{P}\mathbb{N}\}$ (every such series is unconditionally convergent because of the σ-additivity of μ). Observe that Y is the range of a map $\varphi\colon 2^{\mathbb{N}}\to Y$ given by $\varphi(M)=\sum_{n\in M}x_n$. Moreover, φ is continuous with respect to the product topology on $2^{\mathbb{N}}$ which is compact due to Tychonoff's theorem. Therefore, Y is also compact and the proof is completed. □

It is worth mentioning that in the infinite-dimensional case the range of a σ-additive vector measure of bounded variation need not be closed. However, it is not so straightforward to build a counterexample (or, to show that a given vector measure is indeed a counterexample). Below, we present an example due to Uhl [Uhl69] which is a slight modification of one given much earlier by Lyapunov [Lya40]. First, let us quote the main result from [Uhl69]:

Theorem 11.10 (Uhl, 1969)**.** *Let X be a Banach space which is either reflexive or a separable dual space. Assume that $\mu\colon\Sigma\to X$ is a σ-additive vector measure of bounded variation, defined on some σ-algebra Σ. Then, $\mu(\Sigma)$ is relatively* (*norm*) *compact. Moreover, if μ is non-atomic, then the closure of $\mu(\Sigma)$ is convex and compact.*

Example 11.11. Let $\Omega=[0,1]$, Σ be the σ-algebra of all Borel subsets of Ω and λ be the Lebesgue measure on Σ. Let $(\psi_n)_{n=0}^\infty$ be a complete (i.e. linearly dense) orthonormal sequence in $L_2[0,1]$ such that:

- $\psi_0(x)\equiv 1$,
- each ψ_n takes only the values ± 1, for $n\geqslant 1$,
- $\int_0^1\psi_n\,\mathrm{d}\lambda=0$ for each $n\geqslant 1$.

For instance, we may take $(\psi_n)_{n=0}^\infty$ to be the sequence of all *Walsh functions* $\{\varphi_0\}\cup\{\varphi_n^{(j)}\colon n\in\mathbb{N},\,1\leqslant j\leqslant 2^{n-1}\}$ which are defined (up to measure zero sets) recursively as follows:

$$\varphi_0(x)\equiv 1,\quad \varphi_1(x)=\begin{cases}\;\;1, & 0\leqslant x<\frac12,\\ -1, & \frac12<x\leqslant 1,\end{cases}$$

$$\varphi_{n+1}^{(2k-1)}(x) = \begin{cases} \varphi_n^{(k)}(2x), & 0 \leqslant x < \frac{1}{2}, \\ (-1)^{k+1}\varphi_n^{(k)}(2x-1), & \frac{1}{2} < x \leqslant 1 \end{cases}$$

and

$$\varphi_{n+1}^{(2k)}(x) = \begin{cases} \varphi_n^{(k)}(2x), & 0 \leqslant x < \frac{1}{2}, \\ (-1)^{k}\varphi_n^{(k)}(2x-1), & \frac{1}{2} < x \leqslant 1 \end{cases}$$

for $n \in \mathbb{N}$ and $1 \leqslant k \leqslant 2^{n-1}$. In other words, all the functions $\varphi_n^{(j)}$ may be arranged into a binary tree in such a way that $\varphi_n^{(j)}$, for $1 \leqslant j \leqslant 2^{n-1}$, occupy the nth level and every $\varphi_n^{(k)}$ has two direct successors: $\varphi_{n+1}^{(2k-1)}$ and $\varphi_{n+1}^{(2k)}$, whose graphs are formed by two copies of the graph of $\varphi_n^{(k)}$ contracted by the scale $1/2$ so that the former one is even and the latter one is odd with respect to the point $1/2$.

Now, let

$$I_n(E) = \frac{1}{2^n}\int_E \frac{1+\psi_n}{2}\,\mathrm{d}\lambda \quad \text{for } n \in \mathbb{N}_0 \text{ and } E \in \Sigma.$$

We define a vector measure $\mu\colon \Sigma \to \ell_2$ by $\mu(E) = (I_n(E))_{n=0}^{\infty}$ for $E \in \Sigma$. Plainly, μ is finitely additive and non-atomic, whereas the obvious inequality $\|\mu(E)\| \leqslant 2\lambda(E)$ immediately implies that it is also σ-additive and has bounded variation. Hence, if the range of μ was closed, then Theorem 11.10 would imply that it is also convex. However, we will now show that this is not the case (disproving the closedness of $\mu(\Sigma)$ by hand would be a heroic feat). To this end it is enough to show that there is no set $E \in \Sigma$ satisfying $\mu(E) = \frac{1}{2}\mu(\Omega) = \frac{1}{2}(1, 1/4, 1/8, \ldots)$. So, suppose on the contrary that $E \in \Sigma$ satisfies the last equality.

Since $I_0(E) = \int_E \mathrm{d}\lambda$, we have $\lambda(E) = 1/2$. Moreover, for every $n \in \mathbb{N}$ we have

$$\frac{1}{2^{n+2}} = I_n(E) = \frac{1}{2^n}\int_E \frac{1+\psi_n}{2}\,\mathrm{d}\lambda = \frac{\lambda(E \cap U_n)}{2^n},$$

where $U_n = \{s \in \Omega\colon \psi_n(s) = +1\}$. Hence, $\lambda(E \cap U_n) = 1/4$ for $n \in \mathbb{N}$. Since $\lambda(U_n) = \lambda(E) = 1/2$, we obtain also the following three equalities: $\lambda(U_n \backslash E) = 1/4$, $\lambda(E \backslash U_n) = 1/4$ and $\lambda(\Omega \setminus (E \cup U_n)) = 1/4$ for every $n \in \mathbb{N}$. Define a function $\varphi \in L_2[0,1]$ by $\varphi(x) = +1$ for $x \in E$ and $\varphi(x) = -1$ for $x \notin E$. Then, $\int_0^1 \varphi\psi_0\,\mathrm{d}\lambda = 0$ and for every $n \in \mathbb{N}$ we have

$$\begin{aligned} \int_0^1 \varphi\psi_n\,\mathrm{d}\lambda &= \lambda(E \cap U_n) + \lambda(\Omega \setminus (E \cup U_n)) \\ &\quad - \lambda(U_n \setminus E) - \lambda(E \setminus U_n) = 0, \end{aligned}$$

which means that φ is orthogonal to each of the functions $(\psi_n)_{n=0}^{\infty}$. This is impossible as the Walsh system is complete in $L_2[0,1]$.

12 B-convex spaces

In this section, we turn our attention to the notion of B-convex space, which is defined by a Steinitz-type condition, introduced by Beck [Bec62]. The primary motivation for considering B-convex spaces stemmed from the probability theory; this will be briefly expained in what follows. Next, we will come back to Kadets' Theorem 11.6 and convince ourselves that the method based on Steinitz's lemma may be applied to derive an infinite-dimensional analogue of Lyapunov's theorem for vector measures taking values in B-convex spaces. In the next step, the notion of B-convexity will be a pretext for returning to our study of the 'three-space problem' which has been broached in Section 8.

To formulate the definition of B-convexity, and for further discussion, it is convenient to introduce for any Banach space X the following four quantities:

$$a_n = a_n(X) = \sup_{(x_j)_{j=1}^n \subset B_X} \left\| \sum_{j=1}^n x_j \right\|,$$

$$b_n = b_n(X) = \sup_{(x_j)_{j=1}^n \subset B_X} \min_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|,$$

$$c_n = c_n(X) = \sup \left\{ \frac{1}{\sqrt{n}} \cdot \left(\frac{1}{2^n} \sum_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^2 \right)^{1/2} : x_j \in X \text{ and } \sum_{j=1}^n \|x_j\|^2 \leqslant 1 \right\},$$

and

$$d_n = d_n(X) = \sup \left\{ \frac{1}{n} \cdot \left(\frac{1}{2^n} \sum_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^2 \right)^{1/2} : x_j \in X \text{ and } \max_{1 \leqslant j \leqslant n} \|x_j\| \leqslant 1 \right\}.$$

In other words, c_n is the infimum of all those constants $c > 0$ for which the inequality

$$\left(\frac{1}{2^n} \sum_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^2 \right)^{1/2} \leqslant \sqrt{n} \cdot c \cdot \left(\sum_{j=1}^n \|x_j\|^2 \right)^{1/2} \tag{12.1}$$

holds true for all $x_1, \ldots, x_n \in X$, whereas d_n is the infimum of all those constants $d > 0$ for which the inequality

$$\left(\frac{1}{2^n} \sum_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^2 \right)^{1/2} \leqslant n \cdot d \cdot \max_{1 \leqslant j \leqslant n} \|x_j\|$$

holds true for all $x_1, \ldots, x_n \in X$.

Definition 12.1. A Banach space X is called *B-convex*, if $b_n < n$ for at least one $n \in \mathbb{N}$.

Obviously, in every Banach space we have $b_1 = 1$, $0 < b_n \leqslant n$ and $b_n \leqslant b_{n+1}$ for every $n \in \mathbb{N}$. It is also very easy to see that ℓ_1 is not a B-convex space. In fact, if $n \in \mathbb{N}$ and $x_1, \ldots, x_n$ are the standard unit basis vectors, then no matter how we choose the signs $\varepsilon_j = \pm 1$ we always have $\| \sum_j \varepsilon_j x_j \| = n$, thus $b_n = n$. Of course, the essential point lies in the local structure of ℓ_1. Namely, to guarantee that a given Banach space X is not B-convex it suffices to ensure that X contains isomorphic copies of all the finite-dimensional spaces ℓ_1^n ($n \in \mathbb{N}$) which are arbitrarily close to being isometric copies. This is formalised by the notion of *finite representability* which we have already met in Problem 2.12.

Definition 12.2. Let X and Y be Banach spaces. We say that X is *finitely representable* in Y, provided that for every finite-dimensional subspace E of X and every $\varepsilon > 0$ there exists a finite-dimensional subspace F of Y with $\dim E = \dim F$ and such that $d_{\mathsf{BM}}(E, F) < 1 + \varepsilon$, where d_{BM} stands for the *Banach–Mazur distance* defined by the formula

$$d_{\mathsf{BM}}(E, F) = \inf\big\{\|T\| \cdot \|T^{-1}\| \ \mid \ \ T\colon E \to F \text{ is an isomorphism}\big\}$$

(of course, the above definition of d_{BM} makes sense for any two isomorphic Banach spaces E and F).

In this spirit, we say that X *contains* ℓ_1^n's *uniformly*, provided that there is a constant $\lambda \geqslant 1$ and a sequence $(E_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of X such that $E_n \simeq \ell_1^n$ and $d_{\mathsf{BM}}(E_n, \ell_1^n) \leqslant \lambda$ for each $n \in \mathbb{N}$. The following result reveals the mystery behind B-convexity:

Theorem 12.3. *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is B-convex;*
(ii) $\lim_{n\to\infty} n^{-1} b_n = 0$*;*
(iii) *$c_n < 1$ for at least one $n \in \mathbb{N}$;*
(iv) $\lim_{n\to\infty} c_n = 0$*;*
(v) *X does not contain ℓ_1^n's uniformly;*
(vi) *ℓ_1 is not finitely representable in X.*

Before proving this theorem, let us isolate several useful lemmas.

Lemma 12.4. *For every Banach space X we have $b_{mn}(X) \leqslant b_m(X) b_n(X)$ for all $m, n \in \mathbb{N}$ (that is, the sequence $(b_n(X))_{n=1}^{\infty}$ is submultiplicative).*

Proof. For simplicity, we write b_n instead of $b_n(X)$. Fix any set $\{x_{ij}\colon i \in [m],\, j \in [n]\}$ of mn vectors from the unit ball of X. We may find signs $\theta_{ij} = \pm 1$ such that

$$\Big\|\sum_{j=1}^{n} \theta_{ij} x_{ij}\Big\| \leqslant b_n \quad \text{for each } i \in [m].$$

Therefore, applying the formula for b_m to the sequence $(\sum_{j=1}^{n} \theta_{ij} x_{ij})_{i=1}^{m}$ we get some signs $\eta_i = \pm 1$ such that

$$\Big\|\sum_{i=1}^{m} \eta_i \sum_{j=1}^{n} \theta_{ij} x_{ij}\Big\| \leqslant b_m b_n.$$

Consequently, by putting $\varepsilon_{ij} = \eta_i \theta_{ij}$ for $i \in [m]$ and $j \in [n]$ we obtain

$$\Big\|\sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \varepsilon_{ij} x_{ij}\Big\| \leqslant b_m b_n$$

and the result follows. □

Lemma 12.5. *If $(t_n)_{n=1}^{\infty}$ is a submultiplicative, monotone increasing sequence of positive numbers such that $t_k < k$ for some $k \geqslant 2$, then there exist constants $c > 0$ and $p > 1$ such that $t_n \leqslant c n^{1/p}$ for every $n \in \mathbb{N}$.*

Proof. Since $t_k < k$, there is some $p > 1$ such that $t_k \leqslant k^{1/p}$. For any $n \in \mathbb{N}$ write $n = k^r + s$ with some $r \in \mathbb{N}_0$ and $0 \leqslant s < k^{r+1} - k^r$. Then, by the submultiplicativity, we have

$$t_n \leqslant t_{k^{r+1}} \leqslant t_k^{r+1} \leqslant k^{(r+1)/p} = k^{1/p} \cdot k^{r/p} \leqslant k^{1/p} \cdot n^{1/p},$$

whence the result follows with $c = k^{1/p}$. □

Lemma 12.6. *For every Banach space X we have:*

(a) $c_n(X)/c_{n+1}(X) \leqslant \sqrt{(n+1)/n}$ *for each* $n \in \mathbb{N}$*;*
(b) $n^{-1} b_n(X) \leqslant d_n(X) \leqslant c_n(X) \leqslant 1$ *for each* $n \in \mathbb{N}$*;*
(c) $c_n(X) \geqslant n^{-1/2}$ *for each* $n \in \mathbb{N}$*;*
(d) $c_{mn}(X) \leqslant c_m(X) c_n(X)$ *for all* $m, n \in \mathbb{N}$.

Proof. **(a)**: Fix any vectors $x_1, \ldots, x_n \in X$ satisfying $\sum_{j=1}^n \|x_j\|^2 \leqslant 1$ and consider $n+1$ vectors $x_1, \ldots, x_n, 0$. Applying inequality (12.1) to these vectors and replacing c byc_{n+1} and n by $n+1$ we get the upper estimate $\sqrt{n+1} \cdot c_{n+1}$ for the average $(2^{-n} \sum_{\varepsilon_j = \pm 1} \| \sum_j \varepsilon_j x_j \|^2)^{1/2}$ (observe that each combination $\sum_j \varepsilon_j x_j$ appears twice, for $\varepsilon_{n+1} = -1$ and $\varepsilon_{n+1} = +1$, so the corresponding coefficient equals $2 \cdot 2^{-(n+1)} = 2^{-n}$). Hence, by the definition of c_n, we arrive at $\sqrt{(n+1)/n} \cdot c_{n+1} \geqslant c_n$.

(b): The inequalities $d_n \leqslant c_n \leqslant 1$ are obvious. For proving that $n^{-1} b_n \leqslant d_n$ fix any sequence $x_1, \ldots, x_n$ in B_X and recall that by the definition of d_n we have

$$\frac{1}{2^n} \sum_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^2 \leqslant n^2 \cdot d_n^2.$$

Hence, for a certain choice of signs $\varepsilon_j = \pm 1$ it has to be $\| \sum_j \varepsilon_j x_j \|^2 \leqslant n^2 \cdot d_n^2$ which means that $b_n \leqslant n \cdot d_n$.

(c): First, let us make an observation which will be also important in our future investigations. Namely, the square of the left-hand side of inequality (12.1) is nothing else but the expectation value of $\| \sum_j \xi_j x_j \|^2$, where $(\xi_j)_{j=1}^n$ is any sequence of independent random variables such that for every $j \in [n]$ both of the equalities $\xi_j = -1$ and $\xi_j = +1$ hold true with probability $1/2$. Recall that these properties are shared by the sequence $(r_j)_{j=1}^n$ of the Rademacher functions (see Definition 7.2). Therefore,

$$\frac{1}{2^n} \sum_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^2 = \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^2 \mathrm{d}t. \tag{12.2}$$

Now, setting $x_1 = \ldots = x_n = x$ with some $x \in X$ such that $\|x\| = 1$ and using the fact that the Rademacher functions are orthogonal in $L_2[0,1]$ (see formula (7.1)) we see that the square of the left-hand side of (12.1) equals

$$\int_0^1 \left| \sum_{j=1}^n r_j(t) \right|^2 \mathrm{d}t = \left\| \sum_{j=1}^n r_j \right\|_{L_2} = \sum_{j=1}^n \|r_j\|_{L_2} = n,$$

whereas $\sum_j \|x_j\|^2 = n \cdot \|x\|^2 = n$. Hence, c_n^2 must be at least n^{-1} which gives the assertion.

(d): This one is a bit tricky. Fix arbitrary $x_1, \ldots, x_{mn} \in X$ and define

$$y_k(t) = \sum_{(k-1)m < j \leqslant km} r_j(t) x_j \quad \text{for } 1 \leqslant k \leqslant n \text{ and } t \in [0,1].$$

Let also

$$I=\int_0^1\int_0^1\Big\|\sum_{j=1}^n r_j(s)y_j(t)\Big\|^2\,\mathrm{d}s\,\mathrm{d}t.$$

Note that for any fixed $t\in[0,1]$ the definition of c_n combined with formula (12.2) gives

$$\int_0^1\Big\|\sum_{j=1}^n r_j(s)y_j(t)\Big\|^2\,\mathrm{d}s\leqslant nc_n^2\cdot\sum_{j=1}^n\|y_j(t)\|^2.$$

Hence, integrating with respect to t and using the definition of c_m we get

$$I\leqslant nc_n^2\cdot\sum_{j=1}^n\int_0^1\|y_j(t)\|^2\,\mathrm{d}t\leqslant nc_n^2\cdot\sum_{k=1}^n mc_m^2\cdot\sum_{(k-1)m<j\leqslant km}\|x_j\|^2=m\,n\,c_m^2\,c_n^2\cdot\sum_{j=1}^{mn}\|x_j\|^2.$$

On the other hand, for every $s\in[0,1]$ we have

$$\int_0^1\Big\|\sum_{j=1}^n r_j(s)y_j(t)\Big\|^2\,\mathrm{d}t=\int_0^1\Big\|\sum_{k=1}^n r_k(s)\sum_{(k-1)m<j\leqslant km}r_j(t)x_j\Big\|^2\,\mathrm{d}t=\int_0^1\Big\|\sum_{j=1}^{mn}r_j(u)x_j\Big\|^2\,\mathrm{d}u.$$

Indeed, the last but one term is the expectation value of $\|\sum_{j=1}^{mn}\xi_jx_j\|^2$, where each ξ_j is either $+r_j$ or $-r_j$ (depending on whether $r_k(s)=+1$ or $r_k(s)=-1$, where $k\in[n]$ is selected so that $(k-1)m<j\leqslant km$), that is, $(\xi_j)_{j=1}^{mn}$ is a realization of the Rademacher random variables. The same may be said about the last term, so they are both equal to the average $2^{-mn}\sum_{\varepsilon_j=\pm1}\|\sum_{j=1}^{mn}\varepsilon_jx_j\|^2$. Integrating the above equality with respect to s we obtain

$$\int_0^1\Big\|\sum_{j=1}^{mn}r_j(u)x_j\Big\|^2\,\mathrm{d}u=I\leqslant m\,n\,c_m^2\,c_n^2\cdot\sum_{j=1}^{mn}\|x_j\|^2$$

which yields $c_{mn}\leqslant c_mc_n$, as desired. □

Lemma 12.7. *For every finite-dimensional subspace E of ℓ_1 and every $\varepsilon>0$ there exists another finite-dimensional subspace H of ℓ_1 such that $E\subset H$ and $d_{\mathsf{BM}}(H,\ell_1^m)<1+\varepsilon$, where $m=\dim H$.*

Proof. Suppose that $E=\operatorname{span}\{x_j\}_{j=1}^n$, where $\|x_j\|=1$ for $1\leqslant j\leqslant n$. Since all norms in any finite-dimensional linear space are equivalent, there is a constant $k\geqslant1$ such that

$$\Big\|\sum_{j=1}^n\lambda_jx_j\Big\|\geqslant k^{-1}\max_{1\leqslant j\leqslant n}|\lambda_j| \tag{12.3}$$

for any scalars $\lambda_1,\dots,\lambda_n$. Fix any $\delta\in(0,1)$ and define $\varepsilon=\delta/2kn$. Since every element in ℓ_1 is a norm limit of a sequence of step functions, there exists a partition $\mathbb{N}=\bigcup_{j=1}^mN_j$, with some non-empty pairwise disjoint sets $N_1,\dots,N_m$, and a set $\{y_j\}_{j=1}^n\in F:=\operatorname{span}\{\mathbb{1}_{N_j}\colon 1\leqslant j\leqslant m\}$ such that $\|y_j\|=1$ and $\|x_j-y_j\|<\varepsilon$. It is evident that F is isometrically isomorphic to ℓ_1^m. Therefore, the subspace E is 'almost' contained in an isometric copy F of ℓ_1^m. Now, we will show that there is an almost isometric perturbation H of F which contains E and this will complete the proof.

In view of inequality (12.3), for any scalars $\lambda_1, \dots, \lambda_n$ we have

$$\begin{aligned}\Big\|\sum_{j=1}^{n}\lambda_j y_j\Big\| &\geqslant \Big\|\sum_{j=1}^{n}\lambda_j x_j\Big\| - \sum_{j=1}^{n}|\lambda_j|\cdot\|x_j-y_j\| \\ &\geqslant (k^{-1}-\varepsilon n)\max_{1\leqslant j\leqslant n}|\lambda_j| \geqslant (2k)^{-1}\max_{1\leqslant j\leqslant n}|\lambda_j|.\end{aligned}$$

Therefore, by the Hahn–Banach extension theorem there exist functionals $x_j^* \in \ell_1^*$ such that $\|x_j^*\| \leqslant 2k$ and $x_j^* y_i = \delta_{ij}$ for all $i, j \in [n]$. Define an operator $T\colon F \to \ell_1$ by

$$Tz = z + \sum_{j=1}^{n} x_j^*(z)(x_j - y_j).$$

Obviously, $Ty_j = x_j$ for each $j \in [n]$, thus E is contained in the m-dimensional subspace $H := T(F)$ of ℓ_1. Moreover, for every $z \in F$ we have $\|z\| - 2kn\cdot\varepsilon\|z\| \leqslant \|Tz\| \leqslant \|z\| + 2kn\cdot\varepsilon\|z\|$, hence $(1-\delta)\|z\| \leqslant \|Tz\| \leqslant (1+\delta)\|z\|$ which yields $d_{\mathsf{BM}}(H, \ell_1^m) \leqslant (1+\delta)/(1-\delta)$. So, H is the desired perturbation of F and the proof is completed. □

Remark 12.8. The same proof works, with obvious changes, for every ℓ_p-space with any $p \in [1, \infty)$ and, more generally, for every $L_p(\mu)$-space with any non-negative measure μ. The assertion of Lemma 12.7 says that ℓ_1 belongs to the Lindenstrauss–Pedż″czydż″ski class of $\mathscr{L}_1$-spaces (more generally, every $L_p(\mu)$-space is a $\mathscr{L}_p$-space). The interested reader may consult, e.g., Chapter 5 in [LT73].