Combinatorics in Banach space theory

Lecture 15

Definition 16.4. Let $\mathscr{F}$ be a set algebra. By a *submeasure* we mean any function $\varphi\colon \mathscr{F}\to\mathbb{R}$ that satisfies the following three conditions:

(a) $\varphi(\varnothing)=0$;
(b) $\varphi(A)\leqslant\varphi(B)$ for all $A,B\in\mathscr{F}$ with $A\subset B$;
(c) $\varphi(A\cup B)\leqslant\varphi(A)+\varphi(B)$ for all $A,B\in\mathscr{F}$.

Lemma 16.5 (Kalton, Roberts, 1983)**.** *Let $\mathscr{F}$ be an algebra of subsets of Ω and $\varphi\colon\mathscr{F}\to\mathbb{R}$ be a submeasure such that for some constants $\alpha,\beta\geqslant 0$ and arbitrary mutually disjoint sets $A_1,\dots,A_n\in\mathscr{F}$ we have $\sum_{j=1}^n\varphi(A_j)\leqslant\alpha n+\beta$. Then, whenever $B_1,\dots,B_m\in\mathscr{F}$ satisfy $\frac{1}{m}\sum_{j=1}^m\mathbb{1}_{B_j}\geqslant(1-\varepsilon)\mathbb{1}_\Omega$ for some $\varepsilon>0$, we have*

$$\frac{1}{m}\sum_{j=1}^m\varphi(B_j)\geqslant\varphi(\Omega)-\alpha r-\beta\vartheta(r,\varepsilon)\quad\textit{for every } r\in\mathbb{N},\ r\geqslant 3. \tag{16.1}$$

Proof. Fix any $r\in\mathbb{N}$, $r\geqslant 3$, and let $B_1,\dots,B_m\in\mathscr{F}$ satisfy the above condition. First, we assume that there exists an (m,p,q,r)-concentrator $R\colon[m]\to\mathcal{P}[p]$, where $q/m\geqslant\varepsilon$. We shall then show that inequality (16.1) holds true with p/m in the place of $\vartheta(r,\varepsilon)$.

Let $\mathscr{E}=\{E\subset[m]\colon |E|\leqslant q\}$. By the condition (c2) of the Definition 16.1, and the Hall marriage lemma (see Problem 5.5), for every $E\in\mathscr{E}$ there exists a one-to-one map $f_E\colon E\to[p]$ such that $f_E(j)\in R(j)$ for each $j\in E$. For every $E\in\mathscr{E}$ define

$$C_E=\bigcap_{k\in E}(\Omega\setminus B_k)\cap\bigcap_{k\notin E}B_k;$$

note that our assumption on $B_1,\dots,B_m$ yields $\bigcup_{E\in\mathscr{E}}C_E=\Omega$. Indeed, every element of Ω is covered by at least $m(1-\varepsilon)\geqslant m-q$ sets among $B_1,\dots,B_m$, that is, it belongs to at most q sets among $\Omega\setminus B_1,\dots,\Omega\setminus B_m$, which makes it possible to find an $E\in\mathscr{E}$ for which C_E contains the element in question.

For any $i\in[m]$ and $j\in[p]$ define

$$A_{ij}=\bigcup_{\substack{i\in E\\ f_E(i)=j}}C_E;$$

if such a set is non-empty, then necessarily $j\in R(i)$ which, in view of the condition (c1) of Definition 16.1, may happen for at most rm pairs $(i,j)\in[m]\times[p]$. Moreover, notice that for any fixed $j\in[p]$ the sets $A_{1j},\dots,A_{mj}$ are mutually disjoint because whenever $C_E\subset A_{ij}$ and $C_E\subset A_{kj}$ we have $f_E(i)=j=f_E(k)$, thus $i=k$. Let n_j be the number of all non-empty sets among $A_{1j},\dots,A_{mj}$; then

$$\sum_{i=1}^m\varphi(A_{ij})\leqslant\alpha n_j+\beta\quad\text{for each } j\in[p],$$

whence

$$\sum_{j=1}^{p}\sum_{i=1}^{m}\varphi(A_{ij}) \leqslant \alpha\sum_{j=1}^{p}n_j + \beta p \leqslant \alpha m r + \beta p.$$

We are ready to estimate the average of $\varphi(B_i)$'s. To this end, observe that since $\bigcup_{E\in\mathscr{E}}C_E = \Omega$, for each $i\in[m]$ we have $\bigcup_{i\in E}C_E = \Omega\setminus B_i$. Therefore,

$$\varphi(\Omega)-\varphi(B_i) \leqslant \varphi(\Omega\setminus B_i) = \varphi\Big(\bigcup_{i\in E}C_E\Big) = \varphi\Big(\bigcup_{j\in[p]}A_{ij}\Big) \leqslant \sum_{j=1}^{p}\varphi(A_{ij})$$

and hence

$$m\varphi(\Omega) - \sum_{i=1}^{m}\varphi(B_i) \leqslant \alpha m r + \beta p,$$

which gives inequality (16.1) with p/m instead of $\vartheta(r,\varepsilon)$.

Now, for every $\delta > \vartheta(r,\varepsilon)$, the condition $H(r,\delta,\varepsilon)$ holds true which produces a sequence of (m_k,p_k,q_k,r)-concentrators for which $m_k\to\infty$, $p_k/m_k \leqslant \delta$ and $q_k/m_k \geqslant \varepsilon$. So, if $m_k \geqslant m$ we may repeat the above argument for the collection of m_k sets which consists of $s := \lfloor m_k/m\rfloor$ repetitions of $(B_1,\ldots,B_m)$ and $m_k - sm$ copies of Ω. By doing so, we obtain

$$\frac{s}{m_k}\sum_{i=1}^{m}\varphi(B_i) + \frac{m_k - sm}{m_k}\varphi(\Omega) \geqslant \varphi(\Omega) - \alpha r - \beta\frac{p}{m_k} \geqslant \varphi(\Omega) - \alpha r - \beta\delta$$

and since $s/m_k \xrightarrow[k\to\infty]{} 1/m$, we get that $\frac{1}{m}\sum_{i=1}^{m}\varphi(B_i) \geqslant \varphi(\Omega) - \alpha r - \beta\delta$, where δ may be arbitrarily close to $\vartheta(r,\varepsilon)$. This gives the desired inequality. □

Definition 16.6. Let $\mathscr{F}$ be an algebra of subsets of Ω and $\mathcal{A}$ be a non-empty subfamily of $\mathscr{F}$. The *covering index* of $\mathcal{A}$, denoted $\mathsf{C}(\mathcal{A})$, is defined by the formula

$$\mathsf{C}(\mathcal{A}) = \sup\bigg\{\delta \geqslant 0\colon \frac{1}{n}\sum_{j=1}^{n}\mathbb{1}_{A_j} \geqslant \delta\mathbb{1}_\Omega \text{ for some sequence } (A_j)_{j=1}^{n}\subset\mathcal{A}\bigg\}$$

(repetitions are allowed in $(A_j)_{j=1}^n$). Equivalently, $\mathsf{C}(\mathcal{A})$ may be defined as $\gamma(\mathcal{A})^{-1}$, where

$$\gamma(\mathcal{A}) = \bigg\{\sum_{A\in\mathcal{A}}x_A\colon \sum_{A\in\mathcal{A}}x_A\mathbb{1}_A \geqslant \mathbb{1}_\Omega \text{ and } x_A \geqslant 0 \text{ for every } A\in\mathcal{A}\bigg\}$$

(see Problem 5.12).

Theorem 16.7 (Kelley, 1959)**.** *Let $\mathscr{F}$ be a set algebra and $\mathcal{A}$ be a non-empty subfamily of $\mathscr{F}$. Then there exists a finitely additive set function $\mu\colon\mathscr{F}\to[0,1]$ such that $\mu(A)\leqslant\mathsf{C}(\mathcal{A})$ for every $A\in\mathcal{A}$ and $\mu(\Omega)=1$.*

Proof. Of course, we may assume that $\mathsf{C}(\mathcal{A})>0$, i.e. the family $\mathcal{A}$ covers the whole of Ω. Let $\mathcal{X}$ be the linear space of all real-valued bounded functions defined on Ω and let $p\colon\mathcal{X}\to[0,\infty)$ be defined by

$$p(u) = \mathsf{C}(\mathcal{A})\cdot\inf\bigg\{\sum_{j=1}^{n}\alpha_j\colon \sum_{j=1}^{n}\alpha_j\mathbb{1}_{A_j}\geqslant u,\ n\in\mathbb{N},\ \alpha_j\geqslant 0 \text{ and } A_j\in\mathcal{A} \text{ for } j\in[n]\bigg\}.$$

It is evident that $p(u+v) \leqslant p(u)+p(v)$ and $p(\alpha u) = \alpha p(u)$ for all $u, v \in \mathcal{X}$ and $\alpha \geqslant 0$. It also follows from the very definition of p that $p(\mathbb{1}_A) \leqslant \mathsf{C}(\mathcal{A})$ whenever $A \in \mathcal{A}$. Moreover, we claim that $p(\mathbb{1}_\Omega) \geqslant 1$. Suppose not. Then there exist $A_1, \ldots, A_n \in \mathcal{A}$ and rational numbers $\alpha_1, \ldots, \alpha_n > 0$ such that $\sum_{j=1}^n \alpha_j \mathbb{1}_{A_j} \geqslant \mathbb{1}_\Omega$ but $\sum_{j=1}^n \alpha_j < \mathsf{C}(\mathcal{A})^{-1}$; write $\alpha_j = k_j/m$ for $j \in [n]$, where $m, k_1, \ldots, k_n \in \mathbb{N}$. Consider the sequence

$$(D_i)_{i\in I} := \big(\underbrace{A_1, \ldots, A_1}_{k_1 \text{ times}};\ \underbrace{A_2, \ldots, A_2}_{k_2 \text{ times}};\ \ldots;\ \underbrace{A_n, \ldots, A_n}_{k_n \text{ times}}\big)$$

which consists of $|I| = k_1 + \ldots + k_n = m(\alpha_1 + \ldots + \alpha_n) < m\mathsf{C}(\mathcal{A})^{-1}$ terms satisfying $\frac{1}{m}\sum_{i\in I} \mathbb{1}_{D_i} \geqslant \mathbb{1}_\Omega$. Hence, $\frac{1}{|I|}\sum_{i\in I} \mathbb{1}_{D_i} > \mathsf{C}(\mathcal{A}) \cdot \mathbb{1}_\Omega$ which stands in contradiction to the definition of $\mathsf{C}(\mathcal{A})$.

By the Hahn–Banach theorem, there exists a linear functional $h\colon \mathcal{X} \to \mathbb{R}$ such that $h(\mathbb{1}_\Omega) = p(\mathbb{1}_\Omega) \geqslant 1$ and $h(u) \leqslant p(u)$ for every $u \in \mathcal{X}$. Therefore, whenever $A \in \mathcal{A}$ and $B \subset A$, we have $h(\mathbb{1}_B) \leqslant p(\mathbb{1}_B) \leqslant p(\mathbb{1}_A) \leqslant \mathsf{C}(\mathcal{A})$.

The formula $\nu(A) = h(\mathbb{1}_A)$ defines a signed, finitely additive measure on $\mathscr{F}$ which satisfies $\nu(\Omega) \geqslant 1$ and $\nu(A) \leqslant \mathsf{C}(\mathcal{A})$ for every $A \in \mathcal{A}$. Consequently, a function $\mu\colon \mathscr{F} \to \mathbb{R}$ defined by $\mu(A) = \sup_{B\subset A} \nu(B)$ is a finitely additive, non-negative measure which satisfies $\mu(A) \leqslant \mathsf{C}(\mathcal{A})$ for every $A \in \mathcal{A}$ and

$$1 \leqslant \nu(\Omega) \leqslant \mu(\Omega) = \sup_{B\subset\Omega} h(\mathbb{1}_B) \leqslant \sup_{B\subset\Omega} p(\mathbb{1}_B) \leqslant p(\mathbb{1}_\Omega) < \infty.$$

Hence, multiplying μ by $\mu(\Omega)^{-1}$ we get a normalised, finitely additive, non-negative measure on $\mathscr{F}$ that does not exceed $\mathsf{C}(\mathcal{A})$ on any member of $\mathcal{A}$. □

Proof of Theorem 15.1. We start with noticing that it is enough to consider only finite algebras. Indeed, suppose the assertion has been proved for every finite algebra and let $\mathscr{F}$ be an infinite set algebra. Let $\mathscr{E}$ be the collection of all finite subalgebras of $\mathscr{F}$, directed by inclusion. For each $\mathscr{G} \in \mathscr{E}$ there is an additive set function $\mu_\mathscr{G}\colon \mathscr{G} \to \mathbb{R}$ satisfying $|\nu(A) - \mu_\mathscr{G}(A)| \leqslant K$ for each $A \in \mathscr{G}$; extend this map to the whole of $\mathscr{F}$ by putting $\mu_\mathscr{G}(A) = 0$ whenever $A \in \mathscr{F} \setminus \mathscr{G}$, and let us denote this extension again by $\mu_\mathscr{G}$. Then, every term of the net $(\mu_\mathscr{G})_{\mathscr{G}\in\mathscr{E}}$ belongs to the set $\{f \in \mathbb{R}^\mathscr{F} : |f(A)| \leqslant |\nu(A)| + K\}$ which is compact with respect to the topology inherited from the product topology on $\mathbb{R}^\mathscr{F}$. It is now evident that if $\mu\colon \mathscr{F} \to \mathbb{R}$ is the limit of any convergent subnet of $(\mu_\mathscr{G})_{\mathscr{G}\in\mathscr{E}}$, then μ is an additive set function and $|\nu(A) - \mu(A)| \leqslant K$ for every $A \in \mathscr{F}$. Thus, from now on we assume that $\mathscr{F}$ is finite and, in fact, we may also assume that $\mathscr{F} = \mathcal{P}\Omega$ for some finite set Ω.

For any map $f\colon \mathscr{F} \to \mathbb{R}$ we set $V(f) = \max_{A,B\in\mathscr{F}}(f(A) - f(B))$ and we pick an additive set function $\mu\colon \mathscr{F} \to \mathbb{R}$ so that $V(\nu - \mu)$ takes the smallest possible value (see Problem 5.7). Let $g = \nu - \mu$ and set

$$a = \max_{A\in\mathscr{F}} g(A), \quad b = -\min_{A\in\mathscr{F}} g(A).$$

With no loss of generality we may assume that $a \geqslant b$ and then $|g(A)| \leqslant a$ for every $A \in \mathscr{F}$. Therefore, we are to prove that $a < 45$.

Pick a set $S \subset \Omega$ such that $g(S) = a$ and define a map $\varphi\colon \mathcal{P}S \to \mathbb{R}$ by

$$\varphi(A) = \begin{cases} 1 + \sup_{B\subset A} g(B) & \text{if } A \neq \varnothing, \\ 0 & \text{if } A = \varnothing. \end{cases}$$

Obviously, φ is monotone and since g satisfies the same inequality as ν, i.e.

$$|g(A\cup B)-g(A)-g(B)|\leqslant 1\quad\text{for } A,B\in\mathscr{F}\text{ with } A\cap B=\varnothing,$$

the map φ is also subadditive. Hence, φ is a submeasure. Moreover, if $A_1,\ldots,A_n\subset S$ are non-empty and mutually disjoint, then for some sets $B_j\subset A_j$ ($j\in[n]$) we have

$$\sum_{j=1}^{n}\varphi(A_j)=n+\sum_{j=1}^{n}g(B_j)\leqslant n+g(B_1\cup\ldots\cup B_n)+n-1\leqslant 2n+(a-1),$$

which means that φ satisfies the assumption of Lemma 16.5 with $\alpha=2$ and $\beta=a-1$. Note also that $\varphi(S)=a+1$.

Now, define

$$\mathcal{A}=\Big\{A\subset S\colon \varphi(A)\leqslant\frac{9}{2}\Big\}.$$

The essential part of the proof consists in showing that the so-defined collection $\mathcal{A}$ contains a sequence of sets which yields a 'good' covering of S. This will be done in two steps.

Claim 1. $\mathsf{C}(\mathcal{A})\geqslant\frac{1}{2}$.

Suppose this is not true. Then, by Kelley's Theorem 16.7, there exists a finitely additive measure $\lambda\colon\mathcal{P}S\to[0,1]$ such that $\lambda(A)<\frac{1}{2}$ for every $A\in\mathcal{A}$ and $\lambda(S)=1$. If so, consider a map $h\colon\mathscr{F}\to\mathbb{R}$ given by

$$h(A)=g(A)-\lambda(A\cap S)\quad\text{for } A\subset\Omega;$$

of course, this is still a difference between ν and some additive set function. By our choice of μ, we have $V(h)\geqslant V(g)=a+b$. However, as we shall now show,

$$-b-\frac{1}{2}<h(A)<a-\frac{1}{2}\quad\text{for every } A\in\mathscr{F}.$$

First, suppose that $h(A)\geqslant a-\frac{1}{2}$ for some $A\in\mathscr{F}$. Then also $g(A)\geqslant a-\frac{1}{2}$. Moreover,

$$g(A\setminus S)\leqslant 1+g(A\cup S)-g(S)\leqslant 1$$

(recall that $g(S)$ is the maximal value of g) and hence

$$g(A\cap S)\geqslant g(A)-g(A\setminus S)-1\geqslant a-\frac{5}{2}.$$

Therefore, for every $B\subset S\setminus A$ we have

$$g(B)\leqslant g\big((A\cap S)\cup B\big)-g(A\cap S)+1\leqslant a-\Big(a-\frac{5}{2}\Big)+1=\frac{7}{2}.$$

Hence, $\varphi(S\setminus A)\leqslant\frac{9}{2}$ which means that $S\setminus A\in\mathcal{A}$, thus $\lambda(S\setminus A)<\frac{1}{2}$ and so $\lambda(A\cap S)>\frac{1}{2}$. Consequently, $h(A)<a-\frac{1}{2}$ contrary to our supposition.

Now, suppose that $h(A)\leqslant -b-\frac{1}{2}$ for some $A\in\mathscr{F}$. Then $g(A)\leqslant -b+\frac{1}{2}$. If $B\subset A\cap S$, then

$$g(B)\leqslant g(A)-g(A\setminus B)+1\leqslant -b+\frac{1}{2}+b+1=\frac{3}{2},$$

thus $\varphi(A \cap S) \leqslant \frac{5}{2}$. Consequently, $A \cap S \in \mathcal{A}$ and hence $\lambda(A \cap S) < \frac{1}{2}$ which yields $h(A) > -b - \frac{1}{2}$; a contradiction.

In this way, Claim 1 has been proved. Next, we claim that since the collection $\mathcal{A}$ is finite, its covering index is realised by a certain finite sequence of its elements. We shall formulate this assertion taking into account the just proved estimate $\mathsf{C}(\mathcal{A}) \geqslant \frac{1}{2}$:

Claim 2. *There exists a sequence $(A_i)_{i=1}^m \subset \mathcal{A}$ such that $\frac{1}{m}\sum_{i=1}^m \mathbb{1}_{A_i} \geqslant \frac{1}{2}\mathbb{1}_S$.*

Let

$$Z(\mathcal{A}) = \left\{ (x_A)_{A\in\mathscr{A}} \in \mathbb{R}^{|\mathcal{A}|} \colon \sum_{A\in\mathcal{A}} x_A \mathbb{1}_A \geqslant \mathbb{1}_S \text{ and } x_A \geqslant 0 \text{ for every } A \in \mathcal{A} \right\};$$

this is an unbounded polygon in the finite-dimensional linear space $\mathbb{R}^{|\mathcal{A}|}$. The set of all those points $(x_A)_{A\in\mathcal{A}}$ from $Z(\mathcal{A})$ for which $\sum_{A\in\mathcal{A}} x_A$ is minimal is either a singleton or a polygon being a face of $Z(\mathcal{A})$. In each case there is an extreme point $(x_A)_{A\in\mathcal{A}}$ (a vertex) of $Z(\mathcal{A})$ for which $\sum_{A\in\mathcal{A}} x_A$ is minimal. Such a point is a unique solution of a system of linear equations with rational coefficients, hence every its coordinate is rational. Therefore (see Definition 16.6),

$$\mathsf{C}(\mathcal{A}) = \frac{1}{\gamma(\mathcal{A})} = \frac{1}{\sum_{A\in\mathcal{A}} x_A} \in \mathbb{Q}.$$

Since $\sum_{A\in\mathcal{A}} x_A \mathbb{1}_A \geqslant \mathbb{1}_S$ and $\mathsf{C}(\mathcal{A}) \geqslant \frac{1}{2}$, we have

$$\sum_{A\in\mathcal{A}} \mathsf{C}(\mathcal{A}) x_A \mathbb{1}_A \geqslant \frac{1}{2}\mathbb{1}_S,$$

where $\mathsf{C}(\mathcal{A})x_A$ (for $A \in \mathcal{A}$) are rational numbers summing up to one. Let $\mathsf{C}(\mathcal{A})x_A = k_A/m$, where $k_A \in \mathbb{N}$ and $m \in \mathbb{N}$ is a common denominator of those numbers. Then, repeating the sets from $\mathcal{A}$ as required (each $A \in \mathcal{A}$ should be repeated k_A times) we get the desired sequence $(A_i)_{i=1}^m \subset \mathcal{A}$. Claim 2 has been proved.

By virtue of Lemma 16.5, applied to the submeasure φ and the constants $\alpha = 2$, $\beta = a - 1$ and $\varepsilon = \frac{1}{2}$, we get

$$\frac{9}{2} \geqslant \frac{1}{m}\sum_{i=1}^m \varphi(A_i) \geqslant a + 1 - 2r - (a-1)\vartheta\Big(r, \frac{1}{2}\Big) \quad \text{for every } r \geqslant 3.$$

Putting $r = 6$ and appealing to Corollary 16.3 we arrive at $a \leqslant \frac{89}{2} < 45$. $\square$