Combinatorics in Banach space theory

Lecture 3

3 Grothendieck's theorem on weakly compact sets of measures

We will discuss some further applications of Rosenthal's lemma which are of quite a different type than those presented in Section 2, and which characterise non-weakly compact operators on injective Banach spaces. However, before doing this we need some preparations which consist mainly on some measure-theoretic results.

At this moment, let us just announce two main products of our future investigations which lie in the heart of the Banach space theory. Both are originally attributed to Lindenstrauss [Lin67], but our intention is to present their proofs based on the paper [Ros70] and to convince the reader that a crucial part of deriving these result is just Rosenthal's lemma. We say that a Banach space X is *prime* whenever every complemented, infinite-dimensional subspace of X is isomorphic to X.

Theorem 3.1 (Lindenstrauss, 1967)**.** *Let Γ be a non-empty index set and X be a complemented subspace of $\ell_\infty(\Gamma)$. If X contains an isomorphic copy of $c_0(\Gamma)$, then $X \simeq \ell_\infty(\Gamma)$.*

Theorem 3.2 (Lindenstrauss, 1967)**.** *ℓ_∞ is prime.*

One of the main ingredients, which are necessary for proving the announced theorems of Rosenthal on non-weakly compact operators acting on injective spaces, is Pełczyński's characterisation of non-weakly compact operators acting on $C(K)$-spaces. This in turn requires Grothendieck's description of relatively weakly compact subsets of $\mathcal{M}(K)$, the Banach space of all scalar-valued, σ-additive, regular Borel measures on a compact Hausdorff space K, equipped with the total (semi)variation norm. We shall focus on this issue during the present lecture.

Definition 3.3. By saying that a Borel measure μ is *regular* we mean that for every Borel set A from the domain of μ, and every $\varepsilon > 0$, there exist a compact set $K \subset A$ and an open set $V \supset A$ such that $|\mu|(V \setminus K) < \varepsilon$.

A family $\mathcal{A}$ of Borel measures on a topological space is called *uniformly regular* whenever for every open set V, and every $\varepsilon > 0$, there exists a compact set $K \subset V$ such that $|\mu|(V \setminus K) < \varepsilon$ for each $\mu \in \mathcal{A}$.

Recall that the classical Riesz Representation Theorem asserts that $C(K)^* \simeq \mathcal{M}(K)$ via the duality given by $\langle f, \mu \rangle = \int_K f\,\mathrm{d}\mu$ for $f \in C(K)$ and $\mu \in \mathcal{M}(K)$. For simplicity, we will restrict ourselves to the case of real-valued measures (and functions) as in the complex case all the results presented below remain true by simply splitting complex measures (and functions) into their real and imaginary parts. So, for any σ-additive measure $\mu\colon \Sigma \to \mathbb{R}$, defined on a σ-algebra of subsets of K, we have the *Hahn decomposition* $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are non-negative measures on Σ, defined by

$$\mu^+(A) = \mu(A \cap P) \quad \text{and} \quad \mu^-(A) = -\mu(A \cap (K \setminus P)) \quad \text{for } A \in \Sigma,$$

where $\{P, K \setminus P\}$ is a measurable partition of K having the property that

$$\mu(A \cap P) \geqslant 0 \quad \text{and} \quad \mu(A \cap (K \setminus P)) \leqslant 0 \quad \text{for every } A \in \Sigma.$$

Probably the shortest existing proof of the Hahn decomposition theorem is due to Doss [Dos80]. Observe that from the very Definition 2.4 it follows quite easily that for every measure μ as above we have $|\mu| = \mu^+ + \mu^-$, which is also equal to $\|\mu\|$ as μ is scalar-valued (see Problem 3.4). In particular, every measure $\mu \in \mathcal{M}(K)$ is bounded, that is, of finite (semi)variation.

Before proceeding to Grothendieck's theorem we need some measure-theoretic preparations. Everywhere below the letter μ, appearing in the symbol $L_1(\mu)$, stands for a finite*, σ-additive, non-negative measure defined on some σ-algebra Σ of subsets of a set Ω.

Definition 3.4. A bounded set $\mathcal{F} \subset L_1(\mu)$ is called *equi-integrable* whenever

$$\lim_{\mu(E)\to 0} \sup_{f\in\mathcal{F}} \int_E |f| \,\mathrm{d}\mu = 0,$$

that is, for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $E \in \Sigma$ with $\mu(E) < \delta$, and every $f \in \mathcal{F}$, we have $\int_E |f| \,\mathrm{d}\mu < \varepsilon$.

It may be shown that the above condition is equivalent to

$$\lim_{M\to\infty} \sup_{f\in\mathcal{F}} \int_{\{|f|>M\}} |f| \,\mathrm{d}\mu = 0,$$

which is the assertion of Problem 3.14. Now, we will prove that equi-integrability of any set $\mathcal{F} \subset L_1(\mu)$ implies that $\mathcal{F}$ is relatively weakly compact. These two properties are in fact equivalent (which was first proved in [DP40]), but for the time being we focus only on that one announced implication; the converse one will be proved in Section 5. A wider version of this theorem may be found in [AK06, Theorem 5.2.9]. The reader should compare the statement below with the assertion of Problem 1.8(b), which is nothing but an 'atomic' (and compact) version of what we are now going to prove.

Lemma 3.5 (Dunford & Pettis, 1940)**.** *Let $\mathcal{F}$ be a bounded subset of $L_1(\mu)$. Then, the two assertions below are equivalent:*

(i) *$\mathcal{F}$ is relatively weakly compact;*
(ii) *$\mathcal{F}$ is equi-integrable.*

Proof of the implication (ii) $\Rightarrow$ (i). By the Eberlein–Šmulian theorem, it is enough to show that every sequence $(f_n)_{n=1}^\infty \subset \mathcal{F}$ is relatively weakly compact. Since the topology on $\mathbb{R}$, the codomain of each of these functions f_n, has a countable basis, there exists a countable set algebra $\mathscr{F} \subset \Sigma$ such that every f_n is $\mathscr{F}$-measurable. Let Σ' be the σ-algebra generated by $\mathscr{F}$. Since $(f_n)_{n=1}^\infty$ are uniformly bounded, for every set $E \in \mathscr{F}$ we may extract a subsequence $(f_{n_j})_{j=1}^\infty$ such that the limit $F(E) = \lim_{j\to\infty} \int_E f_{n_j} \,\mathrm{d}\mu$ exists. Because $\mathscr{F}$ is countable we may also apply the diagonal procedure and assume that the last equality holds true for every $E \in \mathscr{F}$, with some fixed subsequence $(f_{n_j})_{j=1}^\infty \subset (f_n)_{n=1}^\infty$.

*We some additional effort we could also cover the σ-finite case, but this is not so interesting in our context.

Since μ is a finite measure and $(f_{n_j})_{j=1}^{\infty}$ are uniformly integrable, we conclude that the limit $F(E)=\lim_{j\to\infty}\int_E f_{n_j}\,\mathrm{d}\mu$ exists for every $E\in\Sigma'$ and $F\colon\Sigma'\to\mathbb{R}$ is a σ-additive measure (see Problem 3.6). Plainly, $F\ll\mu$, so the Radon–Nikodým Theorem* implies that there exists a function $f\in L_1(\Sigma',\mu)$ such that

$$\lim_{j\to\infty}\int_E f_{n_j}\,\mathrm{d}\mu=\int_E g\,\mathrm{d}\mu\quad\text{for every } E\in\Sigma',$$

which implies

$$\lim_{j\to\infty}\int_\Omega f_{n_j}g\,\mathrm{d}\mu=\int_\Omega fg\,\mathrm{d}\mu\quad\text{for every } g\in L_\infty(\Sigma',\mu).$$

Hence, $f_{n_j}\xrightarrow{w}f$ in $L_1(\Sigma',\mu)$, and since $L_1(\Sigma',\mu)$ is just a (closed) subspace of $L_1(\mu)$, we have also $f_{n_j}\xrightarrow{w}f$ in $L_1(\mu)$. □

This was a truely neat application of the Radon–Nikodým Theorem. Another proof of the implication (ii) ⇒ (i), given in [AK06], may be worked out by using the characterisation of equi-integrability given in Problem 3.14 and the fact that $\mathcal{F}$ is relatively weakly compact if and only if $\overline{\mathcal{F}}^{w^*}$, the weak* closure of $\mathcal{F}$ in $L_1(\mu)^{**}$, lies inside $L_1(\mu)$. However, this is much more technical and probably not so elegant as the argument given above.

We are now ready to deal with Grothendieck's theorem. It will not be proved completely, since what we essentially need for our purposes is the fact that every non-weakly compact subset of $\mathcal{M}(K)$ contains a subsequence of measures whose values are separated from zero on some sequence of pairwise disjoint open subsets of K. For the complete proof, as well as some additional equivalent clauses, see [AK06, Theorem 5.3.2].

Theorem 3.6 (Grothendieck, 1953). *Let K be a compact Hausdorff space and $\mathcal{A}$ be a bounded subset of $\mathcal{M}(K)$. Then, the following assertions are equivalent:*

(i) *$\mathcal{A}$ is relatively weakly compact;*
(ii) *$\mathcal{A}$ is uniformly regular;*
(iii) *for any sequence $(U_n)_{n=1}^{\infty}$ of pairwise disjoint open subsets of K, and any sequence $(\mu_n)_{n=1}^{\infty}\subset\mathcal{A}$, we have $\lim_{n\to\infty}\mu_n(U_n)=0$.*

Proof of the implications (iii) ⇒ (ii) ⇒ (i). We start with showing that **(iii) ⇒ (ii)**. First of all, the assumption (iii) implies that for any sequence $(U_n)_{n=1}^{\infty}$ of pairwise disjoint open subsets of K, and any sequence $(\mu_n)_{n=1}^{\infty}\subset\mathcal{A}$, we have $\lim_{n\to\infty}|\mu_n|(U_n)=0$. To see this one may proceed by contradiction, using the Hahn decomposition and the regularity of members from $\mathcal{A}$ (see Problem 3.7).

Now, suppose on the contrary, that there is an open set $U\subset K$ and some $\delta>0$ such that

$$\sup_{\mu\in\mathcal{A}}|\mu|(U\setminus H)>\delta\quad\text{for every compact set } H\subset K.$$

*__The Radon-Nikodým Theorem__ says that for any σ-additive, σ-finite, non-negative measure μ, defined on a σ-algebra Σ, and any σ-additive, complex measure λ on Σ, which is *absolutely continuous* with respect to μ (i.e. $\mu(E)=0$ implies $\lambda(E)=0$, for any $E\in\Sigma$), there exists a unique function $f\in L_1(\mu)$ such that $\lambda(E)=\int_E f\,\mathrm{d}\mu$ for every $E\in\Sigma$. We then write $\mathrm{d}\lambda=f\mathrm{d}\mu$ or $f=\mathrm{d}\lambda/\mathrm{d}\mu$, and we call this function f the *Radon-Nikodým derivative* of λ with respect to μ. The fact that λ is absolutely continuous with respect to μ is denoted $\lambda\ll\mu$ and it is equivalent to saying that $\mu(E)=0$ implies $|\lambda|(E)=0$, for each $E\in\Sigma$ (see Problem 3.5)

Let $H_0 = \varnothing$ and pick $\mu_1 \in \mathcal{A}$ such that $|\mu_1|(U \setminus H_0) > \delta$. By the regularity of μ_1, there is a compact set $F_1 \subset U \setminus H_0$ with $|\mu_1|(F_1) > \delta$. To proceed inductively let us pick an open set V_1 with a compact closure which satisfies

$$F_1 \subset V_1 \subset \overline{V_1} \subset U \setminus H_0;$$

the existence of such a set follows from the fact that K, being compact and Hausdorff, is normal. So, by putting $H_1 = \overline{V_1}$ and using our supposition we get $|\mu_2|(U \setminus H_1) > \delta$ for some $\mu_2 \in \mathcal{A}$. Again, there is a compact set $F_2 \subset U \setminus H_2$ with $|\mu_2|(F_2) > \delta$ and by using the T_4-axiom for K we get an open set V_2 with a compact closure satisfying

$$F_2 \subset V_2 \subset \overline{V_2} \subset U \setminus H_1.$$

Now, we put $H_2 = \overline{V_1} \cup \overline{V_2}$ and proceed similarly. Eventually, we get a sequence $(V_n)_{n=1}^\infty$ of pairwise disjoint open subset of K, and a sequence $(\mu_n)_{n=1}^\infty \subset \mathcal{A}$, such that $|\mu_n|(V_n) > \delta$ for each $n \in \mathbb{N}$, which contradicts the conclusion of the previous paragraph.

(ii) $\Rightarrow$ (i) In this part we will combine the implication (ii) $\Rightarrow$ (i) from Lemma 3.5 and a cornerstone of the measure theory, the Radon–Nikodým Theorem. First, observe that in view of the Eberlein–Šmulian theorem, it is enough to show that every sequence contained in $\mathcal{A}$ is relatively weakly compact. So, fix any such $(\mu_n)_{n=1}^\infty \subset \mathcal{A}$.

The formula

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} |\mu_n|$$

defines a σ-additive, bounded, non-negative measure μ on the σ-algebra of all Borel subsets of K. Obviously, $\mu_n \ll \mu$ for each $n \in \mathbb{N}$, thus the Radon–Nikodým Theorem produces a sequence $(f_n)_{n=1}^\infty \subset L_1(\mu)$ such that $\mathrm{d}\mu_n = f_n \mathrm{d}\mu$ for each $n \in \mathbb{N}$. Let

$$\mathcal{AC}(\mu) = \{\lambda \in \mathcal{M}(K) \colon \lambda \ll \mu\}.$$

This is, of course, a linear subspace of $\mathcal{M}(K)$ and, moreover, a closed subspace of $\mathcal{M}(K)$ (see Problem 3.8), that is, a Banach space with the total (semi)variation norm.

The Radon–Nikodým Theorem defines a bijective map $\Phi\colon L_1(\mu) \to \mathcal{AC}(\mu)$, which assigns to every μ-integrable function f the measure $\Phi(f) \in \mathcal{AC}(\mu)$ satisfying $\Phi(f)(E) = \int_E f\,\mathrm{d}\mu$, for every Borel subset E of K. By the definition of (semi)variation, we infer that Φ yields an isometry between $L_1(\mu)$ and $\mathcal{AC}(\mu)$ (see Problem 3.16) and, clearly, satisfies $\Phi(f_n) = \mu_n$ for each $n \in \mathbb{N}$. Consequently, saying that the set $\{\mu_n \colon n \in \mathbb{N}\}$ is relatively weakly compact in $\mathcal{M}(K)$ is equivalent to saying that the set $\{f_n \colon n \in \mathbb{N}\}$ is relatively weakly compact in $L_1(\mu)$. By the implication (ii) $\Rightarrow$ (i) from Lemma 3.5, the last clause would follow from the fact that $\{f_n \colon n \in \mathbb{N}\}$ is equi-integrable. This is what we shall now demonstrate, and this is where we will use our assumption about uniform regularity.

Suppose $\{f_n \colon n \in \mathbb{N}\}$ is not equi-integrable. Then, by using regularity, we get some positive number ε and a sequence $(V_n)_{n=1}^\infty$ of open sets such that

$$\mu(V_n) < 2^{-n} \quad \text{and} \quad \sup_{k\in\mathbb{N}} \int_{V_n} |f_k|\,\mathrm{d}\mu > \varepsilon \quad \text{for each } n \in \mathbb{N}. \tag{3.1}$$

Replacing each V_n be $\bigcup_{k=n+1}^\infty V_k$ we may also assume that $(V_n)_{n=1}^\infty$ is decreasing. For every $n \in \mathbb{N}$ we may apply uniform regularity to the open set V_n and thus we obtain compact sets $K_n \subset V_n$ ($n \in \mathbb{N}$) satisfying

$$\sup_{k\in\mathbb{N}} \int_{V_n \setminus K_n} |f_k|\,\mathrm{d}\mu < \eta_n \quad \text{for each } n \in \mathbb{N}, \tag{3.2}$$

where the numbers $(\eta_n)_{n=1}^\infty$ may be required to be as small as we wish, and they shall be fixed later on. For now, let us just remark that due to the fact that Φ is an isometry we have $|\mu_k|(E) = \int_E |f_k|\,\mathrm{d}\mu$, for every Borel set $E \subset K$ and $k \in \mathbb{N}$, which shows that inequalities (3.2) are indeed a consequence of the uniform regularity of $\mathcal{A}$. Now, to get a desired contradiction from (3.1) and (3.2) we need a little compactness trick.

Observe that $\bigcap_{n=1}^\infty K_n$ is a compact set with $\mu(\bigcap_{n=1}^\infty K_n) = 0$. By uniform *outer* regularity (see Problem 3.13), we may find an open set $W \supset \bigcap_{n=1}^\infty K_n$ such that $\sup_{k\in\mathbb{N}} \int_W |f_k|\,\mathrm{d}\mu < \varepsilon/2$. By compactness, there is some $N \in \mathbb{N}$ satisfying $\bigcap_{n=1}^N K_n \subset W$ and hence

$$\sup_{k\in\mathbb{N}} \int_{\bigcap_{n=1}^N K_n} |f_k|\,\mathrm{d}\mu < \frac{\varepsilon}{2}\,.$$

Since $V_{n+1} \subset V_n$ for $n \in \mathbb{N}$, we have

$$V_{N+1} \subset \bigcup_{n=1}^N (V_n \setminus K_n) \cup \bigcap_{n=1}^N K_n,$$

whence for every $k \in \mathbb{N}$ we have

$$\int_{V_{N+1}} |f_k|\,\mathrm{d}\mu \leqslant \int_{\bigcap_{n=1}^N K_n} |f_k|\,\mathrm{d}\mu + \sum_{n=1}^N \int_{V_n\setminus K_n} |f_k|\,\mathrm{d}\mu < \frac{\varepsilon}{2} + \sum_{n=1}^N \eta_n.$$

Hence, putting $\eta_n = 2^{-(n+1)}\varepsilon$, for $n \in \mathbb{N}$, we obtain a contradiction with (3.1). □