Combinatorics in Banach space theory

Lecture 5

5 Lindenstrauss' theorems on the structure of $\ell_\infty(\Gamma)$-spaces

The goal of this section is to finally prove two Lindenstrauss' Theorems 3.1 and 3.2 stated in Section 3. With the Rosenthal Theorems 4.6 and 4.8 in hand, we are not far away from doing this. However, we still need a couple of auxiliary results, which are certainly important not only for our present purposes.

Definition 5.1. Let $X_1, X_2, \ldots$ be Banach spaces and let $1 \leqslant p < \infty$. We define Banach spaces:

- $\big(\bigoplus_{n=1}^\infty X_n\big)_p = \big\{x = (x_n)_{n=1}^\infty \in \prod_{n=1}^\infty X_n \mid \|x\|_p := \big(\sum_{n=1}^\infty \|x_n\|^p\big)^{1/p} < \infty\big\}$, equipped with the norm $\|\cdot\|_p$ (the the so-called *ℓ_p-sum*),
- $\big(\bigoplus_{n=1}^\infty X_n\big)_\infty = \big\{x = (x_n)_{n=1}^\infty \in \prod_{n=1}^\infty X_n \mid \|x\|_\infty := \sup_{n\in\mathbb{N}} \|x_n\| < \infty\big\}$, equipped with the norm $\|\cdot\|_\infty$ (the so-called *ℓ_∞-sum*),
- $\big(\bigoplus_{n=1}^\infty X_n\big)_0 = \big\{x = (x_n)_{n=1}^\infty \in \prod_{n=1}^\infty X_n \mid \lim_{n\to\infty} \|x_n\| = 0\big\}$, equipped with the norm $\|\cdot\|_\infty$ (the so-called *c_0-sum*).

In the case where $X_1 = X_2 = \ldots = X$ we simply write $\ell_p(X)$, $\ell_\infty(X)$ and $c_0(X)$ instead of those three symbols defined above.

The fact that the so-defined spaces are indeed Banach spaces may be proved in a routine way, so we leave it to a desperate reader. Let us remind that for any Banach spaces X and Y we write $X \simeq Y$ whenever X and Y are isomorphic. Observe that for every Banach space X we have $\ell_p(X) \oplus \ell_p(X) \simeq \ell_p(X)$ and $X \oplus \ell_p(X) \simeq \ell_p(X)$, for any $1 \leqslant p \leqslant \infty$, as well as $c_0(X) \oplus c_0(X) \simeq c_0(X)$ and $X \oplus c_0(X) \simeq c_0(X)$ (see Problem 4.1).

Now, we will show an extremely useful decomposition result.

Theorem 5.2 (The Pełczyński Decomposition Method, 1960)**.** *Let X and Y be Banach spaces, either of which is isomorphic to a complemented subspace of the other. Assume also that at least one of the following conditions holds true:*

(a) *$X \simeq X \oplus X$ and $Y \simeq Y \oplus Y$;*
(b) *$X \simeq c_0(X)$ or $X \simeq \ell_p(X)$ for some $1 \leqslant p \leqslant \infty$.*

Then $X \simeq Y$.

Proof. In both cases, (a) and (b), we will show that both X and Y are isomorphic to $X \oplus Y$, so they are isomorphic to each other. By the assumption, we may write $X \simeq Y \oplus E$ and $Y \simeq X \oplus F$, for some Banach spaces E and F.

(a): We have $X \simeq Y \oplus Y \oplus E \simeq Y \oplus X$ and, similarly, $Y \simeq X \oplus X \oplus F \simeq X \oplus Y$ as desired.

(b): Since X satisfies (b), we have $X \simeq X \oplus X$, thus the argument from part (a) gives $Y \simeq X \oplus Y$. Now, suppose that $X \simeq \ell_p(X)$. Noticing that $\ell_p(Y) \simeq Y \oplus \ell_p(Y)$ we obtain

$$X \simeq \ell_p(X) \simeq \ell_p(Y \oplus E) \simeq \ell_p(Y) \oplus \ell_p(E) \simeq Y \oplus \ell_p(Y) \oplus \ell_p(E) \simeq Y \oplus \ell_p(X) \simeq Y \oplus X.$$

In the case where $X \simeq c_0(X)$ the proof is the same; one only has to replace all the ℓ_p-sums by c_0-sums. $\square$

The first of the two announced Lindenstrauss theorems is already within our reach.

Proof of Theorem 3.1. Suppose X is a complemented subspace of $\ell_\infty(\Gamma)$ which contains an isomorphic copy of $c_0(\Gamma)$. Then, X is injective (being a complemented subspace of an injective space; see Problem 1.7) and hence, of course, it is complemented in its bidual. If so, Corollary 4.9 says that X contains an isomorphic copy of $\ell_\infty(\Gamma)$ and it must be complemented as $\ell_\infty(\Gamma)$ is injective. Moreover, $\ell_\infty(\Gamma) \simeq \ell_\infty(\ell_\infty(\Gamma))$, thus the Pełczyński Decompostion Method yields $X \simeq \ell_\infty(\Gamma)$. $\square$

In order to prove Theorem 3.2 (which says that ℓ_∞ is prime) we need to exclude the existence of Banach spaces that are simultaneously infinite-dimensional, reflexive and injective. This will done in three steps (a)-(c), in Theorem 5.5 below. Let us first recall the definition of Dunford–Pettis property which has already appeared in Problem 1.11. The reader is encouraged to consult this problem, where several equivalent conditions defining Dunford–Pettis operators may be found.

Definition 5.3. A Banach space X is said to have the *Dunford–Pettis property*, provided that every weakly compact operator defined on X (and taking values in an arbitrary Banach space) is a Dunford–Pettis (completely continuous) operator, i.e. it is weak-to-norm sequentially continuous.

Lemma 5.4. *A Banach space X has the Dunford–Pettis property if and only if for every weakly null sequence $(x_n)_{n=1}^\infty \subset X$ and every weakly null sequence $(x_n^*) \subset X^*$ we have* $\lim_{n\to\infty} x_n^* x_n = 0$.

Proof. First, we prove the ‘if’ part. So, suppose Y is a Banach space and $T\colon X \to Y$ is a weakly compact operator that is not Dunford–Pettis. Then, there is a weakly null sequence $(x_n)_{n=1}^\infty \subset X$ such that $\|Tx_n\| \geqslant \delta$ for every $n \in \mathbb{N}$ and some $\delta > 0$. For each $n \in \mathbb{N}$ pick a functional $y_n^* \in B_{Y^*}$ satisfying $y_n^* T x_n = \|Tx_n\|$.

By Gantmacher's Theorem 4.2, the operator $T^*\colon Y^* \to X^*$ is weakly compact, whence by the Eberlein–Šmulian theorem we may assume that the sequence $(T^* y_n^*) \subset T^*(B_{Y^*})$ is weakly convergent to some $x_0^* \in X^*$. Therefore, we have $T^* y_n^* - x_0^* \xrightarrow{w} 0$, so our hypothesis implies that $(T^* y_n^* - x_0^*)(x_n) \to 0$. However, $x_0^* x_n \to 0$, thus $T^* y_n^*(x_n) \to 0$. On the other hand, $T^* y_n^*(x_n) = y_n^* T x_n = \|Tx_n\| \geqslant \delta$ for each $n \in \mathbb{N}$; a contradiction.

Now, in order to prove the ‘only if’ part, fix any weakly null sequences $(x_n)_{n=1}^\infty \subset X$ and $(x_n^*)_{n=1}^\infty \subset X^*$. Define an operator $T\colon X \to c_0$ by $T(x) = (x_n^* x)_{n=1}^\infty$ ($\in c_0$ because $x_n^* \xrightarrow{w} 0$) and let $T^*\colon \ell_1 \to X^*$ be its adjoint. For each $n \in \mathbb{N}$ and $x \in X$ we have $\langle x, T^* e_n\rangle = \langle Tx, e_n\rangle = \langle x, x_n^*\rangle$, whence $T^* e_n = x_n^*$. Consequently, $T^*(B_{\ell_1}) \subset \mathrm{conv}\{x_n^*\colon n \in \mathbb{N}\}$ which is relatively weakly compact by the Krein theorem*. Hence, T^* is a weakly compact operator and, in view of Gantmacher's Theorem 4.2, so is T.

By the assumption (i), T is Dunford–Pettis, whence it is weak-to-norm sequentially continuous (see Problem 1.11). Hence, $\|T(x_n)\|_\infty \to 0$ and our assertion follows, since $|x_n^* x_n| \leqslant \|T(x_n)\|_\infty$ for every $n \in \mathbb{N}$. $\square$

***The Krein theorem** says that if A is a weakly compact subsets of a Banach space, then its closed convex hull $\overline{\mathrm{conv}}(A)$ is weakly compact as well. See, e.g., Theorem 3.133 in [FHH10].

Now, we will need the fact that every relatively weakly compact set $\mathcal{F} \subset L_1(\mu)$ is equi-integrable. The converse statement was proved in Section 3 under the assumption that $\mathcal{F}$ is bounded. Recall that here μ is a finite, σ-additive, non-negative measure defined on a σ-algebra Σ of subsets of Ω.

Proof of the implication (i) $\Rightarrow$ (ii) in Lemma 3.5. It follows readily from Definition 3.4 of equi-integrability that it is enough to prove that every sequence in $\mathcal{F}$ has an equi-integrable subsequence. So, fix any sequence $(f_n)_{n=1}^\infty \subset \mathcal{F}$ and assume (by passing to a subsequence, if necessary) that it is weakly convergent. Then, in particular, for every $A \in \Sigma$ the limit $\lim_{n\to\infty} \int_A f_n \,\mathrm{d}\mu$ exists, since $\mathbb{1}_A \in L_\infty(\mu)$ defines a functional from $L_1(\mu)^*$. We will show that the equi-integrability of $(f_n)_{n=1}^\infty$ is a consequence of this property.

Let $\sim$ be an equivalence relation in Σ defined by saying that $A \sim B$ if and only if $\mu(A \triangle B) = 0$, where $\triangle$ stands for the symmetric difference. It is easy to check that the formula $\rho(A, B) = \mu(A\triangle B)$ defines a pseudometric on Σ, so it generates a metric on the set $\widetilde{\Sigma} = \Sigma/_\sim$ of all equivalence classes. For simplicity, we identify sets from Σ with corresponding equivalence classes from $\widetilde{\Sigma}$ and we use the same symbol ρ for the metric in $\widetilde{\Sigma}$. Notice that

$$\rho(A, B) = \|\mathbb{1}_A - \mathbb{1}_B\|_{L_1(\mu)} \quad \text{for all } A, B \in \widetilde{\Sigma},$$

thus the metric space $(\widetilde{\Sigma}, \rho)$ is isometric to the closed subspace $\{\mathbb{1}_A \colon A \in \Sigma\}$ of $L_1(\mu)$ which is known to be complete. Hence, $(\widetilde{\Sigma}, \rho)$ is a complete metric space (sometimes called the *measure algebra* corresponding to the measure space (Ω, Σ, μ)).

Now, fix any $\varepsilon > 0$ and for each $N \in \mathbb{N}$ define

$$F_N = \left\{ A \in \widetilde{\Sigma} \colon \left| \int_A (f_m - f_n) \,\mathrm{d}\mu \right| \leqslant \varepsilon \ \text{ for all } m, n \geqslant N \right\}.$$

These sets are, of course, closed subsets of $\widetilde{\Sigma}$ and, by our assumption, we have $\widetilde{\Sigma} = \bigcup_{N=1}^\infty F_N$. Hence, the Baire Category Theorem implies that for some $N_0 \in \mathbb{N}$ the set F_{N_0} has a non-empty interior, that is, there is some $A_0 \in F_{N_0}$ and $r > 0$ such that

$$\mu(A \triangle A_0) < r \quad \text{implies} \quad A \in F_{N_0}, \quad \text{for every } A \in \widetilde{\Sigma}. \tag{5.1}$$

For every set $B \in \Sigma$ with $\mu(B) < r$ we have $\mu((A_0 \cup B) \triangle A_0) < r$ and $\mu((A_0 \setminus B) \triangle A_0) < r$. Since $\int_B = \int_{A_0\cup B} - \int_{A_0\setminus B}$, condition (5.1) implies that $|\int_B (f_m - f_n) \,\mathrm{d}\mu| \leqslant 2\varepsilon$ for all $m, n \geqslant N_0$. Applying this argument to the sets $B \cap \{f_m - f_n \geqslant 0\}$ and $B \cap \{f_m - f_n \leqslant 0\}$ we conclude that

$$\int_B |f_m - f_n| \,\mathrm{d}\mu \leqslant 4\varepsilon \quad \text{whenever } \mu(B) < r \text{ and } m, n \geqslant N_0. \tag{5.2}$$

Finally, since $f_{N_0} \in L_1(\mu)$, there exists a positive $s < r$ such that $\int_B |f_{N_0}| \,\mathrm{d}\mu \leqslant \varepsilon$ whenever $\mu(B) < s$. Therefore, for every $n \geqslant N_0$ and every $B \in \Sigma$ with $\mu(B) < s$ inequality (5.2) yields

$$\int_B |f_n| \,\mathrm{d}\mu \leqslant \int_B |f_n - f_{N_0}| \,\mathrm{d}\mu + \int_B |f_{N_0}| \,\mathrm{d}\mu \leqslant 4\varepsilon + \varepsilon = 5\varepsilon,$$

which shows that the set $\{f_n \colon n \geqslant N_0\}$ is equi-integrable, thus so is $\{f_n \colon n \in \mathbb{N}\}$. $\square$

Theorem 5.5 (Grothendieck, 1953)**.** *Let K be a compact Hausdorff space. Then:*

(a) *$C(K)$ has the Dunford–Pettis property;*
(b) *If $T\colon C(K)\to C(K)$ is weakly compact, then T^2 is compact;*
(c) *There is no infinite-dimensional, complemented, reflexive subspace of $C(K)$.*

Proof. **(a):** According to Lemma 5.4 we are to prove that for every weakly null sequence $(f_n)_{n=1}^\infty\subset C(K)$ and every weakly null sequence $(\mu_n)_{n=1}^\infty\subset\mathcal{M}(K)\simeq C(K)^*$ we have $\langle f_n,\mu_n\rangle\to 0$. Since every weakly convergent sequence is bounded, we may safely assume that these sequences lie in the unit balls of $C(K)$ and $\mathcal{M}(K)$, respectively.

Define a non-negative measure $\nu\in\mathcal{M}(K)$ by the formula

$$\nu=\sum_{n=1}^\infty\frac{1}{2^n}|\mu_n|\,.$$

Obviously, $\mu_n\ll\nu$ for each $n\in\mathbb{N}$, so let $g_n\in L_1(\nu)$ be the Radon–Nikodým derivative of μ_n with respect to ν. Then, $\Phi(g_n)=\mu_n$ for $n\in\mathbb{N}$, where $\Phi\colon L_1(\mu)\to\mathcal{AC}(\nu)$ is the isometry described in the proof of Theorem 3.6. Since $(\mu_n)_{n=1}^\infty$ is weakly null in $\mathcal{M}(K)$, the sequence $(g_n)_{n=1}^\infty$ is weakly null in $L_1(\nu)$ and, by the just proved implication (i) $\Rightarrow$ (ii) in Lemma 3.5, it is also equi-integrable. Hence, by the assertion of Problem 3.14, we have

$$\lim_{M\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|g_n|>M\}}|g_n|\,\mathrm{d}\mu=0.$$

For every $M>0$ the Lebesgue Dominated Convergence Theorem implies

$$\lim_{n\to\infty}\int_{\{|g_n|\leqslant M\}}f_ng_n\,\mathrm{d}\nu=0$$

(note that $(f_n)_{n=1}^\infty$ converges pointwise to 0 as it is a weakly null sequence in $C(K)$). Consequently,

$$\limsup_{n\to\infty}|\langle f_n,\mu_n\rangle|=\limsup_{n\to\infty}\left|\int_K f_ng_n\,\mathrm{d}\nu\right|\leqslant\sup_{n\in\mathbb{N}}\int_{\{|g_n|>M\}}|g_n|\,\mathrm{d}\mu\xrightarrow[M\to\infty]{}0$$

as required.

(b): By the assertion (a), every weakly compact operator $T\colon C(K)\to C(K)$ maps (relatively) weakly compact sets into (relatively) norm compact sets. Hence, since $T(B_{C(K)})$ is relatively weakly compact, $T^2(B_{C(K)})$ is relatively norm compact, which means that T^2 is a compact operator.

(c): Suppose $Y\subset C(K)$ is a reflexive and complemented subspace and let $T\colon C(K)\to Y$ be a bounded projection onto Y. Then $T(B_{C(K)})=B_Y$ is weakly compact as Y is reflexive*, thus the assertion (b) says that $T=T^2$ is compact. Therefore, B_Y is compact which may happen only if Y is finite-dimensional. $\square$

*Note that if Y is a subspace of a Banach space X, then the weak topology on Y is exactly the same as the topology inherited by Y from the weak topology on X (why?).

Remark 5.6. The assertion (a) above almost immediately implies that $L_1(\mu)$-spaces, with any σ-finite and non-negative measure μ, also have the Dunford–Pettis property. This fact is due to Dunford, Pettis and Phillips. The only thing one has to know to derive this result from Grothendieck's theorem is that the dual space $L_1(\mu)^* \simeq L_\infty(\mu)$ is isometrically isomorphic to some $C(K)$-space (because with pointwise operations it forms a unital commutative Banach algebra). Hence, if $(f_n)_{n=1}^\infty \subset L_1(\mu)$ and $(g_n)_{n=1}^\infty \in L_\infty(\mu)$ are weakly null, then we may repeat the argument above regarding g_n's as members of $C(K)$ and f_n's as members of $C(K)^*$ (which in fact live in the predual of $C(K)$), which leads to the desired conclusion $\lim_{n\to\infty}\langle f_n, g_n\rangle = 0$.

We are finally in a position to prove that ℓ_∞ is a prime Banach space.

Proof of Theorem 3.2. Let X be an infinite-dimensional complemented subspace of ℓ_∞ and P be any bounded projection from ℓ_∞ onto X. The basic property of the Stone-Čech compactification $\beta\mathbb{N}$ is that every bounded function on $\mathbb{N}$ may be uniquely extended to a continuous function on $\beta\mathbb{N}$, the direct consequence of which is the isometric isomorphism $\ell_\infty \simeq C(\beta\mathbb{N})$. Hence, Theorem 5.5(c) guarantees that X is not reflexive, so the operator P is not weakly compact. Now, Rosenthal's Theorem 4.8 implies that X contains an isomorphic copy of ℓ_∞ and since ℓ_∞ is injective, this copy must be complemented in X. Consequently, we may use the Pełczyński Decomposition Method, Theorem 5.2(b), in the case $p = \infty$ (note that $\ell_\infty \simeq \ell_\infty(\ell_\infty)$) to conclude that $X \simeq \ell_\infty$. □

There are a few facts that should be added to this picture. In 1960, Pełczyński [Peł60] proved that, likewise ℓ_∞, the spaces c_0 and ℓ_p, for $1 \leqslant p < \infty$, are prime. This result was a great application of basic sequences techniques which we will discuss to some extent later. Notably, it is very difficult to construct any other infinite-dimensional prime Banach spaces. One of them was given by Gowers and Maurey [GM97], but its construction is extremely involved. That was a modified version of the original Gowers–Maurey space constructed in 1993 (see [GM93]) which gave a negative solution to the long-standing unconditional basic sequence problem. A distinctive feature of the prime Gowers–Maurey space is that it does not contain any infinite-dimensional complemented subspaces except these which are of finite codimension, so it has very little chance to be a non-prime Banach space. In fact, all its finite-codimensional subspaces are isomorphic to the whole space*.

On the other hand, there is a similar property which is weaker than primeness and shared by much more Banach spaces. Namely, a Banach space X is called *primary*, if for every decomposition $X = X_1 \oplus X_2$ we have $X \simeq X_1$ or $X \simeq X_2$. Every $L_p(0,1)$ space, for $1 \leqslant p < \infty$, is primary. For $1 < p < \infty$ this fact was proved by Alspach, Enflo and Odell [AEO77], whereas Enflo and Starbird [ES79] proved it for $p = 1$. Note that none of these spaces is prime, since it is not difficult to construct a complemented isomorphic copy of ℓ_p inside $L_p(0,1)$ and $\ell_p \not\simeq L_p(0,1)$. Lindenstrauss and Pełczyński [LP71] proved that the space $C[0,1]$ is also primary, while it is not prime as it contains complemented isomorphic copies of c_0.

*Note that in general it is not true that if X is an infinite-dimensional Banach space and $Y \subset X$ is of finite codimension, then $X \simeq Y$. For instance, if X is any HI Banach space (i.e. *hereditarily indecomposable* which means that for any subspace X' of X there is no decomposition $X' = X_1 \oplus X_2$ with both X_1 and X_2 being infinite-dimensional), then X is **not** isomorphic to any of its proper subspaces (this is Theorem 21 in [GM93]). The Gowers–Maurey space without an unconditional basic sequence was the very first example of an HI Banach space; the authors point out that it was observed by W.B. Johnson that their original construction could be modified to give an HI space.