Combinatorics in Banach space theory

Lecture 4

4 Weakly compact operators on $C(K)$-spaces and injective Banach spaces

After some measure-theoretic preparations in Section 3 we are ready to make our way towards the proofs of two structural theorems of Lindenstrauss announced earlier (Theorems 3.1 and 3.2). The main tool for these results are characterisations of non-weakly compact operators on $C(K)$-spaces and injective spaces. Let us start with the basic definition and some introductory remarks.

Definition 4.1. Let X and Y be Banach spaces. An operator $T\colon X \to Y$ is called [*weakly*] *compact* whenever $T(B_X)$, the range of the unit ball of X, is a relatively [weakly] compact subset of Y.

The collection of all (weakly) compact operators forms a two-sided ideal, which means that if S and T are (weakly) compact, then $S+T$ is (weakly) compact and both SR and RS are (weakly) compact for every operator R for which these compositions make sense. These facts are almost obvious; for proving the assertion about $S+T$ one only has to use the fact that an operator R is (weakly) compact if and only for every bounded sequence $(x_n)_{n=1}^{\infty}$ from the domain of R there exists a subsequence $(x_{n_j})_{j=1}^{\infty}$ such that $(Rx_{n_j})_{j=1}^{\infty}$ converges (weakly). In the case of weakly compact operator this argument involves, of course, the Eberlein–Šmulian theorem.

The identity operator I_X, acting on a Banach space X, is compact if and only if X is finite-dimensional. This follows immediately from the fact that the unit ball of an infinite-dimensional normed space is never compact in the norm topology. On the other hand, I_X is weakly compact if and only if X is reflexive, because weak compactness of the unit ball B_X is equivalent to reflexivity of X. For operators which map an arbitrary Banach space X into ℓ_1 (more generally, into any Banach space with Schur's property) we may say something more interesting. Namely, if $T\colon X \to \ell_1$ is a weakly compact operator then it is automatically compact. This follows from the fact that every weakly compact subset of ℓ_1 is compact. Indeed, if $A \subset \ell_1$ is weakly compact and $(x_n)_{n=1}$ is any sequence of elements from A, then by the Eberlein–Šmulian theorem we may extract a subsequence $(x_{n_j})_{j=1}^{\infty}$ that is weakly convergent. But then, Schur's property of ℓ_1 (Theorem 2.9) implies that $(x_{n_j})_{j=1}^{\infty}$ is norm convergent, which proves that A is compact in the norm topology.

The following fact is one of the most fundamental results on weakly compact operators.

Theorem 4.2 (Gantmacher, 1940). *Let X and Y be Banach spaces and $T\colon X \to Y$ be an operator. Then, the following assertions are equivalent:*

(i) *T is weakly compact;*
(ii) *T^* is weakly compact;*
(iii) *$T^{**}(X^{**}) \subset i_Y(Y)$, where $i_Y\colon Y \hookrightarrow Y^{**}$ is the canonical embedding.*

We shall prove Gantmacher's theorem for completeness, starting with the following simple lemma.

Lemma 4.3. *Let X and Y be Banach spaces and $T\colon X \to Y$ be an operator. Let also $i_X\colon X \hookrightarrow X^{**}$ and $i_Y\colon Y \hookrightarrow Y^{**}$ be the canonical embeddings. Then, $T^{**}i_X(X) \subseteq i_Y(Y)$ and $i_Y^{-1}T^{**}i_X = T$.*

Proof. For the first assertion fix any $x \in X$; we shall show that the functional $T^{**}i_X(x) \in Y^{**}$ is w^*-continuous, which would imply that it belongs to $i_Y(Y)$. To this end observe that for every $y^* \in Y^*$ we have

$$\langle y^*, T^{**}i_X(x)\rangle = \langle T^*y^*, i_X(x)\rangle = \langle x, T^*y^*\rangle = \langle Tx, y^*\rangle.$$

Now, let (y^*_α) be any net in Y^*, weak* convergent to some $y^*_0 \in Y^*$. Then,

$$\langle y^*_\alpha, T^{**}i_X(x)\rangle = \langle Tx, y^*_\alpha\rangle \xrightarrow[\alpha]{} \langle Tx, y^*_0\rangle = \langle y^*_0, T^{**}i_X(x)\rangle,$$

which shows that $T^{**}i_X(x)$ is w^*-continuous.

The composition $i_Y^{-1}T^{**}i_X$ is an operator mapping X into Y. By the conclusion of the preceding paragraph, for any $y^* \in Y^*$ we have

$$\langle Tx, y^*\rangle = \langle y^*, T^{**}i_X(x)\rangle = \langle i_Y^{-1}T^{**}i_X(x), y^*\rangle,$$

which completes the proof. □

Proof of Theorem 4.2. Let us start with showing the equivalence **(i)** $\Leftrightarrow$ **(iii)**. We denote as i_X and i_Y the canonical embeddings $X \hookrightarrow X^{**}$ and $Y \hookrightarrow Y^{**}$, respectively.

First, assume that (iii) holds true. Then, since $i_Y^{-1}\colon i(Y) \to Y$ is w^*-to-w continuous and $T^{**}\colon X^{**} \to Y^{**}$ is w^*-to-w^* continuous (as every adjoint operator), $i_Y^{-1}T^{**}\colon X^{**} \to Y$ is w^*-to-w continuous. By the Banach–Alaoglu theorem, the ball $B_{X^{**}}$ is w^*-compact, hence $i_Y^{-1}T^{**}(B_{X^{**}})$ is weakly compact. Therefore, its subset $i_Y^{-1}T^{**}i_X(B_X)$ is relatively weakly compact, but in view of Lemma 4.3 this is just $T(B_X)$.

Now, suppose (i) T is weakly compact. Then, i_YT is weakly compact as well, thus the set $\overline{i_YT(B_X)}^{w}$ is weakly compact, so it is also w^*-compact. Hence,

$$\overline{i_YT(B_X)}^{w^*} = \overline{i_YT(B_X)}^{w} = \overline{i_YT(B_X)},$$

where the last equality follows from the fact that for convex sets weak closure and norm closure are the same. By the w^*-to-w^* continuity of T^{**}, the Banach–Alaoglu theorem and the Goldstine theorem, we have

$$T^{**}(B_{X^{**}}) = \overline{T^{**}i_X(B_X)}^{w^*}.$$

Making use of Lemma 4.3 ($i_Y^{-1}T^{**}i_X = T$) we may thus write

$$T^{**}(B_{X^{**}}) = \overline{T^{**}i_X(B_X)}^{w^*} = \overline{i_YT(B_X)}^{w^*} = \overline{i_YT(B_X)} \subseteq i_Y(Y),$$

as $i_Y(Y)$ is a norm closed subspace of Y^{**}.

Now, we may proceed to the proof of the equivalence **(i)** $\Leftrightarrow$ **(ii)**. Suppose (i) T is weakly compact and fix any $y^{***} \in Y^{***}$. In light of what we have proved so far, we shall show that $T^{***}y^{***} \in i_{X^*}(X^*)$. This would follow if we show that $T^{***}y^{***}$ is w^*-continuous on X^{**}. So, take any net $(x_\alpha^{**}) \subset X^{**}$ which is w^*-convergent to some $x_0^{**} \in X^{**}$. By (iii) we have $T^{**}(X^{**}) \subseteq i_Y(Y)$ and by the w^*-to-w^* continuity of T^{**} we have

$$T^{**}x_\alpha^{**} \xrightarrow[\alpha]{w^*} T^{**}x_0^{**}, \quad \text{whence} \quad T^{**}x_\alpha^{**} \xrightarrow[\alpha]{w} T^{**}x_0^{**},$$

since the relative weak* and the relative weak topologies on $i_Y(Y) \subset Y^{**}$ coincide. Consequently,

$$\langle x_\alpha^{**}, T^{***}y^{***}\rangle = \langle T^{**}x_\alpha^{**}, y^{***}\rangle \xrightarrow[\alpha]{} \langle T^{**}x_0^{**}, y^{***}\rangle = \langle x_0^{**}, T^{***}y^{***}\rangle,$$

which proves our claim.

It remains to prove that (ii) implies (i), but this follows easily from the work we have just done. Namely, if T^* is weakly compact then T^{**} is weakly compact and Lemma 4.3 gives $i_Y^{-1}T^{**}i_X = T$, so T is weakly compact as well. □

Gantmacher's theorem may be also proved with the aid of the factorisation theorem due to Davis, Figiel, Johnson and Pełczyński [DFJ74]. Note that in the case where either the domain or the codomain considered is reflexive, every operator is weakly compact. The aforementioned result says that, in a sense, such a situation is typical for all weakly compact operators. We will not use this result anywhere in the sequel, but it should be at least mentioned; for the proof see, e.g., [Mor01, pp. 214–215].

Theorem 4.4 (Davis, Figiel, Johnson, Pełczyński, 1974)**.** *Let X and Y be Banach spaces and $T\colon X \to Y$ be an operator. Then, T is weakly compact if and only if there exists a reflexive Banach space Z and operators $P\colon Z \to Y$, $Q\colon X \to Z$ such that $T = PQ$. That is, every weakly compact operator factors through a reflexive space.*

Let $T\colon X \to Y$ be an operator between Banach spaces X and Y. We say that T is *bounded below* provided that for some constant $\delta > 0$ we have $\|Tx\| \geqslant \delta\|x\|$ for all $x \in X$. By the Open Mapping Theorem, this is equivalent to T being an isomorphism onto Y. Similarly, if Z is a subspace of X, then T is bounded below on Z if and only if the restriction $T|_Z$ is an isomorphism onto its range $T(Z)$.

Now, suppose we know that T is bounded below on some subspace $Z \subset X$ which is not reflexive. Then, the unit ball B_Z is not weakly compact and the range $T(B_X)$ contains an isomorphic copy of B_Z. This means that T definitely cannot be weakly compact. Regarding the converse, for some classes of operators we may distinguish some 'testing' space, a concrete non-reflexive space Z, such that every non-weakly compact operator from the class considered acts as an isomorphism on some copy of Z. The following result of Pełczyński says that for operators acting on $C(K)$-spaces this role of a 'testing' space is played by c_0. Its proof is based on the three great results we have discussed: Gantmacher's theorem, Grothendieck's theorem and Rosenthal's lemma (not the last time!).

Theorem 4.5 (Pełczyński, 1962)**.** *Let K be a compact Hausdorff space, X be a Banach space and $T\colon C(K) \to X$ be an operator. If T is not weakly compact, then there exists an isomorphic copy $Z \subset C(K)$ of c_0 such that $T|_Z$ is bounded below.*

Proof. Assume $T\colon C(K) \to X$ is not weakly compact. Then, Theorem 4.2 guarantees that neither is $T^*\colon X^* \to \mathcal{M}(K)$. This means that $T^*(B_{X^*})$ is not relatively weakly compact, hence Grothendieck's Theorem 3.6 implies that there exist: a positive number ε, a sequence $(x_n^*)_{n=1}^\infty \subset B_{X^*}$ and a sequence $(U_n)_{n=1}^\infty$ of pairwise disjoint open subsets of K such that if we set $\nu_n = T^* x_n^*$, then $\nu_n(U_n) > \varepsilon$ for each $n \in \mathbb{N}$. Moreover, by Rosenthal's Lemma 2.1 we may (and we do) assume that

$$|\nu_n|\Bigg(\bigcup_{j\neq n} U_j\Bigg) < \frac{\varepsilon}{2} \quad \text{for each } n \in \mathbb{N}.$$

This was the crucial part. The rest is just having fun.

For every $n \in \mathbb{N}$ pick a compact set $F_n \subset U_n$ with $|\nu_n|(U_n \setminus F_n)$ so small that $\nu_n(F_n) - |\nu_n|(U_n \setminus F_n) > \varepsilon$. The Urysohn lemma produces a sequence $(f_n)_{n=1}^\infty \subset C(K)$ satisfying:

- $0 \leqslant f_n \leqslant 1$,
- $f_n = 1$ on F_n,
- f_n vanishes outside U_n,

for each $n \in \mathbb{N}$. Then, we have also

$$\int_K f_n \, \mathrm{d}\nu_n = \int_{U_n} f_n \, \mathrm{d}\nu_n \geqslant \nu_n(F_n) - |\nu_n|(U_n \setminus F_n) > \varepsilon.$$

Let's have a closer look at the measure $\nu_n = T^* x_n^*$. For any $f \in C(K)$ we have

$$\int_K f \, \mathrm{d}\nu_n = \langle f, T^* x_n^* \rangle = \langle Tf, x_n^* \rangle,$$

which shows that ν_n is nothing but the representing measure for the functional $x_n^* T$, stemming from the Riesz Representation Theorem. So, we may add to our list the following conclusion:

- $x_n^* T f_n = \int_K f_n \, \mathrm{d}\nu_n > \varepsilon$ for each $n \in \mathbb{N}$.

Now, let $Y = \{\sum_{n=1}^\infty \alpha_n f_n \colon (\alpha_n)_{n=1}^\infty \in c_0\} \subset C(K)$. Since $\|f_n\| = 1$ for $n \in \mathbb{N}$ and f_n's are disjointly supported, it is easily seen that Y is an isometric copy of c_0 inside $C(K)$. To finish the proof we shall show that $T|_Y$ is bounded below. Take any $f = \sum_{n=1}^\infty \alpha_n f_n \in Y$. Then, for every $n \in \mathbb{N}$ we have

$$\begin{aligned}
|x_n^* T f| &= \left|\int_K f \, \mathrm{d}\nu_n\right| = \left|\alpha_n \int_{U_n} f_n \, \mathrm{d}\nu_n + \int_{\bigcup_{j\neq n} U_j} f \, \mathrm{d}\nu_n\right| \\
&\geqslant |\alpha_n|\varepsilon - \int_{\bigcup_{j\neq n} U_j} |f| \, \mathrm{d}|\nu_n| \\
&\geqslant |\alpha_n|\varepsilon - |\nu_n|\Bigg(\bigcup_{j\neq n} U_j\Bigg)\|f\| \\
&\geqslant |\alpha_n|\varepsilon - \|f\| \cdot \frac{\varepsilon}{2}
\end{aligned}$$

and, consequently, $\|Tf\| \geqslant (\varepsilon/2)\|f\|$ for every $f \in Y$. □

We are now well prepared to prove two Rosenthal's theorems, the latter of which yields a striking improvement of Pełczyński's theorem for injective Banach spaces.

Theorem 4.6 (Rosenthal, 1970). *Let X and Y be Banach spaces, $T\colon X\to Y$ be an operator, and suppose that X is complemented in X^{**}. If there exists a subspace $Z_1\subset X$ isomorphic to $c_0(\Gamma)$, for some non-empty index set Γ, such that $T|_{Z_1}$ is bounded below, then there exists a subspace $Z_2\subset X$ isomorphic to $\ell_\infty(\Gamma)$ such that $T|_{Z_2}$ is also bounded below.*

Proof. First, we want to define an operator $S\colon \ell_\infty(\Gamma)\to X$ such that $S|_{c_0(\Gamma)}$ yields an isomorphism onto Z_1. To this end consider an operator $S_1\colon c_0(\Gamma)\to X$ which is an isomorphism onto Z_1 and let $P\colon X^{**}\to X$ be a projection onto X, that is, an operator satisfying $P\circ i_X=I_X$, where $i_X\colon X\hookrightarrow X^{**}$ is the canonical embedding. Then $S_1^{**}\colon \ell_\infty(\Gamma)\to X^{**}$; set $S=PS_1^{**}$. We shall check that $PS_1^{**}|_{c_0(\Gamma)}=S_1$, so fix any $\xi\in c_0(\Gamma)$. For any $x^*\in X^*$ we have

$$\langle x^*,S_1^{**}\xi\rangle=\langle S_1^*x^*,\xi\rangle=\langle \xi,S_1^*x^*\rangle=\langle S_1\xi,x^*\rangle,$$

which implies that $S_1^{**}\xi=i(S_1\xi)$, hence $PS_1^{**}\xi=S_1\xi$ as desired.

Now, we may apply the following lemma to the operator $\Phi=TS\colon \ell_\infty(\Gamma)\to Y$.

Lemma. Let Γ be an infinite set and Y be a Banach space. Suppose $\Phi\colon \ell_\infty(\Gamma)\to Y$ is an operator such that $\Phi|_{c_0(\Gamma)}$ is bounded below. Then there exists a set $\Delta\subset\Gamma$ with $|\Delta|=|\Gamma|$ and such that $\Phi|_{\ell_\infty(\Delta)}$ is bounded below.

Proof of the Lemma. Define $K=\|(\Phi|_{c_0(\Gamma)})^{-1}\|$; this is a positive and finite number. For each $\gamma\in\Gamma$ we have $\|\Phi\mathbb{1}_{\{\gamma\}}\|\geqslant 1/K$, hence we may pick a functional $\varphi_\gamma\in Y^*$ satisfying $\|\varphi_\gamma\|\leqslant K$ and $\varphi_\gamma(\Phi\mathbb{1}_{\{\gamma\}})=1$.

Now, for each $\gamma\in\Gamma$ we define a scalar-valued set function μ_γ on $\mathcal{P}\Gamma$ by the formula

$$\mu_\gamma(E)=\big(\Phi^*\varphi_\gamma\big)\mathbb{1}_E=\varphi_\gamma\big(\Phi\xi\big)\quad\text{for every } E\subset\Gamma.$$

Plainly, μ_γ is finitely additive. Moreover, for every finite partition $\pi=\{E_1,\dots,E_k\}$ of Γ we have

$$\sum_{E_j\in\pi}|\mu_\gamma(E_j)|=\sum_{E_j\in\pi}|\langle\mathbb{1}_{E_j},\Phi^*\varphi_\gamma\rangle|,$$

which shows that

$$|\mu_\gamma|(\Gamma)=\sup\big\{|\langle\xi,\Phi^*\varphi_\gamma\rangle|\colon \xi\in B_{\ell_\infty(\Gamma)}\text{ is a step function}\big\}=\|\Phi^*\varphi_\gamma\|.$$

Therefore, $\sup_{\gamma\in\Gamma}|\mu_\gamma|\leqslant K\|T\|$ and hence we may apply Rosenthal's Lemma 2.2 to the family $\{|\mu_\gamma|\colon\gamma\in\Gamma\}$ and produce a set $\Delta\subset\Gamma$ satisfying $|\Delta|=|\Gamma|$ and

$$|\mu_\delta|\big(\Delta\setminus\{\delta\}\big)<\frac{1}{2}\quad\text{for each }\delta\in\Delta.$$

Observe that

$$\varphi_\gamma(\Phi\xi)=\int_\Gamma\xi\,\mathrm{d}\mu_\gamma\quad\text{for every }\gamma\in\Gamma\text{ and }\xi\in\ell_\infty(\Gamma),$$

since both sides of this equality, viewed as functions of ξ, are continuous and linear functionals on $\ell_\infty(\Gamma)$ which agree at every characteristic function, so agree everywhere. Hence, for every $\xi\in\ell_\infty(\Delta)$ and each $\delta\in\Delta$ we have

$$\big|\varphi_\delta\big(\Phi\xi\big)\big|=\left|\int_\Gamma\xi\,\mathrm{d}\mu_\delta\right|=\left|\xi(\delta)+\int_{\Delta\setminus\{\delta\}}\xi\,\mathrm{d}\mu_\delta\right|\geqslant|\xi(\delta)|-|\mu_\delta|\big(\Delta\setminus\{\delta\}\big)\|\xi\|\geqslant|\xi(\delta)|-\frac{1}{2}\|\xi\|$$

(we have used the fact that $\mu_\delta\{\delta\} = \varphi_\delta(\Phi\mathbb{1}_{\{\delta\}}) = 1$). Consequently,

$$\|\Phi\xi\| \geqslant \frac{1}{K}\sup_{\delta\in\Delta}\big|\varphi_\delta(\Phi\xi)\big| \geqslant \frac{1}{2K}\|\xi\|,$$

which means that $\Phi|_{\ell_\infty(\Delta)}$ is bounded below.

Proof of Theorem 4.6 (continued). The restriction $TS|_{c_0(\Gamma)}$ is bounded below, so the above Lemma implies that there is a set $\Delta\subset\Gamma$ with $|\Delta| = |\Gamma|$ and such that $TS|_{\ell_\infty(\Delta)}$ is bounded below. Hence T must be bounded below on the subspace $Z_2 = S(\ell_\infty(\Delta))\subset X$ which is obviously isomorphic to $\ell_\infty(\Gamma)$. □

Definition 4.7. A Banach space X is called *injective*, provided that for every Banach space Y, every subspace Z of Y, and every operator $t\colon Z\to X$, there exists an operator $T\colon Y\to X$ which extends t, i.e. $T|_Z = t$.

Theorem 4.8 (Rosenthal, 1970)**.** *Let X be an injective Banach space, Y be a Banach space and $T\colon X\to Y$ be an operator. If T is not weakly compact, then there exists a subspace $Z\subset X$ isomorphic to ℓ_∞ and such that $T|_Z$ is bounded below.*

Proof. Injectivity of X guarantees that there is a compact Hausdorff space K and a surjective operator $S\colon C(K)\to X$. In fact, let $K = B_{X^*}$, equipped with the weak* topology. Then, by the Banach–Alaoglu theorem, K is compact and Hausdorff. The space X embeds isometrically into $C(K)$ via the map $X\ni x\mapsto f_x\in C(K)$ given by $f_x(x^*) = \langle x, x^*\rangle$ (note that $\|f_x\| = \sup_{x^*\in K}\langle x, x^*\rangle = \|x\|$). Hence, $Y = \{f_x\colon x\in X\}$ is a subspace of $C(K)$, isometric to X, for which we have a surjective operator $Y\to X$ acting as $f_x\mapsto x$. By the injectivity of X, we may extend this operator to the desired operator $S\colon C(K)\to X$.

Now, $TS\colon C(K)\to Y$ is not weakly compact. By Pełczyński's Theorem 4.5, there is an isomorphic copy $Z\subset C(K)$ of c_0 such that $TS|_Z$ is bounded below. Hence, $S(Z)\subset X$ is isomorphic to c_0 and $T|_{S(Z)}$ is bounded below. Since X is injective, X is complemented in X^{**} (we may extend the operator $i_X^{-1}\colon i(X)\to X$), so appealing to Theorem 4.6 completes the proof. □

Let us note two quick corollaries. The first one, for $\Gamma = \mathbb{N}$ and X being a dual Banach space, is due to Bessaga and Pełczyński and, in the general form, due to Rosenthal.

Corollary 4.9 (Bessaga & Pełczyński, 1958; Rosenthal, 1970)**.** *Let X be a Banach space complemented in its bidual (for instance, any dual space), and let Γ be an infinite set. If X contains an isomorphic copy of $c_0(\Gamma)$, then it contains an isomorphic copy of $\ell_\infty(\Gamma)$.*

Proof. Apply Theorem 4.6 to $T = I_X$. □

Corollary 4.10 (Amir, 1964)**.** *ℓ_∞/c_0 is not an injective Banach space. It is even not complemented in its bidual.*

Proof. First, notice that ℓ_∞/c_0 contains an isometric copy of $c_0(\mathfrak{c})$ (see Problem 2.4). However, the cardinality of ℓ_∞/c_0 equals $\mathfrak{c}$, so it cannot contain an isomorphic copy of $\ell_\infty(\mathfrak{c})$, since the latter has the cardinality of $2^{\mathfrak{c}}$. The conclusion now follows from Corollary 4.9. □