Combinatorics in Banach space theory

Lecture 13

Proposition 13.6 (Kalton, 1978). *Let X be a quasi-normed space. If $\lim_{n\to\infty} n^{-1}b_n(X)=0$, then $\sup_n n^{-1}a_n(X)<\infty$ and hence X is isomorphic to a B-convex Banach space.*

Proof. First, by the Aoki–Rolewicz Theorem 13.4, we may (and we do) assume that the given norm on X is a p-norm for some $p\in(0,1]$ (that is, $\|x+y\|^p\leqslant\|x\|^p+\|y\|^p$ for all $x,y\in X$) because passing to an equivalent quasi-norm does not affect the condition $\lim_{n\to\infty} n^{-1}b_n(X)=0$.

We claim that the following inequality holds true:

$$a_{2n}^p\leqslant 2^p a_n^p+b_{2n}^p\quad\text{for every } n\in\mathbb{N}. \tag{13.1}$$

For let $(x_j)_{j=1}^{2n}$ be an arbitrary sequence in the unit ball of X and let $(\varepsilon_j)_{j=1}^{2n}\in\{-1,1\}^{2n}$ be chosen so that

$$\Bigg\|\sum_{j=1}^{2n}\varepsilon_j x_j\Bigg\|\leqslant b_{2n}.$$

Define $E^+=\{j\in[2n]\colon \varepsilon_j=+1\}$ and $E^-=\{j\in[2n]\colon \varepsilon_j=-1\}$ and assume, with no loss of generality, that $|E^+|\leqslant n$. Then, we have

$$\sum_{j=1}^{2n}x_j=2\sum_{j\in E^+}x_j-\sum_{j=1}^{2n}\varepsilon_j x_j,$$

hence

$$\Bigg\|\sum_{j=1}^{2n}x_j\Bigg\|^p\leqslant 2^p\cdot\Bigg\|\sum_{j\in E^+}x_j\Bigg\|^p+b_{2n}^p\leqslant 2^p a_n^p+b_{2n}^p$$

and passing to the supremum over all choices of $(x_j)_{j=1}^{2n}$ we get the claimed inequality.

Now, let $\alpha_n=2^{-n}a_{2^n}$ and $\beta_n=2^{-n}b_{2^n}$ for $n\in\mathbb{N}$, so that inequality (13.1) translates into

$$\alpha_{n+1}^p-\alpha_n^p\leqslant\beta_{n+1}^p\quad\text{for every } n\in\mathbb{N}. \tag{13.2}$$

In view of Lemmas 12.4 and 12.5, there are some constant $c>0$ and $q>1$ such that $b_n\leqslant cn^{1/q}$ for $n\in\mathbb{N}$, thus $\beta_n\leqslant c\cdot 2^{-n(1-1/q)}$. Consequently, the series $\sum_{n=1}^\infty\beta_n^p$ converges which, jointly with inequality (13.2), implies that the sequence $(\alpha_n)_{n=1}^\infty$ is bounded. This easily implies that the sequence $(n^{-1}a_n)_{n=1}^\infty$ is also bounded, as we shall now demonstrate.

So, let $d>0$ satisfy $\alpha_n\leqslant d$ for all $n\in\mathbb{N}$. For any natural number m write $m=2^k+r$ with $k\in\mathbb{N}_0$ and $0\leqslant r<2^k$. Then, $n^{-1}a_n\leqslant n^{-1}a_{2^{k+1}}=n^{-1}2^{k+1}\cdot\alpha_{k+1}\leqslant 2d$, so the first part of our assertion has been proved.

In light of Proposition 13.5, X is isomorphic to a Banach space which is B-convex because the condition $\lim_{n\to\infty}n^{-1}b_n(X)=0$ remains true for any equivalent (quasi-)norm on X. □

We are now prepared to prove the Kalton–Giesy theorem.

Theorem 13.7 (Giesy, 1966 & Kalton, 1978). *Being isomorphic to a B-convex Banach space is a* 3SP *property in the category of quasi-normed spaces. In other words, if X is a quasi-normed space and Y is a closed subspace of X such that both Y and X/Y are isomorphic to a B-convex Banach space, then X is also isomorphic to a B-convex Banach space.*

Proof. According to Theorem 12.3, we have $\lim_{n\to\infty} c_n(Y)=0$ and $\lim_{n\to\infty} c_n(X/Y)=0$, where the quasi-norms considered in Y and X/Y may be assumed to be actually norms (these two equalities remain true after passing to any equivalent quasi-norms). In view of Proposition 13.6, our job is to prove that $\lim_{n\to\infty} n^{-1}b_n(X)=0$. To this end, it is enough to show that for at least one $n\in\mathbb{N}$ we have $c_n<1$. Indeed, in such a case for all $x_1,\ldots,x_n\in B_X$ we have

$$\frac{1}{2^n}\sum_{\varepsilon_j=\pm1}\Bigg\|\sum_{j=1}^n \varepsilon_j x_j\Bigg\|^2\leqslant n^2c_n^2<n^2.$$

Hence, for some choice of $(\varepsilon_j)_{j=1}^n\in\{-1,1\}^n$ we must have $\|\sum_j\varepsilon_j x_j\|<n$, so $b_n<n$ and the desired inequality follows now from Lemmas 12.4 and 12.5.

Fix any $m,n\in\mathbb{N}$, $\{x_{ij}\colon i\in[m], j\in[n]\}\subset B_X$ and let $\{\theta_{ij}\colon i\in[m], j\in[n]\}$ be the set of the first mn Rademacher functions $(r_j)_{j=1}^\infty$ on $[0,1]$. We want to find an upper bound for c_{mn}, thus we shall estimate the average

$$\frac{1}{2^{mn}}\sum_{\varepsilon_{ij}=\pm1}\Bigg\|\sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}}\varepsilon_{ij}x_{ij}\Bigg\|^2=\int_0^1\Bigg\|\sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}}\theta_{ij}(t)x_{ij}\Bigg\|^2\,\mathrm{d}t. \tag{13.3}$$

Define

$$u_i(t)=\sum_{j=1}^n\theta_{ij}(t)x_{ij}\quad\text{for } i\in[m]\text{ and } t\in[0,1]$$

and

$$A(t)=\Bigg\{\int_0^1\Bigg\|\sum_{i=1}^m r_i(s)u_i(t)\Bigg\|^2\,\mathrm{d}s\Bigg\}^{1/2}\qquad\text{for } t\in[0,1],$$

and notice that the both sides of equality (13.3) are equal to $\int_0^1|A(t)|^2\,\mathrm{d}t$ (see the proof of Lemma 12.6(d)). Since each $u_i(t)$ is an X-valued step function, there are step functions $v_i\colon[0,1]\to Y$ such that $\|u_i(t)+v_i(t)\|\leqslant 2\|\pi(u_i(t))\|$ for each $i\in[m]$. Hence, denoting by k the modulus of concavity of the underlying quasi-norm on X, we obtain

$$\|v_i(t)\|\leqslant k\cdot\big(\|u_i(t)+v_i(t)\|+\|u_i(t)\|\big)\leqslant 3k\|u_i(t)\|\quad\text{for } i\in[m]\text{ and } t\in[0,1]$$

and

$$\Bigg\|\sum_{i=1}^m r_i(s)u_i(t)\Bigg\|\leqslant k\cdot\Bigg(\Bigg\|\sum_{i=1}^m r_i(s)\big(u_i(t)+v_i(t)\big)\Bigg\|+\Bigg\|\sum_{i=1}^m r_i(s)v_i(t)\Bigg\|\Bigg)\quad\text{for } s,t\in[0,1].$$

Therefore, using Minkowski's inequality and the definition of the numbers $c_m(X)$ and $c_m(Y)$, we get

$$\begin{aligned}
A(t) &\leqslant k \cdot \left\{ \int_0^1 \left\| \sum_{i=1}^m r_i(s)\big(u_i(t)+v_i(t)\big) \right\|^2 \mathrm{d}s \right\}^{1/2} + k \cdot \left\{ \left\| \sum_{i=1}^m r_i(s) v_i(t) \right\|^2 \mathrm{d}s \right\}^{1/2} \\
&\leqslant k\sqrt{m} \cdot \left[c_m(X) \cdot \left(\sum_{i=1}^m \|u_i(t)+v_i(t)\|^2 \right)^{1/2} + c_m(Y) \cdot \left(\sum_{i=1}^m \|v_i(t)\|^2 \right)^{1/2} \right] \\
&\leqslant k\sqrt{m} \cdot \left[2c_m(X) \cdot \left(\sum_{i=1}^m \|\pi(u_i(t))\|^2 \right)^{1/2} + 3kc_m(Y) \cdot \left(\sum_{i=1}^m \|u_i(t)\|^2 \right)^{1/2} \right].
\end{aligned}$$

Applying Minkowski's inequality once again, we get

$$\begin{aligned}
\left\{ \int_0^1 |A(t)|^2 \, \mathrm{d}t \right\}^{1/2} &\leqslant k\sqrt{m} \cdot \left[2c_m(X) \cdot \left(\sum_{i=1}^m \int_0^1 \|\pi(u_i(t))\|^2 \, \mathrm{d}t \right)^{1/2} \right. \\
&\qquad \left. + 3kc_m(Y) \cdot \left(\sum_{i=1}^m \int_0^1 \|u_i(t)\|^2 \, \mathrm{d}t \right)^{1/2} \right] \\
&\leqslant k\sqrt{mn} \cdot \left[2c_m(X)c_n(X/Y) \cdot \left(\sum_{i=1}^m \sum_{j=1}^n \|\pi(x_{ij})\|^2 \right)^{1/2} \right. \\
&\qquad \left. + 3kc_m(Y)c_n(X) \cdot \left(\sum_{i=1}^m \sum_{j=1}^n \|x_{ij}\|^2 \right)^{1/2} \right].
\end{aligned}$$

Consequently,

$$c_{mn}(X) \leqslant k \cdot \big(2c_m(X)c_n(X/Y) + 3kc_m(Y)c_n(X) \big) \quad \text{for all } m, n \in \mathbb{N},$$

whence, in particular,

$$c_{m^2}(X) \leqslant \big(2kc_m(X/Y) + 3k^2 c_m(Y) \big) \cdot c_m(X) \quad \text{for every } m \in \mathbb{N}.$$

By the assumption, the term in the bracket above goes to zero, thus $c_m(X) < 1$ for m's large enough. As it was explained before, this inequality concludes the proof. □

Let us stress once again that the local convexity of X in the above theorem was a part of the assertion, not a part of the assumption. This suggests to ask whether B-convexity of the subspace Y and the quotient X/Y is really essential to derive the local convexity of X. In other words, we may ask whether local convexity is a 3SP property in the category of locally bounded F-spaces. This way, we can smoothly move to our next subject, that is, K-spaces.

14 K-spaces

In order to build an example, which would show that Y and X/Y may be both locally convex despite the fact that X is not locally convex, it is most natural to set Y or X/Y to be ℓ_1, the canonical instance of a non-B-convex Banach space. In fact, we will be looking for a 'non-trivial' exact sequence of the form $0 \to \mathbb{R} \to Z \to \ell_1 \to 0$. To put this into action, we need a few notions from the theory of twisted sums.

Definition 14.1. Let X and Y be linear topological spaces. By a *twisted sum* of Y and X (the order is important) we mean any exact sequence of the form $0 \to Y \to Z \to X \to 0$.

If Z is a linear topological space that contains a subspace Y_1 isomorphic to Y such that $Z/Y_1 \simeq X$, then sometimes Z itself is called a twisted sum, provided that the structure of the underlying exact sequence is known from the context. However, let us stress that what is really hidden behind a twisted sum of Y and X is not only the space Z as above, but also the way in which Y embeds into Z. Plainly, we may always produce the trivial (in fact, direct) twisted sum $Y \oplus X$ together with the exact sequence

$$0 \longrightarrow Y \xrightarrow{\ i\ } Y \oplus X \xrightarrow{\ q\ } X \longrightarrow 0, \tag{14.1}$$

where i is the canonical injection from Y into $Y \oplus X$ and q is the canonical quotient map from $Y \oplus X$ onto X.

Definition 14.2. We say that two exact sequences (twisted sums) of linear topological spaces $0 \to Y \to Z_1 \to X \to 0$ and $0 \to Y \to Z_2 \to X \to 0$ are *equivalent* if there exists a continuous linear operator $T\colon Z_1 \to Z_2$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z_1 \arrow[r] \arrow[d, "T"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z_2 \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

is commutative.

One may easily prove that T in the diagram above must be bijective (the '3-lemma' in [CG97, §1.1]) and hence the Open Mapping Theorem forces T to be an isomorphism, provided that X and Y are F-spaces (then, in view of Proposition 13.1, Z_1 and Z_2 are also F-spaces).

Definition 14.3. We say that an exact sequence $0 \to Y \to Z \to X \to 0$ of linear topological spaces Y and X *splits* (or that this twisted sum is *trivial*), if it is equivalent to the direct sum (14.1).

Proposition 14.4. *Let X and Y be F-spaces and $0 \longrightarrow Y \xrightarrow{\ i\ } Z \xrightarrow{\ q\ } X \longrightarrow 0$ be an exact sequence. Splitting of this sequence is equivalent to any one of the following mutually equivalent conditions:*

(i) *i admits a retraction, that is, a continuous linear map $r\colon Z \to Y$ such that $r \circ i = I_Y$;*
(ii) *q admits a lifting, that is, a continuous linear map $s\colon X \to Z$ such that $q \circ s = I_X$;*
(iii) *the subspace $i(Y)$ is complemented in Z.*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, equal] & Z \arrow[r, "q"] \arrow[dl, dotted, "r"] & X \arrow[r] \arrow[d, equal] \arrow[l, phantom] & 0 \\ & Y & & X \arrow[ul, dotted, "s"'] & \end{tikzcd}$$

Proof. First, note that by the Open Mapping Theorem and the exactness of the sequence above, $i(Y)$ is isomorphic to Y. The clause (i) is nothing else but a precise way of saying that $i(Y)$ is complemented in Z, so clauses (i) and (iii) are tautologically equivalent.

We will show that clause (i) is equivalent to splitting of the given exact sequence; the analogous equivalence for clause (ii) is left to the reader.

Suppose that the exact sequence above splits and let T be an isomorphism which makes the diagram below commute:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, equal] & Z \arrow[r, "q"] \arrow[d, "T"] \arrow[dl, dotted, "r"'] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "j"] & Y \oplus X \arrow[r, "\pi_X"] & X \arrow[r] & 0
\end{tikzcd}$$

where j and π_X are the canonical injection and projection, respectively. Let $\pi_Y \colon Y \oplus X \to Y$ be the canonical projection onto the Y-coordinate and define $r \colon Z \to Y$ as $r = \pi_Y \circ T$. Then, $r \circ i = \pi_Y \circ T \circ i = \pi_Y \circ j = I_Y$, thus r is a retraction of i.

Conversely, assume $r \colon Z \to Y$ is a retraction of i. Then, $z - i(r(z)) \in \ker(r)$ and $i(r(z)) \in i(Y)$ for every $z \in Z$. Hence, we have the decomposition $Z = \ker(r) + i(Y)$ given by $z = i(r(z)) + (z - i(r(z)))$. Moreover, both $\ker(r)$ and $i(Y) = \ker(q)$ are closed subspaces of Z and whenever $z \in i(Y)$, $z = i(y)$ for some $y \in Y$, we have $z - i(r(z)) = i(y) - i(r(i(y))) = i(y) - i(y) = 0$ which proves that $Z = \ker(r) \oplus i(Y)$. Therefore, the restriction $q|_{\ker(r)}$ yields an injective continuous linear map from $\ker(r)$ onto X which, in view of the Open Mapping Theorem, is an isomorphism. Finally, using the identifications $\ker(r) \simeq X$ and $i(Y) \simeq Y$, it is straighforward to define an isomorphism $T \colon Z \to Y \oplus X$ for which the corresponding diagram, like the one above, commutes. $\square$

One more word of warning: a glance at condition (iii) may suggest that a given exact sequence, like the one in Proposition 14.4, splits whenever Y is (isomorphic to) a complemented subspace of Z. This is resoundingly false. One may even construct a non-trivial twisted sum which is of the form $0 \to X \to X \oplus X \to X \to 0$ with a certain, single Banach space X. Such an example follows from a negative answer to Harte's problem; see Appendix 1.10 in [CG97]. Again, while comparing a given exact sequence to the trivial one (14.1), not only is it important that Y is a complemented subspace of X, but also the way in which Y sits inside X.

The first construction of a non-trivial twisted sum of $\mathbb{R}$ and ℓ_1 was given, independently, by Kalton [Kal78], Ribe [Rib79] and Roberts [Rob77]. It involves a brilliant idea of using so-called quasi-linear maps which was later developed by Kalton and Peck [KP79] to give fundamentals of the theory of twisted sums. This is an appropriate moment to define that class of maps.

Definition 14.5. Let X and Y be quasi-normed spaces. We say that a map $f \colon X \to Y$ is *quasi-linear*, if it is homogeneous (i.e. $f(\lambda x) = \lambda f(x)$ for any $x \in X$ and any scalar λ) and there exists a constant $c < \infty$ such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant c\big(\|x\| + \|y\|\big) \quad \text{for all } x, y \in X. \tag{14.2}$$

The map f is called *zero-linear*, if it is homogeneous and satisfies the following, stronger inequality:

$$\left\| f\Big(\sum_{j=1}^{n} x_j\Big) - \sum_{j=1}^{n} f(x_j) \right\| \leqslant C \sum_{j=1}^{n} \|x_j\| \quad \text{for all } n \in \mathbb{N} \text{ and } x_1, \ldots, x_n \in X,$$

with some constant $C < \infty$.

It is easily verified that f is zero-linear if and only if there is a constant $C' < \infty$ such that
$$\Big\| \sum_{j=1}^{n} f(x_j) \Big\| \leqslant C' \sum_{j=1}^{n} \|x_j\| \quad \text{whenever } \sum_{j=1}^{n} x_j = 0,$$
and the the zero-linearity constant C in Definition 14.5 satisfies $C \leqslant 2C'$.

Theorem 14.6 (Kalton, Ribe, Roberts, 1977-79)**.** *There exists a non-locally convex twisted sum of* $\mathbb{R}$ *and* ℓ_1*.*

Proof. Let c_{00} be the linear space of all finitely supported real sequences, considered with the ℓ_1-norm, so that c_{00} becomes a dense subspace of ℓ_1. Our intention is to define a quasi-linear, but not zero-linear, map $F\colon c_{00} \to \mathbb{R}$.

First, define a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ by $\varphi(t) = t \cdot \log|t|$ (with the convention $0 \cdot \log 0 = 0$). For any $s, t > 0$ the concavity of the logarithm yields:
$$\begin{aligned} |\varphi(s+t) - \varphi(s) - \varphi(t)| &= s \cdot \log \frac{s+t}{s} + t \cdot \log \frac{s+t}{t} \\ &= (s+t) \cdot \Big(\frac{s}{s+t} \cdot \log \frac{s+t}{s} + \frac{t}{s+t} \cdot \log \frac{s+t}{t} \Big) \leqslant (s+t) \cdot \log 2. \end{aligned}$$
If $s > 0$, $t < 0$ and $s + t > 0$ then, since φ is an odd function, we get
$$|\varphi(s+t) - \varphi(s) - \varphi(t)| = |\varphi(s) - \varphi(-t) - \varphi(s+t)| \leqslant (s-t) \cdot \log 2.$$
Arguing similarly in any other case, we conclude that
$$|\varphi(s+t) - \varphi(s) - \varphi(t)| \leqslant \big(|s| + |t|\big) \cdot \log 2 \quad \text{for all } s, t \in \mathbb{R}. \tag{14.3}$$
Now, we define a map $f\colon c_{00} \to \mathbb{R}$ by
$$f(x) = \sum_{j=1}^{\infty} x_j \log|x_j| - \Big(\sum_{j=1}^{\infty} x_j \Big) \log \Big| \sum_{j=1}^{\infty} x_j \Big| \quad \text{for every } x = (x_j)_{j=1}^{\infty} \in c_{00}.$$
Fix any $x, y \in c_{00}$ and choose $N \in \mathbb{N}$ so that $x_j = y_j = 0$ for all $j > N$. In view of inequality (14.3), we have
$$\begin{aligned} &|f(x+y) - f(x) - f(y)| \\ &\quad = \Big| \sum_{j=1}^{N} \varphi(x_j + y_j) - \varphi\Big(\sum_{j=1}^{N} (x_j + y_j) \Big) - \sum_{j=1}^{N} \varphi(x_j) - \sum_{j=1}^{N} \varphi(y_j) + \varphi\Big(\sum_{j=1}^{N} x_j \Big) + \varphi\Big(\sum_{j=1}^{N} y_j \Big) \Big| \\ &\quad \leqslant \sum_{j=1}^{N} |\varphi(x_j + y_j) - \varphi(x_j) - \varphi(y_j)| + \Big| \varphi\Big(\sum_{j=1}^{N} (x_j + y_j) \Big) - \varphi\Big(\sum_{j=1}^{N} x_j \Big) - \varphi\Big(\sum_{j=1}^{N} y_j \Big) \Big| \\ &\quad \leqslant \log 2 \cdot \sum_{j=1}^{N} \big(|x_j| + |y_j|\big) + \log 2 \cdot \Bigg(\Big| \sum_{j=1}^{N} x_j \Big| + \Big| \sum_{j=1}^{N} y_j \Big| \Bigg) \\ &\quad \leqslant 2 \log 2 \cdot \sum_{j=1}^{N} \big(|x_j| + |y_j|\big) = 2 \log 2 \cdot \big(\|x\| + \|y\|\big), \end{aligned}$$

which means that f is quasi-linear.

Define $Z_0 = \mathbb{R} \oplus c_{00}$ and let

$$\|(t,x)\|_f := \|x\| + |t - f(x)| \quad \text{for } (t,x) \in Z_0.$$

The so-defined function $\|\cdot\|_f$ is obviously positive for non-zero $(t,x) \in Z_0$ and is positively homogeneous. Moreover, by the just proved quasi-linearity of f, we easily get

$$\|(s+t, x+y)\|_f \leqslant \big(1 + 2\log 2\big) \cdot \big(\|(s,x)\|_f + \|(t,y)\|_f\big) \quad \text{for } (s,x),(t,y) \in Z_0.$$

Therefore, $\|\cdot\|_f$ is a quasi-norm on Z_0 with modulus of concavity $1 + 2\log 2$. Let d be any invariant metric on Z_0 which is equivalent to that quasi-norm (for instance, $d(\boldsymbol{x},\boldsymbol{y}) = |||\boldsymbol{x}-\boldsymbol{y}|||^p$ for $\boldsymbol{x},\boldsymbol{y} \in Z_0$, where $|||\cdot|||$ is a p-norm produced by the Aoki–Rolewicz Theorem 13.4) and let Z be the completion of Z_0 with respect to that metric. It is evident that Z is a quasi-normed space with a quasi-norm (still denoted $\|\cdot\|_f$) defined in the obvious way. Plainly, the continuous and linear map $Z_0 \ni (t,x) \xmapsto{q} x \in c_{00} \subset \ell_1$ extends uniquely to a continuous and linear operator having the range ℓ_1 and the kernel $\{(t,0)\colon t \in \mathbb{R}\}$ which is in turn equal to the range of the map $\mathbb{R} \ni t \xmapsto{i} (t,0) \in Z_0 \subset Z$. Consequently, the quasi-normed space Z gives rise to a twisted sum

$$0 \longrightarrow \mathbb{R} \xrightarrow{i} Z \xrightarrow{q} \ell_1 \longrightarrow 0.$$

Suppose Z is locally convex, that is, there exists a norm $\|\cdot\|$ on Z which is equivalent to the quasi-norm $\|\cdot\|_f$; with no loss of generality we may assume that $\|\boldsymbol{x}\| \leqslant \|\boldsymbol{x}\|_f \leqslant M \cdot \|\boldsymbol{x}\|$ for every $\boldsymbol{x} \in Z$ and some constant $M \geqslant 1$. Then, for all $x_1, \ldots, x_n \in c_{00}$ which satisfy $\sum_j x_j = 0$ we have

$$\left|\sum_{j=1}^n f(x_j)\right| = \left\|\sum_{j=1}^n (f(x_j), x_j)\right\|_f \leqslant M \sum_{j=1}^n \|(f(x_j), x_j)\|_f = M \sum_{j=1}^n \|x_j\|,$$

which would imply that f is zero-linear. However, it is not true because $f(e_j) = 0$ for each $j \in \mathbb{N}$, $\sum_{j=1}^n \|e_j\| = n$ for any $n \in \mathbb{N}$, but $f(\sum_{j=1}^n e_j) = n \log n$. Consequently, Z is not locally convex and the proof is completed. $\square$

Definition 14.7. A Banach space X is called a *K-space*, if every twisted sum of $\mathbb{R}$ and X is trivial.

In other words, X is a K-space if and only if every quasi-normed space Z for which there exists an exact sequence $0 \to \mathbb{R} \to Z \to X \to 0$ is locally convex. Indeed, if that exact sequence splits, then Z is isomorphic to the direct sum $\mathbb{R} \oplus X$ which is locally convex. Conversely, if Z happens to be locally convex, then every finite-dimensional subspace of Z is complemented⋆; in particular, every copy of $\mathbb{R}$ inside Z is complemented which means that every exact sequence $0 \to \mathbb{R} \to Z \to X \to 0$ splits. The Kalton–Giesy Theorem 13.7 implies that every B-convex space (like each ℓ_p with $1 < p < \infty$) is a K-space, whereas Theorem 14.6 says that ℓ_1 is not a K-space. Now, the question arises whether there

⋆See, e.g, [Rud91, Lemma 4.21]. The proof of this fact requires the Hahn–Banach theorem. This is why the local convexity of Z is important. On the other hand, every finite-codimensional subspace of a linear topological space is always complemented.

are any K-spaces which are not B-convex. What springs to mind are the two natural candidates: c_0 and ℓ_∞. In order to prove that they are, in fact, both K-spaces, we need to develop quite sophisticated combinatorial techniques. But before that, we shall see that the method of the proof of Theorem 14.6 carries over to more general situations. The following result comes from [Kal78].

Theorem 14.8 (Kalton, 1978)**.** *Let X and Y be complete quasi-normed spaces and X_0 be a dense subspace of X. Then, the following assertions are equivalent:*

(i) *every twisted sum of Y and X splits;*

(ii) *for every quasi-linear map $f\colon X_0 \to Y$ there exist a linear (not necessarily continuous) map $h\colon X_0 \to Y$ and $L < \infty$ such that*

$$\|f(x) - h(x)\| \leqslant L\|x\| \quad \textit{for every } x \in X_0;$$

(iii) *there is a constant $B < \infty$ such that for every quasi-linear map $f\colon X_0 \to Y$ there exists a linear (not necessarily continuous) map $h\colon X_0 \to Y$ such that*

$$\|f(x) - h(x)\| \leqslant B \cdot \Delta(f)\|x\| \quad \textit{for every } x \in X_0,$$

where $\Delta(f)$ stands for the least possible constant $c \geqslant 0$ for which inequality (14.2) *holds true.*

Proof of the implication (iii) $\Rightarrow$ (i)*.* Assume Z is a (complete) quasi-normed space which produces an exact sequence

$$0 \longrightarrow Y \overset{i}{\longrightarrow} Z \overset{q}{\longrightarrow} X \longrightarrow 0$$

and let $T\colon \ker(q) \to Y$ be an isomorphism. There exists a linear (we do not require any continuity at the moment) lifting $\rho\colon X \to Z$ of q, that is, $q \circ \rho = I_X$. Since q is a surjective operator, the Open Mapping Theorem implies that there is a constant $C < \infty$ so that for any $x \in X$ we may pick $z \in Z$ for which $q(z) = x$ and $\|z\| \leqslant C\|x\|$. Therefore, there is a map $\sigma\colon X \to Z$ satisfying $q \circ \sigma = I_X$, $\sigma(\lambda x) = \lambda\sigma(x)$ and $\|\sigma(x)\| \leqslant C\|x\|$ for any $x \in X$ and any scalar λ. Define a map $f\colon X \to Y$ by $f(x) = T(\sigma(x) - \rho(x))$; note that the difference $\sigma(x) - \rho(x)$ always lies in $\ker(q)$. Plainly, f is homogeneous and

$$\begin{aligned}\|f(x+y) - f(x) - f(y)\| &= \|T(\sigma(x+y) - \sigma(x) - \sigma(y))\| \\ &\leqslant k^2C \cdot \|T\| \cdot (\|x+y\| + \|x\| + \|y\|) \leqslant 2k^3C \cdot \|T\| \cdot (\|x\| + \|y\|),\end{aligned}$$

where k is the modulus of concavity of the quasi-norm on Y. Hence, f is quasi-linear.

By (iii), there exist a constant $L < \infty$ and a linear map $h\colon X_0 \to Y$ such that

$$\|f(x) - h(x)\| \leqslant L\|x\| \quad \text{for } x \in X_0.$$

Define $s\colon X_0 \to Z$ by $s(x) = \rho(x) + T^{-1}h(x)$; this is a linear map. Since

$$s(x) = \rho(x) + T^{-1}f(x) + T^{-1}(h(x) - f(x)) = \sigma(x) + T^{-1}(h(x) - f(x)),$$

for every $x \in X_0$ we have

$$\|s(x)\| \leqslant k \cdot \big(\|\sigma(x)\| + \|T^{-1}\| \cdot \|h(x) - f(x)\|\big) \leqslant k \cdot \big(C + \|T^{-1}\| \cdot L\big) \cdot \|x\|,$$

which shows that s is continuous. Therefore, there is a continuous and linear extension $S\colon X \to Z$ of s. Of course, $q \circ S = I_X$ which means that S is a lifting of q. An appeal to Proposition 14.4 completes the proof. $\square$