Combinatorics in Banach space theory

Lecture 14

15 c_0 and ℓ_∞ are K-spaces

The crucial role in proving that c_0 and ℓ_∞ are K-spaces is played by the following Kalton–Roberts theorem [KR83] on nearly additive set functions.

Theorem 15.1 (Kalton, Roberts, 1983)**.** *There is an absolute constant $K < 45$ with the following property: For every set algebra $\mathscr{F}$ and every function $\nu\colon \mathscr{F} \to \mathbb{R}$ that satisfies*

$$|\nu(A \cup B) - \nu(A) - \nu(B)| \leqslant 1 \quad \textit{for } A, B \in \mathscr{F} \textit{ with } A \cap B = \varnothing$$

there exists a finitely additive measure $\mu\colon \mathscr{F} \to \mathbb{R}$ such that $|\nu(A) - \mu(A)| \leqslant K$ for each $A \in \mathscr{F}$.

We already know from Kalton's Theorem 14.8 that X being a K-space is equivalent to every real-valued quasi linear map on X being approximated, in an appropriate sense, by a linear map. The aim of the present section is to show how this type of stability effect for quasi-linear maps on c_0 and ℓ_∞ follows from Theorem 15.1, whose proof will be postponed until the next section.

As a matter of fact, the Kalton–Roberts theorem implies that every quotient of a $\mathscr{L}_\infty$-space is a K-space (see [KR82, Theorem 6.5]). We will not go into such generality here; however, we shall start with a lemma, which is an analogue of Lemma 12.7, and which may be restated by saying that $C(K)$-spaces, in particular $c_0 \simeq C[0,\omega]$ and $\ell_\infty \simeq C(\beta\mathbb{N})$, are $\mathscr{L}_\infty$-spaces.

Lemma 15.2. *Let K be a compact Hausdorff space. For every finite-dimensional subspace E of $C(K)$ and every $\varepsilon > 0$ there exists another finite-dimensional subspace H of $C(K)$ such that $E \subset H$ and $d_{\mathsf{BM}}(H, \ell_\infty^m) < 1 + \varepsilon$, where $m = \dim H$.*

Proof. Let $E = \operatorname{span}\{f_j\}_{j=1}^n$, where $\|f_j\| = 1$ for $1 \leqslant j \leqslant n$. Let $\{V_i\}_{i=1}^m$ be an open covering of K such that none of V_i's is contained in the union of all the others and every f_j varies by at most ε on each V_i, where $\varepsilon > 0$ is an arbitrarily small fixed number. Let $\{\varphi_i\}_{i=1}^m$ be a partition of unity corresponding to the covering $\{V_i\}_{i=1}^m$, that is, $\varphi_i \in C(K)$, $0 \leqslant \varphi_i \leqslant 1$, $\operatorname{supp}(\varphi_i) \subset V_i$ for each $i \in [m]$ and $\sum_{i=1}^m \varphi_i(x) \equiv 1$ on K. Plainly, for all scalars $\lambda_1, \ldots, \lambda_m$ we have $\|\sum_{i=1}^m \lambda_i\varphi_i\| = \max_{1\leqslant i\leqslant m} |\lambda_i|$, hence the subspace $F := \operatorname{span}\{\varphi_i\}_{i=1}^m$ of $C(K)$ is isometrically isomorphic to ℓ_∞^m. To see that each f_j lies near to H, for every $i \in [m]$ pick any number $\xi_{ij} \in f_j(V_i)$ and define $g_j = \sum_{i=1}^m \xi_{ij}\varphi_i$. For any $x \in K$ let $I_x = \{i \in [m]\colon x \in V_i\}$; then

$$\big|f_j(x) - g_j(x)\big| = \Big|f_j(x) - \sum_{i\in I_x} \xi_{ij}\varphi_i(x)\Big| = \Big|\sum_{i\in I_x} \varphi_i(x)\cdot(f_j(x) - \xi_{ij})\Big| \leqslant \varepsilon.$$

Therefore $\|f_j - g_j\| \leqslant \varepsilon$, so each f_j is 'almost' contained in the isometric copy F of ℓ_∞^m. The rest of the argument, which produces a small perturbation of F containing E, follows the line of the proof of Lemma 12.7. □

Remark 15.3. To be precise, the property described in the assertion of Lemma 15.2 defines $\mathscr{L}_{\infty,1+\varepsilon}$-spaces (with any $\varepsilon > 0$). If we replace $1+\varepsilon$ by some fixed number $\lambda \geqslant 1$, then the resulting condition defines $\mathscr{L}_{\infty,\lambda}$-spaces, and $\mathscr{L}_{\infty}$-spaces are defined as those being $\mathscr{L}_{\infty,\lambda}$ for some $\lambda \geqslant 1$. Since c_0 is not *isometrically* isomorphic to $C[0,\omega]$, we cannot conclude directly from Lemma 15.2 that it is $\mathscr{L}_{\infty,1+\varepsilon}$ for every $\varepsilon > 0$. Nonetheless, it is easily seen that the 'partition' + 'perturbation' argument goes through for every $C_0(K)$-space* with K being locally compact and Hausdorff. Therefore c_0, being isometrically isomorphic to $C_0[0,\omega)$, is a $\mathscr{L}_{\infty,1+\varepsilon}$-space, with every $\varepsilon > 0$.

Now, using the Kalton–Roberts theorem, we prove that quasi-linear maps defined on ℓ_∞^n-spaces may be approximated by linear ones. Subsequently, as a quick corollary, we will prove that $\mathscr{L}_\infty$-spaces are K-spaces.

Proposition 15.4 (Kalton, Roberts, 1983)**.** *Let Ω be a finite, non-empty set and let $f\colon \ell_\infty(\Omega) \to \mathbb{R}$ be a quasi-linear map with quasi-linearity constant c (that is, f and c satisfy inequality* (14.2)*). Then, there exists a linear map $h\colon \ell_\infty(\Omega) \to \mathbb{R}$ such that*

$$|f(x) - h(x)| \leqslant 188\, c \cdot \|x\| \quad \textit{for every } x \in \ell_\infty(\Omega).$$

Proof. Define $\nu\colon \mathcal{P}\Omega \to \mathbb{R}$ by $\nu(A) = f(\mathbb{1}_A)$. Then, for all disjoint $A, B \subset \Omega$ we have

$$|\nu(A\cup B) - \nu(A) - \nu(B)| = |f(\mathbb{1}_A + \mathbb{1}_B) - f(\mathbb{1}_A) - f(\mathbb{1}_B)| \leqslant c\cdot(\|\mathbb{1}_A\| + \|\mathbb{1}_B\|) \leqslant 2c$$

and hence Kalton–Roberts Theorem 15.1 produces a finitely additive measure $\mu\colon \mathcal{P}\Omega \to \mathbb{R}$ such that $|\nu(A) - \mu(A)| \leqslant 90c$ for every $A \subset \Omega$. Let $h\colon \ell_\infty(\Omega) \to \mathbb{R}$ be the natural linear extension of ν and let $g = f - h$. Of course, g is quasi-linear with the quasi-linearity constant c and, moreover, $|g(\mathbb{1}_A)| \leqslant 90c$ for every $A \subset \Omega$. We shall show that this implies $|g(x)| \leqslant 188\, c \cdot \|x\|$, for every $x \in \ell_\infty(\Omega)$, which will complete the proof.

Every $x \in \ell_\infty(\Omega)$ may be written as $x = \sum_{\alpha\in\Omega} x(\alpha) e_\alpha$, where $e_\alpha = \mathbb{1}_{\{\alpha\}}$. Notice that for every $A \subset \Omega$ we have $\|\sum_{\alpha\in A} x(\alpha)e_\alpha\| \leqslant \|x\|$, thus applying the quasi-linearity inequality $(|\Omega| - 1)$ times, we arrive at

$$\Big| g(x) - \sum_{\alpha\in\Omega} x(\alpha) g(\mathrm{e}_\alpha)\Big| \leqslant 2(|\Omega| - 1)\, c \cdot \|x\|,$$

thus

$$|g(x)| \leqslant 2\big(|\Omega| + 45|\Omega| - 1\big)\, c\cdot\|x\|. \tag{15.1}$$

Now, for $0 \leqslant x \leqslant \mathbb{1}_\Omega$ and each $m \in \mathbb{N}$ we may find $A_1, \ldots, A_m \subset \Omega$ such that

$$\Big\| x - \sum_{k=1}^m \frac{1}{2^k}\mathbb{1}_{A_k}\Big\| \leqslant 2^{-m}. \tag{15.2}$$

Moreover, using quasi-linearity of g recursively (see Problem 5.4), we get

$$\Big| g\Big(\sum_{k=1}^m \frac{1}{2^k}\mathbb{1}_{A_k}\Big) - \sum_{k=1}^m \frac{1}{2^k} g(\mathbb{1}_{A_k})\Big| \leqslant c\cdot\sum_{k=1}^m \frac{k}{2^k} \leqslant 2c,$$

*We say that a scalar-valued function f, defined on a locally compact Hausdorff space K, *vanishes at infinity*, provided that for every $\varepsilon > 0$ there exists a compact set $H \subset K$ such that $|f(x)| < \varepsilon$ for every $x \in K \setminus H$. The space $C_0(K)$ consists of all continuous functions vanishing at infinity and is equipped with the supremum norm.

hence

$$\left|g\left(\sum_{k=1}^{m}\frac{1}{2^k}\mathbb{1}_{A_k}\right)\right| \leqslant 2c + 90c = 92c. \tag{15.3}$$

Consequently, combining (15.1), (15.2) and (15.3) we get

$$\begin{aligned}|g(x)| &\leqslant \left|g(x) - g\left(\sum_{k=1}^{m}\frac{1}{2^k}\mathbb{1}_{A_k}\right) - g\left(x - \sum_{k=1}^{m}\frac{1}{2^k}\mathbb{1}_{A_k}\right)\right| \\ &\qquad + \left|g\left(\sum_{k=1}^{m}\frac{1}{2^k}\mathbb{1}_{A_k}\right)\right| + \left|g\left(x - \sum_{k=1}^{m}\frac{1}{2^k}\mathbb{1}_{A_k}\right)\right| \\ &\leqslant \big(1+2^{-m}\big)c + 92c + \big(|\Omega| + 45|\Omega| - 1\big)\cdot 2^{-m+1}\, c\end{aligned}$$

and, letting $m\to\infty$, we obtain $|g(x)| \leqslant 93c$.

Finally, for every $x\in \ell_\infty(\Omega)$ we may write $x = x^+ - x^-$, where $x^+, x^- \geqslant 0$ and $\|x^+\|, \|x^-\| \leqslant \|x\|$ and then we have

$$|g(x)| \leqslant c\cdot(\|x^+\| + \|x^-\|) + |g(x^+)| + |g(x^-)| \leqslant (2c + 2\cdot 93c)\cdot\|x\| = 188\,c\cdot\|x\|,$$

as required. □

Theorem 15.5 (Kalton, Roberts, 1983)**.** *Every $\mathscr{L}_\infty$-space is a K-space. In particular, c_0 and ℓ_∞ are K-spaces.*

Proof. Let X be an arbitrary Banach space which is a $\mathscr{L}_\infty$-space, that is, there is a constant $\lambda\geqslant 1$ such that for every finite-dimensional subspace E of X we may find a further finite-dimensional subspace H of X such that $E\subset H$ and $d_{\mathsf{BM}}(H,\ell_\infty^m) < \lambda$, where $m = \dim H$, which implies that there is an isomorphism $T\colon H\to\ell_\infty^m$ with $\|T\|\cdot\|T^{-1}\|\leqslant\lambda$. In view of (a part of) Kalton's Theorem 14.8, we are to prove that for every quasi-linear map $f\colon X\to\mathbb{R}$ there exists a linear map $h\colon X\to\mathbb{R}$ such that $|f(x)-h(x)|\leqslant k\,c\|x\|$ for $x\in X$, where $k<\infty$ is some constant and c is the quasi-linearity constant of f.

Fix any finite-dimensional space $E\subset X$ and let H and T be as above. Then, $f\circ T^{-1}$ is a quasi-linear map on ℓ_∞^m with the quasi-linearity constant not exceeding $\|T^{-1}\|\cdot c$. By Proposition 15.4, there is a linear map $g\colon \ell_\infty^m\to\mathbb{R}$ satisfying

$$\big|f\circ T^{-1}(y) - g(y)\big| \leqslant 188\,c\,\|T^{-1}\|\cdot\|y\| \quad \text{for every } y\in\ell_\infty^m.$$

Define a linear map $h_E\colon E\to\mathbb{R}$ by $h_E(x) = g(Tx)$. Then

$$\big|f(x) - h_E(x)\big| \leqslant 188\,c\,\|T\|\cdot\|T^{-1}\|\cdot\|x\| \leqslant 188\,c\,\lambda\|x\| \quad \text{for every } x\in E.$$

Now, let $\mathscr{E}$ be the collection of all finite-dimensional subspaces of X, directed by inclusion and consider the net $(\widetilde{h}_E)_{E\in\mathscr{E}}$, where $\widetilde{h}_E(x) = h_E(x)$ if $x\in E$ and $\widetilde{h}_E(x)=0$ otherwise. Notice that for every $E\in\mathscr{E}$ and every $x\in X$ we have

$$\big|\widetilde{h}_E(x)\big| \leqslant \big|f(x)\big| + 188\,c\,\lambda\|x\| =: \rho_x,$$

which means that each $\widetilde{h}_E$ belongs to the (compact) Cartesian product $\prod_{x\in X}[-\rho_x, +\rho_x]$. Let h be a limit of any convergent subnet of $(\widetilde{h}_E)_{E\in\mathscr{E}}$. Obviously, h is linear and satisfies

$$\big|f(x) - h(x)\big| \leqslant 188\,c\lambda\|x\| \quad \text{for every } x\in X,$$

which completes the proof. □

16 The Kalton–Roberts theorem on nearly additive set functions

This whole section is devoted to the proof of Theorem 15.1. We begin with introducing the notion of *concentrator* and proving an existence result due to Pippenger [Pip77] with the aid of the probabilistic method. In the sequel, Pippenger's theorem will produce some useful estimates for submeasures. For any $m, p \in \mathbb{N}$ and any multi-valued mapping $R\colon [m] \to \mathcal{P}[p]$ we set

$$R[E] = \bigcup_{j\in E} R(j) \quad \text{for } E \subset [m].$$

Definition 16.1. Let $m, p, q, r \in \mathbb{N}$ and $m \geqslant p \geqslant q$. Then, a map $R\colon [m] \to \mathcal{P}[p]$ is called an (m, p, q, r)-*concentrator*, provided that:

(c1) $\displaystyle\frac{1}{m}\sum_{j=1}^{m} |R(j)| \leqslant r$;

(c2) $|R[E]| \geqslant |E|$ for every $E \subset [m]$ with $|E| \leqslant q$.

An (m, p, q, r)-concentrator may be also regarded as a bipartite graph with m inputs and p outputs; the edges go only between the set of all inputs, identified with $[m]$, and the set of all outputs, identified with $[p]$. The set $R(j)$ is simply the set of all outputs that are connected to the input j. Condition (c1) says that we do not have more than r outputs on average, whereas conditon (c2) says that every at most q-element subcollection of $(R(1), \dots, R(m))$ satisfies Hall's condition (see Problem 5.5), thus for every $k \leqslant q$ and every set of k inputs there exists a k-*flow* (a set of pairwise disjoint edges) from the given inputs into a certain set of k outputs.

Theorem 16.2 (Pippenger, 1977)**.** *For every $m \in \mathbb{N}$ there exists a $(6m, 4m, 3m, 6)$-concentrator.*

Proof. In fact, we will show that there exists a bipartite graph having all the desired properties and for which every input has out degree at most 6. Of course, this will guarantee that the condition corresponding to (c1) is fulfilled.

Let $\mathcal{M} = \{0, 1, \dots, 36m - 1\}$ and let π be any permutation of $\mathcal{M}$ to which we attach a bipartite graph $\mathsf{G}(\pi)$ defined as follows. The sets of inputs and outputs are $\{0, 1, \dots, 6m - 1\}$ and $\{0, 1, \dots, 4m - 1\}$, respectively, and for every $x \in \mathcal{M}$ we join $(x \bmod 6m)$ with $(\pi(x) \bmod 4m)$. Since every residue class modulo $6m$ has exactly 6 elements in $\mathcal{M}$, each input has out-degree at most 6. Therefore, $\mathsf{G}(\pi)$ will always satisfy the condition corresponding to (c1). Similarly, since every residue class modulo $4m$ has exactly 9 elements in $\mathcal{M}$, each output has in-degree at most 9.

The graph $\mathsf{G}(\pi)$ is called 'bad', whenever there exists $k \leqslant 3m$ and a set E of k inputs such that the set of all outputs connected to E has at most k elements; $\mathsf{G}(\pi)$ is called 'good' in the opposite case. Any good graph would do the job, so we shall prove that the fraction of all permutations π, for which $\mathsf{G}(\pi)$ is bad, is less than 1.

For any $n, r \in \mathbb{N}$ we denote $[n]_r$ the descending product $n(n-1)\cdot\ldots\cdot(n-r+1)$. Fix, for a moment, any set A of k inputs and any set B of k outputs. They correspond to appropriate sets $\mathcal{A}, \mathcal{B} \subset \mathcal{M}$, where $|\mathcal{A}| = 6k$ and $|\mathcal{B}| = 9k$. Given a permutation π of $\mathcal{M}$, every edge in $\mathsf{G}(\pi)$ that is directed out of A hits B if and only if π sends every member of $\mathcal{A}$ into $\mathcal{B}$. There are exactly $[9k]_{6k} \cdot (36m-6k)!$ such permutations. For any fixed $k \leqslant 3m$ there are also $\binom{6m}{k}$ choices for A and $\binom{4m}{k}$ choices for B. Consequently, when picking randomly a permutation π of $\mathcal{M}$, the probability that π will produce a bad graph $\mathsf{G}(\pi)$ is at most

$$I_m = \frac{1}{(36m)!}\sum_{k=1}^{3m}\binom{6m}{k}\binom{4m}{k}\cdot[9k]_{6k}\cdot(36m-6k)! = \sum_{k=1}^{3m}\frac{\binom{6m}{k}\binom{4m}{k}\binom{9k}{6k}}{\binom{36m}{6k}}.$$

In order to estimate I_m, observe that

$$\binom{36m}{6k} \geqslant \binom{6m}{k}\binom{4m}{k}\binom{26m}{4k}$$

because there are more ways of choosing $6k$ elements out of $36m$ than the ways of choosing k out of the first $6m$, k out of the next $4m$ and $4k$ out of the last $26m$. Therefore,

$$I_m \leqslant J_m := \sum_{k=1}^{3m}\frac{\binom{9k}{64}}{\binom{26m}{4k}}\;;$$

let L_k stand for the kth factor in the product above. We claim that the largest factor, among $L_1,\dots,L_{3m}$, is either the first one or the last one. To see this, write L_{k+1}/L_k in the form

$$\frac{(9k+9)\cdot\ldots\cdot(9k+7)(9k+6)\cdot\ldots\cdot(9k+1)(4k+4)(4k+3)\cdot\ldots\cdot(4k+1)}{(6k+6)\cdot\ldots\cdot(6k+1)\qquad(3k+3)\cdot\ldots\cdot(3k+1)(26m-4k)\cdot\ldots\cdot(26m-4k-3)}$$

and notice that each fraction, with numerator and denominator being vertically aligned, is an increasing function of the variable k. Hence, L_{k+1}/L_k is increasing as well, which means that $L_{k-1}L_{k+1} \geqslant L_k^2$. Consequently, the largest factor in J_m is, indeed, either L_1 or L_{3m}. Let us distinguish these two cases.

Case 1. If $L_1 = \max\{L_1,\dots,L_{3m}\}$, then

$$J_m \leqslant 3mL_1 = 3m\frac{\binom{9}{6}}{\binom{26m}{4}} = \frac{3024}{13(26m-1)(26m-2)(26m-3)} < 1$$

for every $m \in \mathbb{N}$.

Case 2. If $L_{3m} = \max\{L_1,\dots,L_{3m}\}$, then

$$J_m \leqslant 3mL_{3m} = 3m\frac{\binom{27m}{18m}}{\binom{26m}{12m}} = 3m\cdot\frac{(27m)!(12m)!(14m)!}{(18m)!(9m)!(26m)!}\,.$$

Now, Stirling's formula

$$(2\pi n)^{1/2}e^{-n}n^n \leqslant n! \leqslant e^{1/12n}(2\pi n)^{1/2}e^{-n}n^n,$$

jointly with the inequality $e^x \leqslant (1-x)^{-1}$, for $x \in (0,1)$, gives

$$(2\pi n)^{1/2} e^{-n} n^n \leqslant n! \leqslant \Big(\frac{12n}{12n-1}\Big) \cdot (2\pi n)^{1/2} e^{-n} n^n.$$

Therefore,

$$3mL_{3m} \leqslant 3m \cdot \Big(\frac{324m}{324m-1}\Big)\Big(\frac{144m}{144m-1}\Big)\Big(\frac{168m}{168m-1}\Big)\Big(\frac{27 \cdot 12 \cdot 14}{18 \cdot 9 \cdot 26}\Big)^{1/2}\Big(\frac{27^{27} \cdot 12^{12} \cdot 14^{14}}{18^{18} \cdot 9^9 \cdot 26^{26}}\Big)^m,$$

which is less than 1 for $m = 3$. Moreover, if m is increased by 1, then the first factor gets multiplied by at most $4/3$, whereas the next three factors decrease and the last factor decreases by a factor which exceeds 2. Hence, the right-hand side of the above inequality is a decreasing function of m and, consequently, $I_m < 1$ for all $m \geqslant 3$. It may be easily verified that also $I_1 < 1$ and $I_2 < 1$. □

For any $r \in \mathbb{N}$ and $\delta, \eta \in (0,1)$ we shall say that $H(r,\delta,\eta)$ *holds true* if and only if there exist sequences $(m_k)_{k=1}^\infty$, $(p_k)_{k=1}^\infty$ and $(q_k)_{k=1}^\infty$ of natural numbers such that $m_k \to \infty$ and for each $k \in \mathbb{N}$ we have: $p_k/m_k \leqslant \delta$, $q_k/m_k \geqslant \eta$ and there exists an (m_k, p_k, q_k, r)-concentrator. For $r \in \mathbb{N}$ and $\eta \in (0,1)$ we set

$$\vartheta(r,\eta) = \inf\big\{\delta \in (0,1)\colon H(r,\delta,\eta) \text{ holds true}\big\}.$$

Now, the following inequality follows immediately from Pippenger's theorem.

Corollary 16.3. $\vartheta(6, \frac{1}{2}) \leqslant \frac{2}{3}$.