Combinatorics in Banach space theory

Lecture 12

The next lemma considerably strengthens the assertion of Lemma 12.6(b).

Lemma 12.9. *For every Banach space X and any $n \in \mathbb{N}$, either all the numbers $n^{-1}b_n(X)$, $c_n(X)$ and $d_n(X)$ are equal to 1, or they are all less than 1.*

Proof. In view of Lemma 12.6(b), we shall only prove that $c_n = 1$ implies $d_n = 1$ and $d_n = 1$ implies $n^{-1}b_n = 1$, for every $n \in \mathbb{N}$.

First, assume $c_n = 1$ and fix any $\delta > 0$. Then, there exist $x_1, \ldots, x_n \in X$ such that $\sum_{j=1}^n \|x_j\|^2 = n$ and

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|^2 \,\mathrm{d}t \geqslant (1-\delta)^2 \cdot n^2. \tag{12.1}$$

To show that $d_n = 1$ we need to give an upper estimate for $\max_j \|x_j\|$. Namely, we will show that

$$\max_{1 \leqslant j \leqslant n} \|x_j\| \leqslant 1 + 2(\delta n)^{1/2}. \tag{12.2}$$

To this end, choose $k_0 \in [n]$ such that $\|x_{k_0}\| = \max_j \|x_j\|$ and apply Minkowski's inequality to get

$$n^{1/2} \cdot \|x_{k_0}\| = \Big(\sum_{j=1}^n \|x_{k_0}\|^2 \Big)^{1/2} \leqslant \Big(\sum_{j=1}^n \big(\|x_{k_0}\| - \|x_j\|\big)^2 \Big)^{1/2} + \Big(\sum_{j=1}^n \|x_j\|^2 \Big)^{1/2}. \tag{12.3}$$

Moreover, by (12.1), we have

$$\begin{aligned}
\sum_{j=1}^n \big(\|x_{k_0}\| - \|x_j\|\big)^2 &\leqslant \sum_{1 \leqslant i,j \leqslant n} \big(\|x_i\| - \|x_j\|\big)^2 = 2n \cdot \sum_{j=1}^n \|x_j\|^2 - 2 \cdot \Big(\sum_{j=1}^n \|x_j\| \Big)^2 \\
&= 2n^2 - 2 \cdot \Big(\sum_{j=1}^n \|x_j\| \Big)^2 \leqslant 2n^2 - 2 \cdot \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|^2 \,\mathrm{d}t \\
&\leqslant 2n^2 - 2(1-\delta)^2 n^2 \leqslant 4\delta n^2,
\end{aligned}$$

which, jointly with (12.3), yields the claimed inequality (12.2). Combining this inequality with (12.1) we get

$$\Big(\int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|^2 \,\mathrm{d}t \Big)^{1/2} \geqslant \frac{1-\delta}{1 + 2(\delta n)^{1/2}} \cdot n \cdot \max_{1 \leqslant j \leqslant n} \|x_j\|,$$

whence

$$d_n \geqslant \frac{1-\delta}{1 + 2(\delta n)^{1/2}} \xrightarrow[\delta \to 0]{} 1.$$

In the second part of the proof, we assume $d_n = 1$ and we want to show that $n^{-1}b_n = 1$. By the definition of d_n, we have

$$\sup_{(x_j)_{j=1}^n \subset B_X} \frac{1}{2^n} \sum_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^2 = n^2. \tag{12.4}$$

For any sequence $(x_j)_{j=1}^n \subset B_X$ pick a sequence $(\hat{\varepsilon}_j)_{j=1}^n \in \{-1,1\}^n$ which minimises the norm of $\sum_j \hat{\varepsilon}_j x_j$, that is, $\|\sum_j \hat{\varepsilon}_j x_j\| \leqslant b_n$. Then, we have

$$\frac{1}{2^n} \sum_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^2 = \frac{1}{2^n} \left\| \sum_{j=1}^n \hat{\varepsilon}_j x_j \right\|^2 + \frac{1}{2^n} \sum_{\substack{\varepsilon_j = \pm 1 \\ (\varepsilon_j) \neq (\hat{\varepsilon}_j)}} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^2 \leqslant \frac{1}{2^n} \big(b_n^2 + (2^n - 1) n^2 \big).$$

Taking the supremum over all sequences $(x_j)_{j=1}^n \subset B_X$, and making use of (12.4), we obtain $n^2 \leqslant 2^{-n}(b_n^2 + (2^n - 1)n^2)$ which forces b_n to be equal to n and completes the proof. $\square$

Proof of Theorem 12.3. The equivalence **(i)** $\Leftrightarrow$ **(ii)** follows directly from Lemmas 12.4 and 12.5. Hence, both **(i)** and **(ii)** are in turn equivalent to **(iii)**, in view of Lemma 12.9. Now, assuming **(iii)** choose $k \in \mathbb{N}$ so that $\sigma := c_k < 1$. By the submultiplicativity of $(c_n)_{n=1}^\infty$ (Lemma 12.6(d)), we have $c_{k^r} \leqslant \sigma^r$ for each $r \in \mathbb{N}$. For an arbitrary $n \in \mathbb{N}$ we may write $n = k^r + s$ with some $r \in \mathbb{N}_0$ and $0 \leqslant s < k^{r+1} - k^r$. Then, by Lemma 12.6(a), we have

$$\frac{c_n}{c_{k^{r+1}}} \leqslant \sqrt{\frac{k^r + s + 1}{k^r + s} \cdot \ldots \cdot \frac{k^{r+1}}{k^{r+1} - 1}} = \sqrt{\frac{k^{r+1}}{k^r + s}} \leqslant \sqrt{\frac{k^{r+1}}{k^r}} = \sqrt{k},$$

whence $c_n \leqslant \sqrt{k} \cdot \sigma^{r+1}$ for each $n \in \mathbb{N}$, which **implies (iv)**. We have proved so far that all conditions **(i)-(iv)** are pairwise equivalent.

Now, assume that any of the conditions **(i)-(iv)** is satisfied (then, all of them are satisfied). We will show that if X contains ℓ_1^n's uniformly and $\lambda \geqslant 1$ is the uniformity constant, then $b_n \geqslant \lambda^{-1} n$ for every $n \in \mathbb{N}$ which will give a contradiction.

So, suppose there is a sequence $(E_n)_{n=1}^\infty$ of finite-dimensional subspaces of X such that for every $n \in \mathbb{N}$ there exists an isomorphism $U_n\colon \ell_1^n \to E_n$ such that $\|U_n\| = 1$ and $\|U_n^{-1}\| \leqslant \lambda$. Fix any $n \in \mathbb{N}$ and let $(\varepsilon_j)_{j=1}^n \in \{-1,1\}^n$ be a sequence of signs that minimises the norm of $\sum_{j=1}^n \varepsilon_j U_n(e_j)$. Then,

$$n = \left\| \sum_{j=1}^n \varepsilon_j e_j \right\|_{\ell_1^n} \leqslant \|U_n^{-1}\| \cdot \left\| \sum_{j=1}^n \varepsilon_j U_n(e_j) \right\|_X \leqslant \lambda \cdot \left\| \sum_{j=1}^n \varepsilon_j U_n(e_j) \right\|_X = \lambda b_n$$

and so the proof of the implication **(i)** $\Rightarrow$ **(v)** has been completed.

The implication **(v)** $\Rightarrow$ **(vi)** is trivial.

Finally, to complete the proof we shall show that **(vi)** $\Rightarrow$ **(i)**. Again, we prove it by contraposition. So, suppose that X is not B-convex. In light of Lemma 12.7, to guarantee ℓ_1 being finitely representable in X we shall merely show that X contains all ℓ_1^n's λ-uniformly, for every $\lambda > 1$ (i.e. for every $n \in \mathbb{N}$ and $\lambda > 1$ there exists an n-dimensional subspace E of X such that $d_{\mathsf{BM}}(E, \ell_1^n) \leqslant \lambda$).

Fix any $n \in \mathbb{N}$ and $\lambda > 1$, and set $\delta = 1 - \lambda^{-1} \in (0,1)$. Since X is not B-convex, there exist $x_1, \ldots, x_n \in B_X$ such that

$$\Big\| \sum_{j=1}^{n} \varepsilon_j x_j \Big\| \geqslant n - \delta \quad \text{for every } (\varepsilon_j)_{j=1}^{n} \in \{-1,1\}^n. \tag{12.5}$$

As shall be expected, the vectors $x_1, \ldots, x_n$ span an 'almost' isometric copy of ℓ_1^n inside X. In order to show this, we shall prove that the inequality

$$\frac{1}{\lambda} \cdot \sum_{j=1}^{n} |a_j| \leqslant \Big\| \sum_{j=1}^{n} a_j x_j \Big\| \leqslant \sum_{j=1}^{n} |a_j| \tag{12.6}$$

holds true for arbitrary scalars $a_1, \ldots, a_n$. In fact, this would mean that the sequence $(x_j)_{j=1}^n$ is equivalent to the canonical basis $(e_j)_{j=1}^n$ of ℓ_1^n in the sense that there is an isomorphism $T\colon \mathrm{span}\{x_j\}_{j=1}^n \to \ell_1^n$ satisfying $Tx_j = e_j$ for each $j \in [n]$. Moreover, the estimates (12.6) would also indicate that $\|T\| \leqslant \lambda$ and $\|T^{-1}\| \leqslant 1$, thus $d_{\mathsf{BM}}(\mathrm{span}\{x_j\}_{j=1}^n, \ell_1^n) \leqslant \lambda$, as desired. After these explanations we may proceed to the proof of inequalities (12.6).

The second inequality in (12.6) is obvious, since $x_j \in B_X$ for each $j \in [n]$. For proving the first one, we may assume with no loss of generality that $\sum_j |a_j| = 1$. Then, in view of (12.5), we have

$$\begin{aligned} n - \delta \leqslant \Big\| \sum_{j=1}^{n} \mathrm{sgn}(a_j) x_j \Big\| &= \Big\| \sum_{j=1}^{n} \big(\mathrm{sgn}(a_j) \cdot (1 - |a_j|)\big) x_j \Big\| \\ &\leqslant \sum_{j=1}^{n} (1 - |a_j|) + \Big\| \sum_{j=1}^{n} a_j x_j \Big\| = n - 1 + \Big\| \sum_{j=1}^{n} a_j x_j \Big\|, \end{aligned}$$

which gives the first inequality in (12.6). The proof is completed. □

Now, when we understand the essence of being B-convex pretty well, we shall quote the main result from Beck's paper [Bec62]. We will not prove it here, because it is beyond the main scope of our interest. However, it must be recalled, since it is a pioneering result which initiated the study of B-convex spaces.

Recall that if (Ω, Σ, μ) is a measure space and X is a Banach space, then a map $f\colon \Omega \to X$ is called *μ-measurable* (or *μ-strongly measurable*), provided that there exists a sequence $(g_n)_{n=1}^\infty$ of Σ-measurable step functions such that $\lim_{n\to\infty} \|f - g_n\| = 0$ holds true μ-almost everywhere⋆. The map f is Bochner integrable if and only if $\int_\Omega \|f\| \,\mathrm{d}\mu < \infty$; see [DU77, §II.2]. Now, if $(\Omega, \Sigma, \mathbb{P})$ is a probabilistic space, then any $\mathbb{P}$-measurable function $\xi\colon \Omega \to X$ is called a *random variable.* If it is Bochner integrable, then the integral $\int_\Omega \xi \,\mathrm{d}\mathbb{P}$ is called its *expectation value* and is denoted as $\mathbb{E}\xi$. We also define the *variation* $\mathbb{D}^2\xi$ of the random variable ξ by $\mathbb{D}^2\xi = \int_\Omega \|\xi - \mathbb{E}\xi\|^2 \,\mathrm{d}\mathbb{P}$. We say that ξ is *symmetric*, if there exists a measure preserving map $\phi\colon \Omega \to \Omega$ such that $\xi \circ \phi = -\xi$. A sequence $(\xi_n)_{n=1}^\infty$ of X-valued random variables is called *independent*, whenever for every finite sequence

⋆By the classical **Pettis' measurability theorem**, a function $f\colon \Omega \to X$ is μ-measurable if and only if it is *essentially separably valued* (i.e. for some μ-measure zero set $E \in \Sigma$ the range $f(\Omega \setminus E)$ is a separable subset of X) and *weakly μ-measurable* (i.e. for every $x^* \in X^*$ the scalar function $x^* \circ f$ is Σ-measurable); see [DU77, §II.1].

$n_1 < \ldots < n_k$ of natural numbers, and every choice of Borel sets $B_1, \ldots, B_k \subset X$, we have

$$\mathbb{P}(\xi_{n_1} \in B_1 \wedge \ldots \wedge \xi_{n_k} \in B_k) = \prod_{j=1}^{k} \mathbb{P}(\xi_{n_j} \in B_j).$$

The main results of Beck's paper [Bec62] may be summarised in the following theorem which, roughly speaking, asserts that the strong law of large numbers for X-valued random variables is valid if and only if X is B-convex.

Theorem 12.10 (Beck, 1962)**.** *Let $(\Omega, \Sigma, \mathbb{P})$ be a probabilistic space and X be a Banach space. If X is B-convex, then for every independent sequence $(\xi_n)_{n=1}^{\infty}$ of Bochner integrable random variables $\xi_n \colon \Omega \to X$ satisfying $\mathbb{E}\xi_n = 0$ for $n \in \mathbb{N}$ and $\sup_n \mathbb{D}^2 \xi_n < \infty$ we have*

$$\Big\| \frac{1}{n} \sum_{j=1}^{n} \xi_j \Big\| \xrightarrow[n\to\infty]{} 0 \qquad \mathbb{P}\text{-almost everywhere.}$$

Moreover, if X fails to be B-convex, then there exists an independent sequence $(\xi_n)_{n=1}^{\infty}$ of Bochner integrable, symmetric random variables $\xi_n \colon \Omega \to X$ satisfying $\mathbb{E}\xi_n = 0$ and $\|\xi_n\| \leqslant 1$ almost everywhere $(n \in \mathbb{N})$ and such that

$$\operatorname*{ess\,sup}_{\Omega} \ \limsup_{n\to\infty} \Big\| \frac{1}{n} \sum_{j=1}^{n} \xi_j \Big\| = 1.$$

At the end of this section, we derive another infinite-dimensional counterpart of the Lyapunov convexity theorem for vector measures taking values in B-convex Banach spaces. The proof is essentially the same as the proof of the quantitative Theorem 11.6; just instead of Steinitz's lemma we use the basic property of B-convex spaces.

Theorem 12.11 (V. Kadets, 1991)**.** *If $\mu \colon \Sigma \to X$ is a σ-additive, non-atomic vector measure of bounded variation, defined on a σ-algebra Σ and taking values in a B-convex Banach space X, then the closure of $\mu(\Sigma)$ is convex.*

Proof. Of course, it is enough to show that $\mathsf{co}(\mu(\Sigma)) = 0$. This will follow from Lemma 11.8, if we prove that for every $A \in \Sigma$ and $\varepsilon > 0$ there exists $A_\varepsilon \subset A$, $A_\varepsilon \in \Sigma$, such that $\|\mu(A_\varepsilon) - \frac{1}{2}\mu(A)\| \leqslant \varepsilon$. So, fix any $A \in \Sigma$ and $\varepsilon > 0$.

Let $M = |\mu|(\Omega)$ be the total variation of μ. In view of Proposition 11.2, applied to the variation of μ, there exists a partition $\{A_1, \ldots, A_n\} \in \Pi(A)$ such that $|\mu|(A_j) \leqslant \varepsilon$ for each $j \in [n]$ and n is the least natural number for which $n\varepsilon \geqslant |\mu|(A)$. Hence, $(n-1)\varepsilon < |\mu|(A) \leqslant M$, so $n < \varepsilon^{-1}M + 1 < 2\varepsilon^{-1}M$ for ε small enough. Now, repeating the calculation from the proof of Theorem 11.6, with the notation $A_J = \bigcup_{j\in J} A_j$ for $J \subset [n]$, we obtain

$$\min_{J\subset[n]} \Big\| \mu(A_J) - \frac{1}{2}\mu(A) \Big\| = \frac{1}{2} \min_{\varepsilon_j = \pm 1} \Big\| \sum_{j=1}^{n} \varepsilon_j \mu(A_j) \Big\| \leqslant \frac{1}{2}\, b_n \cdot \max_{1\leqslant j\leqslant n} \|\mu(A_j)\| \leqslant \frac{1}{2}\, b_n \varepsilon < M n^{-1} b_n.$$

By the characterisation of B-convexity (see Theorem 12.3), the last expression goes to zero as $n \to \infty$, thus the proof is completed. □

Remark 12.12. It is by no means straightforward to state what precisely the relation is between Theorem 12.11 and Uhl's Theorem 11.10. While it is easy to see that the latter does not follow from the former (for instance, ℓ_1 is a separable dual space which is not B-convex), it appears to be quite difficult to show that there exists any B-convex Banach space which is not reflexive (if it was not true, then Theorem 12.11 would be a direct consequence of Theorem 11.10). In 1964, after discovering that B-convexity and reflexivity have much in common, James conjectured that every B-convex space is necessarily reflexive. Some partial confirmation of this conjecture was given in terms of (k,ε)-convexity. We say that a Banach space X is (k,ε)-*convex* (for some $k \geqslant 2$ and $\varepsilon > 0$), if

$$\sup_{(x_j)_{j=1}^k \subset B_X} \ \min_{\varepsilon_j = \pm 1} \ \Big\| \sum_{j=1}^k \varepsilon_j x_j \Big\| \leqslant (1-\varepsilon)k.$$

James [Jam64] showed that $(2,\varepsilon)$-convexity implies reflexivity, for every $\varepsilon > 0$. In 1973, Giesy confirmed James' conjecture for Banach lattices ([Gi73]) and then he proved that for every $k \geqslant 3$ and $\varepsilon > 1 - \frac{9}{4}k$ every (k,ε)-convex Banach space must be reflexive. Finally, in 1974, James [Jam74] himself disproved his conjecture by constructing a non-reflexive Banach space $X_{\mathcal{J}}$ such that for some $\lambda > 1$ there is no isomorphism T from ℓ_1^3 into $X_{\mathcal{J}}$ satisfying

$$\frac{1}{\lambda} \cdot \|x\| \leqslant \|Tx\| \leqslant \lambda \|x\| \quad \text{for } x \in \ell_1^3$$

(hence, ℓ_1 is not finitely representable in $X_{\mathcal{J}}$ and therefore $X_{\mathcal{J}}$ is B-convex). He improved his construction later in [Jam78]. Consequently, Theorems 11.10 and 12.11 are incomparable.

13 B-convexity is a 3SP property

In Section 8 we described the general framework for the three-space problem. Now, we turn our attention to the question whether B-convexity is a 3SP property, to which an affirmative answer, in the category of Banach spaces, was obtained by Giesy [Gie66]. Precisely saying, if X, Y and Z are all Banach spaces which form a short exact sequence

$$0 \to Y \to Z \to X \to 0$$

and if X and Y are B-convex, then Z also must be B-convex. Giesy's result was later expanded by Kalton [Kal78] who showed that it is not essential to assume that Z is a Banach space (which is, in fact, equivalent to assuming that Z is locally convex; see Proposition 13.1 below and recall the classical result, e.g. [Rud91, Theorem 1.39], which says that a linear topological space is normed iff it is locally bounded and locally convex). Instead, we may merely assume that Z is a *locally bounded F-space* (that is, a linear topological space having a bounded zero neighbourhood and being completely metrisable). Therefore, for the time being, we move to the world of linear topological spaces (not necessarily locally convex) and we aim to prove in this section the Kalton–Giesy theorem, in particular to show how local convexity of Z follows from B-convexity of X and Y in the exact sequence above.

Let us start with a simple result, due to Roelcke and Dierolf [RD81], which justifies our interest in locally bounded F-spaces.

Proposition 13.1 (Roelcke, Dierolf, 1981)**.** *Let X be a linear topological space and Y be a closed subspace of X.*

(a) *If both Y and X/Y are F-spaces, then so is X.*
(b) *If both Y and X/Y are locally bounded, then so is X.*

Proof. **(a):** Recall that a linear topological space Z is metrisable if and only if it has a countable basis of zero neighbourhoods and in such a case there exists an invariant metric d on Z (i.e. $d(x+z, y+z) = d(x, y)$ for all $x, y, z \in Z$) which is consistent with the given topology on Z (see [Rud91, Theorem 1.24]). So, there exists a decreasing sequence $(V_n)_{n=1}^\infty$ of open subsets of X such that $(Y \cap V_n)_{n=1}^\infty$ is a basis of zero neighbourhoods in Y and $(\pi(V_n))_{n=1}^\infty$ is a basis of zero neighbourhoods in X/Y, where $\pi\colon X \to X/Y$ is the canonical map. Fix any zero neighbourhood $U \subset X$; we are to prove that $V_m \subset U$ for some $m \in \mathbb{N}$.

Pick any zero neighbourhood $W \subset X$ with $W + W \subset U$. There exist $m, n \in \mathbb{N}$ with $m > n$ such that
$$Y \cap V_n \subset W \quad \text{and} \quad \pi(V_m) \subset \pi(W \cap V_{n+1}),$$
hence $V_m \subset (W \cap V_{n+1}) + Y$. Fix any $v \in V_m$; $v = w + y$ for some $w \in W \cap V_{n+1}$ and $y \in Y$. Then
$$y = v - w \in Y \cap [V_m - (W \cap V_{n+1})] \subset Y \cap (V_m - V_{n+1})$$
and, consequently,
$$\begin{aligned} V_m &\subset (W \cap V_{n+1}) \cap [Y \cap (V_m - V_{n+1})] \subset (W \cap V_{n+1}) + [Y \cap (V_{n+1} - V_{n+1})] \\ &\subset (W \cap V_{n+1}) + (Y \cap V_n) \subset W + W \subset U. \end{aligned}$$
Therefore, $(V_n)_{n=1}^\infty$ is a (countable) basis of zero neighbourhoods in X.

Now, assume that Y and X/Y are F-spaces and let d_X be an invariant metric on X. Then, the formula
$$d_{X/Y}(\pi(x), \pi(y)) = \inf\{d_X(x - y, z)\colon z \in Y\} \tag{13.1}$$
defines an invariant metric on X/Y which is consistent with the quotient topology on X/Y (see [Rud91, Theorem 1.41]). It follows that if $(x_n)_{n=1}^\infty$ is a Cauchy sequence in X, then $(\pi(x_n))_{n=1}^\infty$ is a Cauchy sequence in X/Y. Hence, there exists $\xi \in X/Y$ such that $\lim_{n\to\infty} d_{X/Y}(\pi(x_n), \xi) = 0$. Choose any $x_0 \in X$ with $\xi = \pi(x_0)$. Then, in view of formula (13.1), there exists a sequence $(z_n)_{n=1}^\infty \subset Y$ satisfying $\lim_{n\to\infty} d_X(x_n - x_0, z_n) = 0$. The inequality
$$d_X(z_n, z_m) \leqslant d_X(x_n - x_0, z_n) + d_X(x_m - x_0, z_m) + d_X(x_n, x_m)$$
shows that $(z_n)_{n=1}^\infty$ is a Cauchy sequence in Y and hence it is convergent to some $z_0 \in Y$. Then $\lim_{n\to\infty} d_X(x_n - x_0, z_0) = 0$, thus the sequence $(x_n)_{n=1}^\infty$ converges to $x_0 + z_0$.

(b): Assume that both Y and X/Y are locally bounded. Then, there is a zero neighbourhood $U \subset X$ such that both $\pi(U)$ and $Y \cap U$ are bounded. Let $V \subset X$ be a balanced zero neighbourhood such that $V + V \subset U$. We claim that V is bounded. Assume not and let $(x_n)_{n=1}^\infty$ be a sequence in V such that $(\frac{1}{n}x_n)_{n=1}^\infty$ is unbounded. Then $\frac{1}{n}\pi(x_n) \to 0$ and hence $\frac{1}{n}x_n + y_n \to 0$ for some sequence $(y_n)_{n=1}^\infty \subset Y$. Therefore, for n's large enough we have $\frac{1}{n}x_n + y_n \in V$, so $y_n \in V - \frac{1}{n}V \subset V + V \subset U$, which means that $(y_n)_{n=1}^\infty$ is bounded and forces the sequence $(\frac{1}{n}x_n)_{n=1}^\infty$ to be bounded as well; a contradiction. □

To prepare for the proof of the Kalton–Giesy theorem we need to recall some material on geometric properties of locally bounded spaces.

Definition 13.2. Let X be a real or complex vector space. By a *quasi-norm* we mean any function $X \ni x \mapsto \|x\|$ that satisfies the following three conditions:

(i) $\|x\| > 0$ for every $x \in X$, $x \neq 0$;
(ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for every $x \in X$ and every scalar λ;
(iii) $\|x + y\| \leqslant k \cdot (\|x\| + \|y\|)$ for all $x, y \in X$, where $k \geqslant 1$ is a constant independent of x and y. Any such constant is called the *modulus of concavity* of X.

The class of quasi-normed spaces and the class of locally bounded spaces coincide. In fact, if X is quasi-normed by a quasi-norm $\|\cdot\|$, then the collection $\{\varepsilon B_X \colon \varepsilon > 0\}$, where $B_X = \{x \in X \colon \|x\| \leqslant 1\}$, is a base of bounded zero neighbourhoods for some linear topology on X. Conversely, if X is a locally bounded linear topological space, then the Minkowski functional of any bounded, balanced zero neighbourhood defines a quasi-norm on X which is consistent with the original topology.

For quasi-normed spaces the notions of equivalent norms, quotient space, operator norm *etc.* are defined in a very the same way as for normed spaces.

Definition 13.3. Let X be a quasi-normed space and $0 < p \leqslant 1$. We say that X is *locally p-convex*, provided that there exists a quasi-norm $\|\cdot\|$ on X, equivalent to the original one, such that

$$\|x + y\|^p \leqslant \|x\|^p + \|y\|^p \quad \text{for all } x, y \in X.$$

In such a case, the quasi-norm $\|\cdot\|$ is called a *p-norm.*

Let us now quote the fundamental result proved independently by Aoki and Rolewicz. The proof of the version presented below is explained in details in the hint to [Gra08, Exercise 1.4.6].

Theorem 13.4 (Aoki, 1942 & Rolewicz, 1957)**.** *Let $\|\cdot\|$ be a quasi-norm on a vector space X with modulus of concavity $k \geqslant 1$ and let $0 < p \leqslant 1$ be the solution of $(2k)^p = 2$. Then, there exists a p-norm $|\cdot|$ on X which is equivalent to $\|\cdot\|$. More precisely, the quasi-norm $\|\cdot\|$ satisfies the inequality*

$$\|x_1 + \ldots + x_n\|^p \leqslant 4\big(\|x_1\|^p + \ldots \|x_n\|^p\big) \quad \textit{for all } x_1, \ldots, x_n \in X$$

and the quasi-norm $|\cdot|$ given by the formula

$$|x| = \inf\left\{ \Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p} \colon x_1, \ldots, x_n \in X \textit{ and } x = \sum_{j=1}^n x_j \right\}$$

is a p-norm equivalent to $\|\cdot\|$.

Let X be a quasi-normed space and $\|\cdot\|$ be a quasi-norm on X. Certainly, all the formulas defining the quantities: $a_n(X)$, $b_n(X)$, $c_n(X)$ and $d_n(X)$ (see Section 12) make sense in this more general setting. We are ready to make the first essential step towards the proof of the Kalton–Giesy theorem.

Proposition 13.5 (Kalton, 1978)**.** *A quasi-normed space X is locally convex if and only if $\sup_n n^{-1} a_n(X) < \infty$.*

Proof. The 'only if' part is obvious. For the 'if' part the only thing to be proved is that the convex hull of the unit ball $\{x \in X\colon \|x\| \leqslant 1\}$ is bounded. So, assume there is a constant $C < \infty$ such that $a_n \leqslant Cn$ for each $n \in \mathbb{N}$ and fix arbitrary vectors $x_1, \ldots, x_n \in X$ with $\|x_j\| \leqslant 1$ (for $j \in [n]$) and arbitrary non-negative numbers $\alpha_1, \ldots, \alpha_n$ with $\sum_{j=1}^n \alpha_j = 1$.

For every $j \in [n]$ and $m \in \mathbb{N}$ set $k_{j,m} = \lfloor m\alpha_j \rfloor$. Plainly, $\lim_{m\to\infty} m^{-1}k_{j,m} = \alpha_j$. For every $m \in \mathbb{N}$ we have

$$\Big\|\sum_{j=1}^n k_{j,m}x_j\Big\| \leqslant a_{\sum_{j=1}^n k_{j,m}} \leqslant C \cdot \sum_{j=1}^n k_{j,m},$$

hence

$$\Big\|\sum_{j=1}^n \frac{k_{j,m}}{m}x_j\Big\| \leqslant C.$$

Letting $m \to \infty$, and denoting k the modulus of concavity, we arrive at

$$\begin{aligned}\Big\|\sum_{j=1}^n \alpha_j x_j\Big\| &\leqslant k \cdot \left(\Big\|\sum_{j=1}^n \frac{k_{j,m}}{m}x_j\Big\| + \Big\|\sum_{j=1}^n \Big(\alpha_j - \frac{k_{j,m}}{m}\Big)x_j\Big\|\right)\\ &\leqslant k \cdot C + k \cdot \Big\|\sum_{j=1}^n \Big(\alpha_j - \frac{k_{j,m}}{m}\Big)x_j\Big\| \xrightarrow[m\to\infty]{} k \cdot C,\end{aligned}$$

so the result follows. $\square$