Combinatorics in Banach space theory

Lecture 2

Now, we will derive the promised corollaries from Rosenthal's lemma. First, following [Ros70], we will show how this lemma implies Nikodým's uniform boundedness principle for bounded vector measures. Before doing this we need to recall some definitions.

Definition 2.4. Let $\mathscr{F}$ be a set algebra. By a *partition* of a given set $E \in \mathscr{F}$ we mean a finite collection $\{E_1, \ldots, E_k\}$ of pairwise disjoint members of $\mathscr{F}$ such that $\bigcup_{j=1}^k E_j = E$. We denote $\Pi(E)$ the set of all partitions of E.

Let also X be a Banach space and $\mu\colon \mathscr{F} \to X$ be a finitely additive set function (called a *vector measure*). The *variation* of μ is a function $|\mu|\colon \mathscr{F} \to [0, \infty]$ given by

$$|\mu|(E) = \sup\left\{\sum_{A\in\pi} \|\mu(A)\| \colon \pi \in \Pi(E)\right\}.$$

The *semivariation* of μ is a function $\|\mu\|\colon \mathscr{F} \to [0, \infty]$ given by

$$\|\mu\| = \sup\big\{|x^*\mu|(E)\colon x^* \in B_{X^*}\big\}.$$

By a straightforward calculation, one may check that the variation $|\mu|$ is always finitely additive, whereas the semivariation $\|\mu\|$ is monotone and finitely subadditive. Of course, we always have $\|\mu\| \leqslant |\mu|$. Moreover, since we know how the functionals on the scalar space ($\mathbb{R}$ or $\mathbb{C}$) look like, it is easily seen that for scalar-valued measures the notions of variation and semivariation coincide.

By saying that a vector measure $\mu\colon \mathscr{F} \to X$ is *bounded* we mean that $\|\mu\|$ is finitely valued. This is in turn equivalent to saying that the range of μ is a bounded subset of X. More precisely, for every $E \in \mathscr{F}$ we have

$$\sup_{E\supset F\in\mathscr{F}} \|\mu(F)\| \leqslant \|\mu\|(E) \leqslant 4 \sup_{E\supset F\in\mathscr{F}} \|\mu(F)\|. \tag{2.1}$$

The first inequality is obvious, as we have

$$\sup_{E\supset F\in\mathscr{F}} \|\mu(F)\| = \sup_{E\supset F\in\mathscr{F}}\ \sup_{x^*\in B_{X^*}} |x^*\mu(F)| \leqslant \sup_{E\supset F\in\mathscr{F}} \|\mu\|(F) = \|\mu\|(E).$$

The second one really says that there are not too many direction on the real line. In fact, fix any $\pi = \{E_1, \ldots, E_k\}$ from $\Pi(E)$ and any $x^* \in B_{X^*}$, and let

$$\pi^+ = \{i \in [k]\colon x^*\mu(E_i) \geqslant 0\}, \quad \pi^- = \{j \in [k]\colon x^*\mu(E_j) < 0\}.$$

Then, in the case where X is a *real* Banach space, we have

$$\begin{aligned}
\sum_{j=1}^k |x^*\mu(E_j)| &= \sum_{i\in\pi^+} x^*\mu(E_i) - \sum_{j\in\pi^-} x^*\mu(E_j)\\
&= x^*\mu\Bigg(\bigcup_{i\in\pi^+} E_i\Bigg) - x^*\mu\Bigg(\bigcup_{j\in\pi^-} E_j\Bigg) \leqslant 2 \sup_{E\supset F\in\mathscr{F}} \|\mu(F)\|.
\end{aligned}$$

In the case where X is a *complex* Banach space, we get the same estimate with 4 instead of 2, by simply splitting x^* into its real and imaginary parts.

Theorem 2.5 (Nikodým boundedness principle, 1930). *Let Σ be a σ-algebra of subsets of Ω and X be a Banach space. Suppose $\{\mu_\gamma\colon \gamma\in\Gamma\}$ is a family of X-valued, bounded vector measures defined on Σ such that*

$$\sup_{\gamma\in\Gamma}\|\mu_\gamma(E)\|<\infty \quad \textit{for every } E\in\Sigma.$$

Then, this family is uniformly bounded, that is, $\sup_{\gamma\in\Gamma}\|\mu_\gamma\|(\Omega)<\infty$.

Proof. First, we may get rid of the Banach space X, just by replacing the original family of vector measures by the family of scalar measures given by

$$\big\{x^*\mu_\gamma\colon \gamma\in\Gamma,\, x^*\in B_{X^*}\big\}.$$

Next, if we suppose that our assertion fails to hold, then there would be a sequence of measures from this family with total semivariations increasing to infinity. Consequently, we may work only with a sequence $(\mu_n)_{n=1}^\infty$ of bounded scalar measures satisfying $\sup_n\|\mu_n(E)\|<\infty$, for $E\in\mathscr{F}$, and supposing on the contrary to our claim that $\sup_n\|\mu_n\|(\Omega)=\infty$.

This was a kind of formality. What really makes the result difficult is that the measures μ_n are sign-changing, or even complex-valued.

By our supposition and inequality (2.1), there is a subsequence $(\mu_{n_j})_{j=1}^\infty$ of $(\mu_n)_{n=1}^\infty$ and a sequence $(E_j)_{j=1}^\infty\subset\Sigma$ such that

$$|\mu_{n_j}(E_j)|\geqslant\frac{1}{5}\|\mu_{n_j}\|(\Omega)\geqslant n+2\sum_{i=1}^{j-1}\sup_{n\in\mathbb{N}}|\mu_n(E_i)|\quad \text{for each } j\in\mathbb{N}.$$

Put $F_1=E_1$ and $F_j=E_j\setminus\bigcup_{i=1}^{j-1}E_i$ for $j\geqslant 2$. Then $(F_j)_{j=1}^\infty$ are pairwise disjoint and the above inequality implies that $|\mu_{n_j}(F_j)|\geqslant\|\mu_{n_j}\|/10$ for each $j\in\mathbb{N}$.

Now, observe that the measures $|\mu_{n_j}|/|\mu_{n_j}(F_j)|$ are non-negative, finitely additive and uniformly bounded by 10. Thus, in view of Rosenthal's Lemma 2.1, we may pass to an appropriate subsequence and assume that

$$|\mu_{n_j}|\Bigg(\bigcup_{i\neq j}F_i\Bigg)<\frac{1}{3}|\mu_{n_j}(F_j)|\quad \text{for each } j\in\mathbb{N}.$$

Then, putting $F=\bigcup_{j\in\mathbb{N}}F_j\in\Sigma$, we obtain

$$|\mu_{n_j}(F)|\geqslant\frac{2}{3}|\mu_{n_j}(F_j)|\geqslant\frac{1}{15}\|\mu_{n_j}\|\xrightarrow[j\to\infty]{}\infty,$$

which contradicts the assumption about pointwise boundedness. □

Now, we proceed to the beautiful lemma proved by Phillips [Phi40], which has some truely deep consequences in the Banach space theory. Its original proof was technical and rather complicated. However, as it was shown in [Ros70], Phillips' lemma is an easy consequence of Rosenthal's lemma. One may compare the proof presented below with the one in Morrison's book, [Mor01, pp. 270–274].

The complex case will require the following well-known property of the complex plane, which we prove here for completeness.

Lemma 2.6. *For any $z_1, \ldots, z_n \in \mathbb{C}$ there exists a finite set $J \subset [n]$ such that*

$$\Big| \sum_{j \in J} z_j \Big| \geqslant \frac{1}{6} \sum_{j=1}^{n} |z_j|.$$

Proof. Let $w = \sum_{j=1}^{n} |z_j|$. Divide the complex plane on four sectors bounded by the lines $y = \pm x$. For at least one sector, say Q, the sum of the absolute values of all the numbers from the set $\{z_1, \ldots, z_n\}$ which belong to Q is at least $w/4$. With no loss of generality we may assume that Q is the sector given by $|y| \leqslant x$. Let $J = \{j \in [n] \colon z_j \in Q\}$. Then, for every $z \in Q$ we have $\mathrm{Re} z \geqslant |z|/\sqrt{2}$, thus

$$\Big| \sum_{j \in J} z_j \Big| \geqslant \sum_{j \in J} \mathrm{Re} z_j \geqslant \frac{1}{\sqrt{2}} \sum_{j \in J} |z_j| \geqslant \frac{w}{4\sqrt{2}} \geqslant \frac{w}{6}\,. \qquad \square$$

Remark 2.7. By using the so-called *isoperimetric inequality*, which asserts that the perimeter of any convex polygon does not exceed π times the diameter of this polygon, one may show that the assertion of Lemma 2.6 holds true with $1/\pi$ instead of $1/6$, and this cannot go any better.

Lemma 2.8 (Phillips' lemma, 1940)**.** *Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of bounded, finitely additive, scalar-valued measures defined on the σ-algebra $\mathcal{P}\mathbb{N}$. If for every $E \subset \mathbb{N}$ we have $\lim_{n\to\infty} \mu_n(E) = 0$, then*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |\mu_n(\{k\})| = 0. \tag{2.2}$$

Proof. First, by Nikodým's Theorem 2.5, we infer that $\sup_n \|\mu_n\| < \infty$. Suppose that our assertion fails to hold. Then, by a standard 'sliding-hump' argument, we may construct a sequence $(F_j)_{j=1}^{\infty}$ of pairwise disjoint subsets of $\mathbb{N}$ and a subsequence $(\mu_{n_j})_{j=1}^{\infty}$ of $(\mu_n)_{n=1}^{\infty}$ such that $|\mu_{n_j}(F_j)| > \delta$ for each $j \in \mathbb{N}$, with some $\delta > 0$. Indeed, the negation of (2.2) produces a subsequence $(\nu_n)_{n=1}^{\infty}$ of $(\mu_n)_{n=1}^{\infty}$ such that $\sum_{k=1}^{\infty} |\nu_n(\{k\})| > 7\delta$ for every $n \in \mathbb{N}$, with some $\delta > 0$. Using Lemma 2.6 choose μ_{n_1} as one of the measures from $(\nu_n)_{n=1}^{\infty}$ such that $|\mu_{n_1}(F_1)| > \delta$ with some finite set $F_1 \subset \mathbb{N}$. Next, let $m_1 \in \mathbb{N}$ be so large that

$$\sum_{k=1}^{\max F_1} |\nu_n(\{k\})| < \delta, \quad \text{hence} \quad \sum_{k=\max F_1 + 1}^{\infty} |\nu_n(\{k\})| > 6\delta, \quad \text{for every } n \geqslant m_1.$$

Again, using Lemma 2.6, we may pick μ_{n_2}, where $n_2 > n_1$, as one of the measures from $(\nu_n)_{n=m_1}^{\infty}$ such that $|\mu_{n_2}(F_2)| > \delta$ with some finite set $F_2 \subset \mathbb{N} \setminus F_1$. Continuing this procedure we obtain the desired subsequences.

By Rosenthal's Lemma 2.1, we may suppose that

$$|\mu_{n_j}|\Big(\bigcup_{i \neq j} F_i \Big) < \delta/2 \quad \text{for each } j \in \mathbb{N}.$$

Then, setting $F = \bigcup_{j=1}^{\infty} F_j$, we would get $|\mu_{n_j}(F)| > \delta/2$; a contradiction. $\square$

This is the time to hit the first target in Banach space theory, with the aid of Phillips' lemma. Let us recall that a Banach space X is said to have the *Schur property* whenever every weakly null sequence in X is norm convergent.

Theorem 2.9 (Schur, 1921)**.** *ℓ_1 has the Schur property.*

Before proving Schur's theorem, let us briefly remind how the dual space of ℓ_∞ is represented in terms of measures. For any σ-algebra Σ the symbol $\mathsf{ba}(\Sigma)$ will stand for the Banach space of all bounded, finitely additive, scalar-valued measures on Σ, equipped with the total (semi)variation norm.

Proposition 2.10. *ℓ_∞^* is isometrically isomorphic to the space $\mathsf{ba}(\mathcal{P}\mathbb{N})$, via an identification $\ell_\infty^* \ni \varphi \mapsto m_\varphi$ such that*

$$\varphi(x) = \int_{\mathbb{N}} x \,\mathrm{d}m_\varphi \quad \textit{for every } x \in \ell_\infty. \tag{2.3}$$

Remark 2.11 (before proof)**.** The integration in formula (2.3) is with respect to a finitely additive measure. Such an integral is defined in a very the same way as the ordinary Lebesgue integral. Namely, let $m\colon \Sigma \to X$ be a finitely additive vector measure, defined on a σ-algebra of subsets of Ω. For every Σ-measurable, scalar-valued step function f, defined on Ω, we may write $f = \sum_{j=1}^k \alpha_j \mathbb{1}_{A_j}$ with some scalars α_j, and some partition $\{A_1, \ldots, A_k\} \in \Pi(\Omega)$. Then, we set

$$\int_\Omega f \,\mathrm{d}m = \sum_{j=1}^k \alpha_j m(A_j).$$

This defines a linear map on the space of all Σ-measurable, scalar-valued step functions and a glance at formula (2.4) below shows that its norm equals $\|m\|(\Omega)$ (we consider the supremum norm on the space of all those step functions). Hence, this operator has a (unique) norm preserving extension to the space $B(\Sigma)$ of all bounded, Σ-measurable, scalar-valued functions on Ω (with the supremum norm). Its value at any $f \in B(\Sigma)$ we denote, of course, by $\int_\Omega f \,\mathrm{d}m$.

If we replace Σ by an algebra of sets $\mathscr{F}$, then such a construction defines an integral on the space $B(\mathscr{F})$ of all scalar-valued functions on Ω which are uniform limits of step functions on $\mathscr{F}$.

Proof of Proposition 2.10. For any $\varphi \in \ell_\infty^*$ one may simply define $m_\varphi \in \mathsf{ba}(\mathcal{P}\mathbb{N})$ by putting $m_\varphi(E) = \varphi(\mathbb{1}_E)$ for every $E \subset \mathbb{N}$. Then formula (2.3) is valid for every $x \in \ell_\infty$ which is a step function on $\mathbb{N}$. Hence, by the construction in Remark 2.11, it must hold for every $x \in \ell_\infty$. On the other hand, every measure $m \in \mathsf{ba}(\mathcal{P}\mathbb{N})$ defines an element of ℓ_∞^* given by the integral with respect to m. What is left to be proved is that the correspondence $\varphi \mapsto m_\varphi$ is an isometry. This will follow from the following useful formula for semivariation:

$$\|m\|(\Omega) = \sup\Bigg\{\Bigg\|\sum_{E_j \in \pi} \varepsilon_j m(E_j)\Bigg\| \colon \pi \in \Pi(\Omega) \text{ and } |\varepsilon_j| \leqslant 1\Bigg\}, \tag{2.4}$$

which is true for any vector measure $m\colon \Sigma \to X$ (X being an arbitrary Banach space).

The inequality '$\geqslant$' is rather straightforward as for any $\pi \in \Pi(\Omega)$ and any scalars (ε_j)

with $|\varepsilon_j| \leqslant 1$ we have:

$$\begin{aligned}\Big\|\sum_{E_j\in\pi}\varepsilon_j m(E_j)\Big\| &= \sup\Big\{\Big|x^*\sum_{E_j\in\pi}\varepsilon_j m(E_j)\Big| \colon x^*\in B_{X^*}\Big\}\\ &\leqslant \sup\Big\{\sum_{E_j\in\pi}\big|x^* m(E_j)\big| \colon x^*\in B_{X^*}\Big\} \leqslant \|m\|(\Omega).\end{aligned}$$

For the reverse inequality, fix any $x^*\in B_{X^*}$, any $\pi=\{E_1,\dots,E_k\}$ from $\Pi(\Omega)$, and take any scalars $(\alpha_j)_{j=1}^k$ with $|\alpha_j|=1$ and $\alpha_j x^* m(E_j)=|x^* m(E_j)|$ for $j\in[k]$. Then,

$$\sum_{E_j\in\pi}\big|x^* m(E_j)\big| = \sum_{E_j\in\pi}\alpha_j x^* m(E_j) = x^*\sum_{E_j\in\pi}\alpha_j m(E_j) \leqslant \Big\|\sum_{E_j\in\pi}\alpha_j m(E_j)\Big\|,$$

which does not exceed the right-hand side of (2.4).

Finally, formulas (2.3) and (2.4) show that for any $\varphi\in\ell_\infty^*$ the semivariation of the measure m_φ equals the supremum of all $|\varphi(x)|$, where x runs through the set of all step functions from the unit ball of ℓ_∞, and this is nothing else but the norm of φ. ☐

Proof of Theorem 2.9. Suppose $\xi^{(n)}\xrightarrow{w}0$ in ℓ_1. Each of $\xi^{(n)}$'s may be identified with the functional $\varphi_n\in\ell_\infty^*$ given

$$\varphi_n(x)=\sum_{j=1}^\infty \xi_j^{(n)} e_j^*(x),\quad \text{for } x\in\ell_\infty,$$

which is in turn identified via Proposition 2.10 with the measure

$$m_n=\sum_{j=1}^\infty \xi_j^{(n)}\delta_j\in \mathsf{ba}(\mathcal{P}\mathbb{N})$$

(δ_j being Dirac's measure concentrated at $\{j\}$). By the assumption, we have $\varphi_n(x)\to 0$ for every $x\in\ell_\infty$, thus for any $E\subset\mathbb{N}$ by taking $x=\mathbb{1}_E$ we get $m_n(E)\to 0$. Consequently, Phillips' Lemma 2.8 implies that

$$\lim_{n\to\infty}\sum_{j=1}^\infty\big|\xi_j^{(n)}\big|=0,$$

which means that $\xi^{(n)}\to 0$ in norm. ☐

Remark 2.12. An inspection of this proof shows that the assumption about weak convergence of the sequence $(\xi^{(n)})_{n=1}^\infty$ was not fully exploited and it was enough to assume only that for every $\{0,1\}$-valued sequence $x\in\ell_\infty$ we have

$$\lim_{n\to\infty}\sum_{j=1}^\infty \xi_j^{(n)} e_j^*(x)=0.$$

In other words, in the Banach space ℓ_1 *very weak convergence implies strong convergence.*

Observe that the identification $\xi^{(n)}\longleftrightarrow m_n$ in the above proof is nothing else but the identification between an element $\xi\in\ell_1$ and an element from the bidual space $\ell_1^{**}\simeq \mathsf{ba}(\mathcal{P}\mathbb{N})$ corresponding to ξ via the canonical embedding $\ell_1\hookrightarrow\ell_1^{**}$. Note also that this proof works without formally appealing to Proposition 2.10. Nonetheless it is good to keep it in mind in this context.