Combinatorics in Banach space theory

Lecture 6

6 Grothendieck spaces

Having in hand Phillips' Lemma 2.8 and Grothendieck's Theorem 3.6 it would be a crime not to discuss another Grothendieck's result that expresses a highly non-trivial property of $C(K)$-spaces, with K being an extremally disconnected* compact Hausdorff space. Motivated by his characterisation of non-weakly compact subsets of $\mathcal{M}(K)$, in 1953 Grothendieck isolated a certain property, which is tautologically true for all reflexive Banach spaces, and which was later accepted by the community as a definition of Banach spaces named in honour of Alexandre Grothendieck.

Definition 6.1. A Banach space X is called a *Grothendieck space*, provided that every sequence $(x_n^*)_{n=1}^\infty \subset X^*$ that converges to 0 in the weak* topology converges to 0 in the weak topology.

Of course, this property may be equivalently stated by saying that every weak*-convergent sequence in X^* is weakly convergent (and its weak limit is the same as the weak* limit). If X is reflexive, then so is X^*, hence the weak* and weak topologies on X^* coincide. Consequently, every reflexive Banach space is, obviously, a Grothendieck space. It turns out that in the class of separable Banach spaces there are no other Grothendieck spaces except reflexive spaces. This is an easy consequence of the following fact:

Proposition 6.2. *Let X be a Banach space. Then, the following assertions are equivalent:*

(i) *X is a Grothendieck space;*
(ii) *for every separable Banach space Y every operator $T\colon X\to Y$ is weakly compact;*
(iii) *every operator $T\colon X\to c_0$ is weakly compact.*

Proof. First, we prove that **(i) ⇒ (ii)**. Let Y be a separable Banach space and $T\colon X\to Y$ be an operator. In view of Gantmacher's Theorem 4.2, we shall only prove that $T^*\colon Y^*\to X^*$ is weakly compact, which by Eberlein–Šmulian theorem is equivalent to saying that for every sequence $(y_n^*)_{n=1}^\infty \subset B_{Y^*}$ there exists a weakly convergent subsequence of $(T^*(y_n^*))_{n=1}^\infty$. Since Y is separable, the unit dual ball B_{Y^*} is metrisable in its weak* topology, hence there exists a weak*-convergent subsequence $(y_{n_j}^*)_{j=1}^\infty$ of $(y_n^*)_{n=1}^\infty$. Then, $(T^*(y_{n_j}^*))_{j=1}^\infty$ is also weak*-convergent as T^* is w^*-to-w^* continuous. Now, our assumption (i) implies that $(T^*(y_{n_j}^*))_{j=1}^\infty$ is in fact weakly convergent, which was to be proved.

The implication **(ii) ⇒ (iii)** is trivial.

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*Recall that a topological space is called **extremally disconnected** (also called a **Stonean** space) whenever the closure of any open set is still open. Equivalently, the closures of any two disjoint open sets are still disjoint. Every discrete space is extremally disconnected, as well as the Stone–Čech compactification βX of any discrete space X. The latter statement follows from the well-known (and easy to prove) formula $\overline{U}^{\beta X} = U\cup U^*$, which is true for every $U\subset X$, where $U^* = \{\mathsf{p}\in\beta X\colon U\in\mathsf{p},\ \text{and}\ \bigcap\mathsf{p}=\varnothing\}$: Indeed, observe that if $V\subset\beta X$ is open, then by the fact that X is dense in βX we have $\overline{V} = \overline{X\cap V} = (X\cap V)^*\cup(X\cap V)$ which is still an open set.

Now, in order to prove that **(iii) ⇒ (i)** let $(x_n^*)_{n=1}^\infty \subset X^*$ be a sequence that converges to 0 in the weak* topology. Fix, for a moment, any $x \in X$. Since the functional $i(x) \in X^{**}$, given by $X^* \ni x^* \mapsto \langle x, x^* \rangle$, is weak* continuous, we have $\lim_{n\to\infty} x_n^* x = 0$. Therefore, we may define an operator $T\colon X \to c_0$ by the formula $T(x) = (x_n^* x)_{n=1}^\infty$ and, by our assumption (iii), T is a weakly compact operator. By Gantmacher's Theorem 4.2(iii), we have $T^{**}(x^{**}) \in c_0$ for every $x^{**} \in X^{**}$. But observe that $\langle x, T^* e_n^* \rangle = \langle Tx, e_n^* \rangle = \langle x, x_n^* \rangle$ for every $n \in \mathbb{N}$ and $x \in X$, thus $T^* e_n^* = x_n^*$, and hence $\langle e_n^*, T^{**}(x^{**}) \rangle = \langle T^* e_n^*, x^{**} \rangle = \langle x_n^*, x^{**} \rangle$ for each $n \in \mathbb{N}$. Consequently, $c_0 \ni T^{**}(x^{**}) = (x^{**} x_n^*)_{n=1}^\infty$ for every $x^{**} \in X^{**}$, which implies that $(x_n^*)_{n=1}^\infty$ is weakly null. □

There are several other equivalent conditions defining Grothendieck spaces. We will not discuss them here; the interested reader may consult, e.g, [DU77, p. 179] or [Mor01, Theorem 4.9]. Just observe that if X is a separable Grothendieck space, then the identity operator I_X has a separable range, so the assertion (ii) implies that it is weakly compact which may happen only if X is reflexive. The following theorem is historically the first result which gives a non-trivial (non-reflexive) example of a Grothendieck space.

Theorem 6.3 (Grothendieck, 1953)**.** *If K is an extremally disconnected compact Hausdorff space, then $C(K)$ is a Grothendieck space. In particular, ℓ_∞ is a Grothendieck space.*

Proof. By the Riesz Representation Theorem we have $C(K)^* \simeq \mathcal{M}(K)$, so let us fix any sequence $(\mu_n)_{n=1}^\infty \subset \mathcal{M}(K)$ that is weak* convergent to 0. Plainly, it is enough to show that this sequence is relatively weakly compact. Assume not. Then, Grothendieck's Theorem 3.6 produces a sequence $(O_n)_{n=1}^\infty$ of pairwise disjoint open subsets of K and a subsequence $(\nu_n)_{n=1}^\infty$ of $(\mu_n)_{n=1}^\infty$ such that $|\nu_n(O_n)| > \varepsilon$ for each $n \in \mathbb{N}$ and some $\varepsilon > 0$. Moreover, since K is extremally disconnected and all the measures ν_n's are regular, we may (and we do) assume that each O_n is clopen.

Now, we define a sequence $(\widetilde{\nu}_n)_{n=1}^\infty$ of bounded, finitely additive, scalar-valued measures on $\mathcal{P}\mathbb{N}$ in the following way: For any $\Delta \subset \mathbb{N}$ let

$$V_\Delta = \overline{\bigcup_{k\in\Delta} O_k} \quad \text{and} \quad \varphi_\Delta = \mathbb{1}_{V_\Delta} \in C(K)$$

(observe that since V_Δ is clopen, the map φ_Δ is continuous), and define $\widetilde{\nu}_n(\Delta) = \nu_n(V_\Delta)$. Observe that for every $\Delta \subset \mathbb{N}$ we have

$$\lim_{n\to\infty} \widetilde{\nu}_n(\Delta) = \lim_{n\to\infty} \nu_n(V_\Delta) = \lim_{n\to\infty} \int_K \varphi_\Delta \,\mathrm{d}\nu_n = \lim_{n\to\infty} \langle \varphi_\Delta, \nu_n \rangle = 0,$$

because $(\nu_n)_{n=1}^\infty$ converges to 0 in the weak* topology. Hence, Phillips' Lemma 2.8 implies that

$$\lim_{n\to\infty} \sum_{k=1}^\infty |\nu_n(O_k)| = \lim_{n\to\infty} \sum_{k=1}^\infty |\widetilde{\nu}_n(\{k\})| = 0,$$

which is impossible, as $|\nu_n(O_n)| > \varepsilon$ for each $n \in \mathbb{N}$.

Finally, recall that ℓ_∞ is (isometrically) isomorphic to the Banach space of continuous functions defined on $\beta\mathbb{N}$, which is extremally disconnected. □

It is a well-known property of extremally disconnected Hausdorff spaces that they do not contain any non-trivial (that is, not eventually constant) convergent sequences. Indeed, let K be such a space and suppose that $(x_n)_{n=1}^{\infty} \subset K$ is a non-trivial sequence converging to some $x \in K$. We may assume that x_n's are pairwise distinct and $x_n \neq x$ for $n \in \mathbb{N}$. Then, choose a sequence $(O_n)_{n=1}^{\infty}$ of pairwise disjoint open sets such that $x_n \in O_n$ for $n \in \mathbb{N}$ and consider the open sets: $U = \bigcup_{k=1}^{\infty} O_{2k-1}$ and $V = \bigcup_{k=1}^{\infty} O_{2k}$. Obviously, $U \cap V = \varnothing$, but $x \in \overline{U} \cap \overline{V}$ which contradicts that K is extremally disconnected. In light of Theorem 6.3, the following fact generalises this observation:

Proposition 6.4. *Let K be a compact Hausdorff space such that $C(K)$ is a Grothendieck space. Then K does not contain any non-trivial convergent sequences.*

Proof. Suppose that a sequence $(x_n)_{n=1}^{\infty} \subset K$ converges to some $x \in K$, and $x_m \neq x_n$ for all $m \neq n$. For any $y \in K$ let $\delta_y \in \mathcal{M}(K)$ be Dirac's measure concentrated at the point y. It is easy to check that the map $K \ni y \mapsto \delta_y$ yields a homeomorphism between K and the set $\{\delta_y \colon y \in K\}$ equipped with the weak* topology. Hence, $\delta_{x_n} \xrightarrow{w*} \delta_x$ and by our assumption we have $\delta_{x_n} \xrightarrow{w} \delta_x$. However, this is impossible because the set $D = \{\delta_{x_n} \colon n \in \mathbb{N}\}$ is discrete in the weak topology. To see this, note that for every Borel set $E \subset K$ the map $\mathcal{M}(K) \ni \nu \mapsto \nu(E)$ is a continuous linear functional, whence for each $n \in \mathbb{N}$ the set $\{\nu \in \mathcal{M}(K) \colon \nu\{x_n\} > 0\}$ is weakly open. But δ_{x_n} is the only one element of D that belongs to this set. □

Now, our intention is to present an example which shows that the implication converse to that of Proposition 6.4 is false, but before doing this we need to give a brief account of the definition and basic properties of the Stone space corresponding to a Boolean algebra[⋆].

First, recall that the Stone Representation Theorem says that every Boolean algebra $(\mathcal{B}, \vee, \wedge, \neg, \mathbf{0}, \mathbf{1})$ is isomorphic to the Boolean algebra of all clopen (= closed and open) subsets of a compact, totally disconnected[⋆⋆], Hausdorff space $\mathsf{St}(\mathcal{B})$, called the *Stone space* of $\mathcal{B}$. The space $\mathsf{St}(\mathcal{B})$ consists of all ultrafilters contained in $\mathcal{B}$ and its topology is generated by the family $\{\overline{x} \colon x \in \mathcal{B}\}$, where $\overline{x} = \{\mathsf{p} \in \mathsf{St}(\mathcal{B}) \colon x \in \mathsf{p}\}$ for $x \in \mathcal{B}$. Basic properties of ultrafilters imply that the operation $\mathcal{B} \ni x \mapsto \overline{x} \in \mathcal{P}(\mathsf{St}(\mathcal{B}))$ satisfies:

(a) $\overline{x \vee y} = \overline{x} \cup \overline{y}$,
(b) $\overline{x \wedge y} = \overline{x} \cap \overline{y}$,
(c) $\overline{\neg x} = \mathsf{St}(\mathcal{B}) \setminus \overline{x}$.

Therefore, the collection $\{\overline{x} \colon x \in \mathcal{B}\}$ forms a basis for the topology on $\mathsf{St}(\mathcal{B})$ and consists exclusively of clopen sets. Moreover, every clopen subset H of $\mathsf{St}(\mathcal{B})$ is compact, so it is a sum of finitely many basic open sets, whence the assertion (a) above implies that $H = \overline{x}$ for some $x \in \mathcal{B}$. Consequently, the family of all clopen subsets of $\mathsf{St}(\mathcal{B})$ is exactly equal to the family $\{\overline{x} \colon x \in \mathcal{B}\}$.

[⋆]A reader not familiar with the notions of *Boolean algebra*, *filter*, *ultrafilter* etc. may consult, e.g., Chapter 7 in [Jec00].

[⋆⋆]A topological space is *totally disconnected* whenever every its connected component is a one-point set. Every (locally) compact, totally disconnected, Hausdorff space must be *zero-dimensional*, i.e. it has a basis consisting of clopen sets (see [Wil70, Theorem 29.7]; the fact that every zero-dimensional Hausdorff space is totally disconnected is trivial). Of course, every extremally disconnected space is totally disconnected, but there are much more examples: the space $\mathbb{Q}$ of rationals, the space $\mathbb{P}$ of irrationals, the Cantor set $\{0,1\}^{\omega}$ (more generally, every product of totally disconnected spaces), the space $\beta X \setminus X$ for any infinite discrete space X. However, none of these spaces is extremally disconnected.

Note that in any Boolean algebra $\mathcal{B}$ there are two types of ultrafilters: *principal* ultrafilters $\mathsf{p} \in \mathsf{St}(\mathcal{B})$, which are generated by a single element $x \in \mathcal{B}$ and are of the form $\mathsf{p} = \{y \in \mathcal{B}\colon x \leqslant y\}$, and *free* ultrafilters $\mathsf{p} \in \mathsf{St}(\mathcal{B})$ (called also *non-principal*) for which $\bigwedge \mathsf{p} = \mathbf{0}$. Every principal ultrafilter $\mathsf{p} \in \mathsf{St}(\mathcal{B})$, being generated by some $x \in \mathcal{B}$, is a discrete point of the Stone space $\mathsf{St}(\mathcal{B})$. Indeed, since x generates an ultrafilter, it has to be a minimal non-zero element of the Boolean algebra $\mathcal{B}$, so the open set $\overline{x}$ is a singleton consisting only of p.

Suppose $\mathbb{D}$ is a discrete set and consider the Boolean algebra $\mathcal{P}\mathbb{D}$ of all subsets of $\mathbb{D}$. Let also $\mathsf{p} \subset \mathcal{P}\mathbb{D}$ be an ultrafilter. Then, p is principal if and only if there is a single point $x \in \mathbb{D}$ such that $\mathsf{p} = \{A \subset \mathbb{D}\colon x \in A\}$ (in this case p is generated by the element $\{x\}$ of the considered Boolean algebra), whereas p is non-principal if and only if $\bigcap \mathsf{p} = \varnothing$. Observe that the space $\beta\mathbb{D}$ is nothing else but the Stone space of the Boolean algebra $\mathcal{P}\mathbb{D}$, which follows straight from the definition of the Stone–Čech compactification*.

Similarly, the *remainder* space $\beta\mathbb{D} \setminus \mathbb{D}$ is homeomorphic to the Stone space of the quotient** Boolean algebra $\mathcal{P}\mathbb{D}/\mathcal{F}\mathbb{D}$, where $\mathcal{F}\mathbb{D}$ is the ideal of all finite subsets of $\mathbb{D}$. To see this, let us change our notation a bit and for every $A \in \mathcal{P}\mathbb{D}$ set

$$A^* = \big\{\mathsf{p} \subset \mathcal{P}\mathbb{D}\colon \mathsf{p} \text{ is a non-principal ultrafilter with } A \in \mathsf{p}\big\}$$

(note that for finite sets A we have $A^* = \varnothing$). This differs from the operation $x \mapsto \overline{x}$, defined earlier, by the requirement that each member of A^* must be non-principal. Nonetheless, this new operation still satisfies the conditions analogous to (a)-(c) above:

(a)′ $(A \cup B)^* = A^* \cup B^*$,
(b)′ $(A \cap B)^* = A^* \cap B^*$,
(c)′ $(\mathbb{D} \setminus A)^* = \mathbb{D}^* \setminus A^*$

(notice that $\mathbb{D}^*$ is just the collection of all non-principal ultrafilters on $\mathbb{D}$). All these conditions again follow easily from basic properties of ultrafilters. Now, note that every principal ultrafilter on the set $\mathbb{D}$ may be identified with a unique point of $\mathbb{D}$, so the set $\beta\mathbb{D}$ of all ultrafilters on $\mathbb{D}$ may be identified with $\mathbb{D} \cup \mathbb{D}^*$, while $\beta\mathbb{D} \setminus \mathbb{D}$ stands, of course, for the subspace of $\beta\mathbb{D}$ consisting only of non-principal ultrafilters. But observe that the conditions (a)′-(c)′ imply that the collection $\{A \cup A^*\colon A \in \mathcal{P}\mathbb{D}\}$ forms a basis for a certain topology on $\mathbb{D} \cup \mathbb{D}^*$. However, it is quite evident that the so-defined topological space is homeomorphic to $\beta\mathbb{D}$ via the homeomorphisms $\varphi\colon \mathbb{D} \cup \mathbb{D}^* \to \beta\mathbb{D}$ $(= \mathsf{St}(\mathcal{P}\mathbb{D}))$ given by

$$\begin{cases} \varphi(x) = \{A \subset \mathbb{D}\colon x \in A\} \ \text{ (the ultrafilter generated by } \{x\}), & \text{for } x \in \mathbb{D} \\ \varphi(\mathsf{p}) = \mathsf{p}, & \text{for } \mathsf{p} \in \mathbb{D}^* \end{cases}$$

*Formally, we **define** the Stone-Čech compactification of a discrete set $\mathbb{D}$ to be the Stone space of the Boolean algebra $\mathcal{P}\mathbb{D}$, that is, the elements of $\beta\mathbb{D}$ are ultrafilters on $\mathbb{D}$. However, many people like to define $\beta\mathbb{D}$ as the disjoint union of $\mathbb{D}$ and the set of all non-principal ultrafilters on $\mathbb{D}$, with a topology described in the text. These two definitions are equivalent, that is, they give homeomorphic topological spaces.

**Let $(\mathcal{B}, +, \cdot, \mathbf{0}, \mathbf{1})$ be a Boolean algebra, i.e. a unital ring satisfying $x^2 = x$ for every $x \in \mathcal{B}$ (which already implies that $x + x = 0$ and $xy = yx$ for all $x, y \in \mathcal{B}$) and let $\mathscr{I} \subset \mathcal{B}$ be an ideal. Then the quotient ring $\mathcal{B}/\mathscr{I}$ is also a Boolean algebra (see [Fre02, §312K]). In the case where $\mathcal{B} = \mathcal{P}X$ is a Boolean algebra of all subsets of some set X, the addition $+$ is the symmetric difference $\triangle$, while the multiplication $\cdot$ is the set intersection $\cap$. Therefore, if $\mathscr{I}$ is an ideal in $\mathcal{P}X$, then the quotient algebra $\mathcal{P}X/\mathscr{I}$ consists of equivalence classes $[A]_{\mathscr{I}}$ (for $A \in \mathcal{P}X$) with respect to the relation $\sim$ given by $A \sim B \Leftrightarrow A \triangle B \in \mathscr{I}$.

(see Problem 4.2). The conditions (a)$'$-(c)$'$ imply also that the collection $\{A^*\colon A\in\mathcal{P}\mathbb{D}\}$ forms a basis for the topology on $\beta\mathbb{D}\setminus\mathbb{D}$ that is inherited from the Stone topology on $\beta\mathbb{D}$. Moreover, by (a)$'$, every clopen subset of $\beta\mathbb{D}\setminus\mathbb{D}$ is of the form A^* for some $A\in\mathcal{P}\mathbb{D}$. Hence, we may define a map $\psi\colon \mathsf{CO}(\beta\mathbb{D}\setminus\mathbb{D})\to\mathcal{P}\mathbb{D}/\mathcal{F}\mathbb{D}$, from the Boolean algebra of all clopen subsets of $\beta\mathbb{D}\setminus\mathbb{D}$ into the Boolean algebra of all subsets of $\mathbb{D}$ modulo finite sets, by the formula

$$\psi(A^*)=[A]_{\mathcal{F}\mathbb{D}},$$

which is well-posed and defines a homomorphism between these two Boolean algebras (see Problem 4.3). On the other hand, since the Stone space associated with a given Boolean algebra is uniquely determined up to a homeomorphism (in fact we have a one-to-one correspondence between Boolean algebras and zero-dimensional compact Hausdorff spaces, provided that we identify Boolean algebras via isomorphisms and topological spaces via homeomorphisms; see [Fre02, §311J]), we conclude that

$$\beta\mathbb{D}\setminus\mathbb{D}\simeq\mathsf{St}\big(\mathcal{P}\mathbb{D}/\mathcal{F}\mathbb{D}\big)\quad\text{(homeomorphically).}$$

There is another (equivalent) way of constructing the Stone space $\mathsf{St}(\mathcal{B})$ of a given Boolean algebra $\mathcal{B}$, in which the elements of $\mathsf{St}(\mathcal{B})$ are non-zero ring homomorphisms from $\mathcal{B}$ into the two-element algebra $\mathbb{Z}_2$. The interested reader should consult Chapter 31 in [Fre02] for a detailed study of this approach.

Example 6.5. Define a set algebra $\mathscr{F}\subset\mathcal{P}\mathbb{N}$ by

$$\mathscr{F}=\big\{A\subset\mathbb{N}\colon |A\cap\{2k-1,2k\}|\in\{0,2\}\ \text{ for all but finitely many } k\in\mathbb{N}\big\}$$

and let $K=\mathsf{St}(\mathscr{F})$. Then, $C(K)$ is not a Grothendieck space, although K does not contain any non-trivial convergent sequences.

First of all, observe that every singleton $\{n\}$ (for $n\in\mathbb{N}$) belongs to $\mathscr{F}$, whence the Stone space $\mathsf{St}(\mathscr{F})$ contains the principal ultrafilter generated by $\{n\}$, for each $n\in\mathbb{N}$. For simplicity, every such ultrafilter will be denoted as n and we will identify the set $\mathbb{N}$ with the collection of all such ultrafilters, which is discrete subset of $\mathsf{St}(\mathscr{F})$.

In order to disprove that $C(K)$ is a Grothendieck space, consider the sequence $(\delta_{2k-1}-\delta_{2k})_{k=1}^\infty\subset C(K)^*$. Fix any function $f\in C(K)$ and suppose that there exists $\delta>0$ and a strictly increasing sequence $(k_j)_{j=1}^\infty$ of natural numbers such that $|f(2k_j-1)-f(2k_j)|\geqslant\delta$ for each $j\in\mathbb{N}$. Put

$$A=\{2k_j-1\colon j\in\mathbb{N}\}\cup\{2k_j\colon j\in\mathbb{N}\}\in\mathscr{F}$$

and let $\mathsf{p}\in\mathsf{St}(\mathscr{F})\setminus\mathbb{N}$ be such that $A\in\mathsf{p}$. Then, for every neighbourhood V of p there exist infinitely many $j\in\mathbb{N}$ with $2k_j-1\in V$ and $2k_j\in V$, which means that for every $\varepsilon>0$ there are infinitely many j's satisfying

$$|f(2k_j-1)-f(\mathsf{p})|<\varepsilon\quad\text{ and }\quad|f(2k_j)-f(\mathsf{p})|<\varepsilon.$$

Taking any $\varepsilon<\delta/2$ we get the contradiction: $|f(2k_j-1)-f(2k_j)|<\delta$ (for some j's). Consequently, we have proved that $\delta_{2k-1}-\delta_{2k}\xrightarrow{w*}0$. However, as we have seen in the proof of Proposition 6.4, every set of disjointly supported Dirac's measures (or finite combinations of thereof) is discrete in the weak topology of $C(K)^*$. Therefore, the sequence $(\delta_{2k-1}-\delta_{2k})_{k=1}^\infty$ is not weakly null and hence $C(K)$ is not a Grothendieck space.

Now, we are going to show that K does not contain any non-trivial convergent sequences. First, notice that the discrete set $\mathbb{N} \subset K$ obviously does not contain such sequences, so if we prove that the remainder $K \setminus \mathbb{N}$ is homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$ (which does not contain such sequences either, as being a subspace of the extremally disconnected space $\beta\mathbb{N}$), we will be done.

Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be the surjection given by $\sigma(2k-1) = \sigma(2k) = k$, for every $k \in \mathbb{N}$. The map σ induces a homeomorphism $\varphi\colon K \setminus \mathbb{N} \to \beta\mathbb{N} \setminus \mathbb{N}$ in the following way: For every $A \in \mathscr{F}$ let $A' \in \mathscr{F}$ be any set containing almost all numbers from A and satisfying $|A' \cap \{2k-1, 2k\}| \in \{0, 2\}$ for each $k \in \mathbb{N}$ (of course, these sets A' are not uniquely determined but this will not impact the definition of φ). Now, for any $\mathsf{p} \in K \setminus \mathbb{N}$ define $\varphi(\mathsf{p})$ to be the unique principal ultrafilter on $\mathbb{N}$ that contains all the A''s, for $A \in \mathsf{p}$. It is left as an exercise (Problem 4.4) to show that $\varphi(\mathsf{p})$ is indeed uniquely determined and that φ yields the desired homeomorphism.

Now, we would like to broaden our set of examples of Grothendieck spaces. To start, let us observe that in the proof of Theorem 6.3 we have actually proved a bit more than stated. In fact, it was not indispensable that K was extremally disconnected; what was really used is that the closure of countably many open sets remained open.

Definition 6.6. We call a topological space K *basically disconnected* (or *σ-Stonean*), provided that the closure of any open F_σ subset of K is open. Similarly, when κ is a cardinal number, K is called *κ-basically disconnected*, provided that the closure of any subset of K that is a sum of not more than κ open sets is open.

Definition 6.7. We call a Boolean algebra $\mathcal{B}$ *Dedekind complete*, provided that every subset of $\mathcal{B}$ that has an upper bound has a least upper bound (equivalently: every subset of $\mathcal{B}$ that has a lower bound has a greatest lower bound). Similarly, when κ is a cardinal number, $\mathcal{B}$ is called *Dedekind κ-complete*, provided that every subset of $\mathcal{B}$ with cardinality at most κ that has an upper bound has a least upper bound (equivalently: every subset of $\mathcal{B}$ with cardinality at most κ that has a lower bound has a greatest lower bound).

The latter definition is a perfect analogue to the former, which is expressed in the following theorem (for the proof, see [Sto49] and [Fre02, §314S]):

Theorem 6.8 (Stone, 1949)**.** *A Boolean algebra $\mathcal{B}$ is Dedekind κ-complete if and only if its Stone space $\mathsf{St}(\mathcal{B})$ is κ-basically disconnected. Consequently, $\mathcal{B}$ is Dedekind complete if and only if $\mathsf{St}(\mathcal{B})$ is extremally disconnected.*

For any algebra $\mathscr{F}$ of subsets of a set Ω let $B(\mathscr{F})$ be the Banach space consisting of all scalar-valued functions defined on Ω, that are uniform limits of sequences of $\mathscr{F}$-measurable step functions, and equipped with the supremum norm. It follows almost immediately from the Stone–Weierstrass theorem that $B(\mathscr{F})$ is isometrically isomorphic to the Banach space $C(\mathsf{St}(\mathscr{F}))$ (see Problem 4.5). As we have mentioned, the proof of Theorem 6.3 goes through in the case where K is a σ-Stonean compact Hausdorff space, so in view of Stone's Theorem 6.8 we conclude that for every σ-algebra Σ the Banach space $B(\Sigma)$ is a Grothendieck space. But $B(\mathscr{F})$ may be also a Grothendieck space, provided only that the set algebra $\mathscr{F}$ behaves similarly to a σ-algebra. The following definition was introduced independently by Haydon [Hay81] and Schachermayer [Sch82]:

Definition 6.9. A Boolean algebra $\mathcal{B}$ is said to have the *subsequential completeness property* (SCP for short) whenever for every disjoint sequence $(x_n)_{n=1}^{\infty} \subset \mathcal{B}$ there exists a subsequence $(x_{n_j})_{j=1}^{\infty}$ that has a least upper bound in $\mathcal{B}$.

The next result strengthens our conclusion about $B(\Sigma)$-spaces, with Σ being a σ-algebra. This is a delightful application of the machinery that we have developed so far.

Theorem 6.10 (Haydon, 1981)**.** *If $\mathscr{F}$ is a set algebra having the* SCP, *then $B(\mathscr{F})$ is a Grothendieck space.*

Proof. Let $K = \mathsf{St}(\mathscr{F})$. Then K is a zero-dimensional compact Hausdorff space and, since $\mathscr{F}$ has the SCP, K has the following property: For every sequence $(O_n)_{n=1}^{\infty}$ of pairwise disjoint clopen subsets of K there exists a subsequence $(O_{n_j})_{j=1}^{\infty}$ such that $\overline{\bigcup_{j=1}^{\infty} O_{n_j}}$ is open (see Problem 4.6).

Likewise in the proof of Theorem 6.3, suppose there is a weak* null sequence $(\nu_n)_{n=1}^{\infty} \subset C(K)^*$ and a sequence $(O_n)_{n=1}^{\infty}$ of pairwise disjoint clopen subsets of K such that $\nu_n(O_n) > \varepsilon$ for each $n \in \mathbb{N}$ and some $\varepsilon > 0$. By Rosenthal's Lemma 2.1, we may also assume that

$$|\nu_n|\Big(\bigcup_{j\neq n} O_j\Big) < \frac{\varepsilon}{2} \quad \text{for each } n \in \mathbb{N}.$$

Now, let $\{A_i\}_{i\in I} \subset \mathcal{P}_{\infty}\mathbb{N}$ be an uncountable almost disjoint family. For every $i \in I$ use the SCP to choose an infinite set $B_i \subset A_i$ such that the set $F_i := \overline{\bigcup_{m\in B_i} O_m}$ is open. Then $\mathbb{1}_{F_i} \in C(K)$ for $i \in I$ and the sets $R_i := F_i \setminus \bigcup_{m\in B_i} O_m$ $(i \in I)$ are pairwise disjoint, as the family $\{B_i\}_{i\in I}$ is almost disjoint. Hence, there exists some $i_0 \in I$ such that $|\nu_n|(R_{i_0}) = 0$ for every $n \in \mathbb{N}$. Now, for each $n \in B_{i_0}$ we have

$$\langle \mathbb{1}_{F_{i_0}}, \nu_n\rangle = \nu_n\Big(R_{i_0} \cup \bigcup_{m\in B_{i_0}} O_m\Big) \geqslant \nu_n(O_n) - |\nu_n|\Big(\bigcup_{\substack{m\in B_{i_0}\\ m\neq n}} O_m\Big) > \frac{\varepsilon}{2}.$$

But this is impossible, since $\nu_n \xrightarrow{w*} 0$. □

Haydon constructed a set algebra $\mathscr{F} \subset \mathcal{P}\mathbb{N}$ which has the SCP but is quite far away from being a σ-algebra. It also has the property that $C(\mathsf{St}(\mathscr{F}))$ is a Grothendieck space that does not contain any isomorphic copy of ℓ_∞ (see Proposition 1E and Theorem 1F in [Hay81]). The problem of characterising those set algebras $\mathscr{F}$ for which $B(\mathscr{F})$ is a Grothendieck space is still not completely solved and it is closely related to some vector measure-theoretic properties of $B(\mathscr{F})$, such as the validity of the Nikodým Boundedness Principle for measures defined on $\mathscr{F}$. The interested reader should consult [Sch82] and the references therein. Here, let us just note the following elegant necessary condition ([Sch82, Proposition 4.6]):

Theorem 6.11 (Schachermayer, 1982)**.** *Suppose $\mathscr{F}$ is a set algebra such that $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$, where $\mathscr{F}_n$'s are subalgebras of $\mathscr{F}$ and $\mathscr{F}_n \subsetneq \mathscr{F}_{n+1}$ for each $n \in \mathbb{N}$. Then, $B(\mathscr{F})$ is not a Grothendieck space.*

Consequently, since for every σ-algebra Σ the space $B(\Sigma)$ is a Grothendieck space, no σ-algebra may be represented as a sum of strictly increasing sequence of subalgebras.

We close this section by collecting a few results that identify Grothendieck spaces among $C(K)$-spaces (where K, as usual, is a compact Hausdorff space):

- As it was already mentioned, for every basically disconnected (σ-Stonean) compact Hausdorff space K the space $C(K)$ is a Grothendieck space. This was first observed by Andô [And61].
- Seever [See68] proved that $C(K)$ is a Grothendieck space whenever K is an F-*space*, that is, for every open F_σ sets $U, V \subset K$ with $U \cap V = \varnothing$ we have $\overline{U} \cap \overline{V} = \varnothing$.
- Cembranos [Cem84] proved that $C(K)$ is a Grothendieck space if and only if it does not contain a complemented isomorphic copy of c_0.
- The reader familiar with von Neumann algebras would like to recall that the spectrum of a commutative von Neumann algebra is extremally disconnected, thus Grothendieck's Theorem 6.3 automatically implies that every such algebra has the Grothendieck property as a Banach space. But this carries on to the non-commutative case as well: Pfitzner [Pfi94] showed that every von Neumann algebra is a Grothendieck space.