Combinatorics in Banach space theory

Lecture 7

7 Khintchine's inequality and reflexive quotients of ℓ_∞

Since ℓ_∞ is a Grothendieck space and the class of Grothendieck spaces is closed under quotients (Problem 3.9), whenever there exists a surjective operator from ℓ_∞ onto a Banach space X, that space X must be a Grothendieck space. In particular, every separable quotient of ℓ_∞ is automatically reflexive (see the remarks after Definition 6.1). The separable Hilbert space ℓ_2 is by a clear mile the most classical infinite-dimensional reflexive space, so the following question arises: Is ℓ_2 actually a quotient of ℓ_∞? In other words, is there any operator from ℓ_∞ onto ℓ_2, or maybe even onto $\ell_2(\mathfrak{c})$? The positive answer is contained in the following result by Rosenthal [Ros68]:

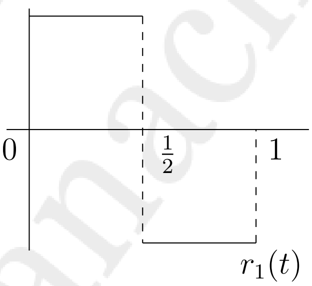
Theorem 7.1 (Rosenthal, 1968)**.** *For any infinite cardinal number Γ the space $\ell_2(2^\Gamma)$ is a quotient of $\ell_\infty(\Gamma)$.*

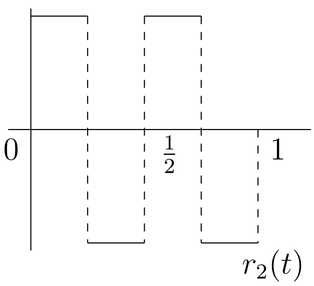
Before proving this theorem we need to derive a widely used inequality usually attributed to Khintchine who first proved its special case. The general version of that inequality is due to Littlewood, Paley and Zygmund.

Definition 7.2. For any $n \in \mathbb{N}$ the nth *Rademacher function* $r_n \in L_1[0,1]$ is defined by $r_n(t) = \mathrm{sgn}(\sin(2^n \pi t))$ or, equivalently,

$$r_n(t) = \begin{cases} 1 & \text{for } t \in \bigcup_{j=0}^{2^{n-1}-1} \left[\frac{2j}{2^n}, \frac{2j+1}{2^n}\right) \\ -1 & \text{for } t \in \bigcup_{j=0}^{2^{n-1}-1} \left[\frac{2j+1}{2^n}, \frac{2j+2}{2^n}\right) \end{cases}$$

(we treat r_n's as random variables on the probabilistic space $[0,1]$ with the Lebesgue measure $\mathbb{P}$).





Plainly, for all sequences $n_1 < \ldots < n_k$ and $(\varepsilon_j)_{j=1}^k \in \{-1,1\}^k$ we have

$$\mathbb{P}(r_{n_1} = \varepsilon_1 \wedge \ldots \wedge r_{n_k} = \varepsilon_k) = \prod_{j=1}^{k} \mathbb{P}(r_{n_j} = \varepsilon_j)$$

which means that $(r_n)_{n=1}^\infty$ is a sequence of independent random variables. Therefore, $(r_n)_{n=1}^\infty$ is just a concrete example of a *Rademacher system* which is defined to be any sequence $(X_n)_{n=1}^\infty$ of independent random variables on some probabilistic space $(\Omega,\mathbb{P})$ satisfying $\mathbb{P}(X_n=1)=\mathbb{P}(X_n=-1)=1/2$ for each $n\in\mathbb{N}$. Consequently, the expectation values of Rademacher's functions satisfy $\mathbb{E}(r_{n_1}\cdot\ldots\cdot r_{n_k})=\mathbb{E}(r_{n_1})\cdot\ldots\cdot\mathbb{E}(r_{n_k})$ whenever $n_1<\ldots<n_k$. In particular, $\int_0^1 r_i(t)r_j(t)\,\mathrm{d}t=\delta_{ij}$ which means that $(r_n)_{n=1}^\infty$ is an orthonormal sequence in the Hilbert space $L_2[0,1]$. Therefore, by the Pythagorean theorem we get

$$\Big\|\sum_{j=1}^n a_jr_j\Big\|_{L_2}=\Big(\sum_{j=1}^n|a_j|^2\Big)^{1/2} \tag{7.1}$$

for any complex scalars $a_1,\ldots,a_n$. We thus see that the Rademacher system $(r_n)_{n=1}^\infty$ in $L_2[0,1]$ behaves likewise the standard basis $(e_n)_{n=1}^\infty$ in ℓ_2. More precisely, these two sequences are *equivalent* in the sense that there exists an isomorphism T from ℓ_2 onto the subspace $\overline{\mathrm{span}}\{r_n\colon n\in\mathbb{N}\}$ of $L_2[0,1]$ such that $T(e_n)=r_n$ for each $n\in\mathbb{N}$ (we will discuss this notion of *equivalence* later when we talk about bases in Banach spaces). Khintchine's inequality asserts that the sequence $(r_n)_{n=1}^\infty$ remains 'almost orthonormal' in $L_p[0,1]$ for $p\in[1,\infty)$.

Theorem 7.3 (Khintchine's inequality)**.** *For every $p\in[1,\infty)$ there exist positive (and finite) constants A_p and B_p such that*

$$A_p\Big(\sum_{j=1}^n|a_j|^2\Big)^{1/2}\leqslant\Big\|\sum_{j=1}^n a_jr_j\Big\|_{L_p}\leqslant B_p\Big(\sum_{j=1}^n|a_j|^2\Big)^{1/2} \tag{7.2}$$

for any real scalars $a_1,\ldots,a_n$.

Proof. For any $p\in[1,\infty)$ let A_p and B_p be the best possible constants in inequality (7.5). As we have already observed, we have $A_2=B_2=1$. Notice that for $1\leqslant p<r$ and for every function $g\in L_r[0,1]$ Hölder's inequality applied to the exponents $q=r/p>1$ and q' satisfying $1/q+1/q'=1$ gives

$$\|g\|_{L_p}=\Big(\int_0^1|g(t)|^p\,\mathrm{d}t\Big)^{1/p}\leqslant\Big(\int_0^1|g(t)|^r\,\mathrm{d}t\Big)^{1/r}\cdot\Big(\int_0^1 1^{q'}\,\mathrm{d}t\Big)^{1/q'}=\|g\|_{L_q}.$$

Therefore, $1\leqslant p<r$ implies $A_p\leqslant A_r$ and $B_p\leqslant B_r$, so if we show that $A_1>0$ and $B_{2k}<\infty$ for each $k\in\mathbb{N}$, we will be done.

We start with estimating B_{2k} by using the multinomial expansion. Fix $n\in\mathbb{N}$ and $a_1,\ldots,a_n\in\mathbb{R}$. For any $m\in\mathbb{N}$ let $\mathcal{S}_m$ be the set of all multi-indices $(\alpha_1,\ldots,\alpha_n)$ such that each α_j is a non-negative integer and $\sum_{j=1}^n\alpha_j=m$. Let also

$$\binom{m}{\alpha_1\ldots\alpha_n}=\frac{m!}{\alpha_1!\cdot\ldots\cdot\alpha_n!}$$

be the multinomial coefficient. Since $(r_j)_{j=1}^\infty$ is the Rademacher system, for every multi-index $(\alpha_1,\ldots,\alpha_n)$ we have

$$\int_0^1 r_1^{\alpha_1}(t)\cdot\ldots\cdot r_n^{\alpha_n}(t)\,\mathrm{d}t=\begin{cases}1 & \text{if each of } \alpha_j\text{'s is even}\\ 0 & \text{otherwise.}\end{cases}$$

Therefore,

$$\begin{aligned}\int_0^1 \Big|\sum_{j=1}^n a_j r_j(t)\Big|^{2k} \mathrm{d}t &= \sum_{(\alpha_1,\ldots,\alpha_n)\in\mathcal{S}_{2k}} \binom{2k}{\alpha_1 \ldots \alpha_n} a_1^{\alpha_1}\cdot\ldots\cdot a_n^{\alpha_n} \int_0^1 r_1^{\alpha_1}(t)\cdot\ldots\cdot r_n^{\alpha_n}(t)\,\mathrm{d}t \\ &= \sum_{(\beta_1,\ldots,\beta_n)\in\mathcal{S}_{k}} \binom{2k}{2\beta_1 \ldots 2\beta_n} a_1^{2\beta_1}\cdot\ldots\cdot a_n^{2\beta_n}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Big(\sum_{j=1}^n |a_j|^2\Big)^k &= \sum_{(\beta_1,\ldots,\beta_n)\in\mathcal{S}_{k}} \binom{k}{\beta_1 \ldots \beta_n} a_1^{2\beta_1}\cdot\ldots\cdot a_n^{2\beta_n} \\ &= \sum_{(\beta_1,\ldots,\beta_n)\in\mathcal{S}_{k}} \frac{\binom{k}{\beta_1 \ldots \beta_n}}{\binom{2k}{2\beta_1 \ldots 2\beta_n}} \binom{2k}{2\beta_1 \ldots 2\beta_n} a_1^{2\beta_1}\cdot\ldots\cdot a_n^{2\beta_n}.\end{aligned}$$

Consequently, setting

$$b_k = \min\left\{\binom{k}{\beta_1 \ldots \beta_n}\cdot\binom{2k}{2\beta_1 \ldots 2\beta_n}^{-1} : (\beta_1,\ldots,\beta_n)\in\mathcal{S}_k\right\}$$

(which is a minimum over a finite set of positive numbers, so $b_k > 0$), we infer that

$$\Big(\sum_{j=1}^n |a_j|^2\Big)^k \geqslant b_k\cdot\Big\|\sum_{j=1}^n a_j r_j(t)\Big\|_{L_{2k}}^{2k},$$

whence $B_{2k} \leqslant b_k^{-1/2k} < \infty$.

In order to show that $A_1 > 0$, we will combine what we have found so far with Hölder's inequality. We claim that $A_1 \geqslant B_4^{-2}$. For simplicity, denote $f(t) = \sum_{j=1}^n a_j r_j(t)$. We have already learned that $(\sum_{j=1}^n |a_j|^2)^{1/2} = (\int_0^1 |f(t)|^2\,\mathrm{d}t)^{1/2}$, thus if we show the inequality

$$\Big(\int_0^1 |f(t)|^2\,\mathrm{d}t\Big)^{1/2} \leqslant B_4^2 \int_0^1 |f(t)|\,\mathrm{d}t, \tag{7.3}$$

then our claim will follow. To this end, we write 2 as $2/3 + 4/3$ and we use Hölder's inequality with the exponents $q = 3/2$ and $q' = 3$:

$$\begin{aligned}\int_0^1 |f(t)|^2\,\mathrm{d}t &= \int_0^1 |f(t)|^{2/3}|f(t)|^{4/3}\,\mathrm{d}t \leqslant \Big(\int_0^1 |f(t)|\,\mathrm{d}t\Big)^{2/3}\Big(\int_0^1 |f(t)|^4\,\mathrm{d}t\Big)^{1/3} \\ &\leqslant B_4^{4/3}\Big(\int_0^1 |f(t)|\,\mathrm{d}t\Big)^{2/3}\Big(\sum_{j=1}^n |a_j|^2\Big)^{2/3} = B_4^{4/3}\Big(\int_0^1 |f(t)|\,\mathrm{d}t\Big)^{2/3}\Big(\int_0^1 |f(t)|^2\,\mathrm{d}t\Big)^{2/3}\end{aligned}$$

which implies (7.3) and completes the proof. □

Remark 7.4. Although the above proof used the fact that the scalars $a_1,\ldots,a_n$ are reals, it is not difficult to see that Khintchine's inequality (7.2) is valid also for complex scalars with possibly different constants A_p and B_p which we shall denote in this case as $A_p^{\mathbb{C}}$ and $B_p^{\mathbb{C}}$, respectively. Indeed, observe that for all real scalars a_j and b_j ($1 \leqslant j \leqslant n$) we have

$$\Big\|\sum_{j=1}^n (a_j + \mathrm{i}b_j) r_j\Big\|_{L_p} \leqslant \Big\|\sum_{j=1}^n a_j r_j\Big\|_{L_p} + \Big\|\sum_{j=1}^n b_j r_j\Big\|_{L_p}$$

$$\leqslant B_p\Big(\sum_{j=1}^n a_j^2\Big)^{1/2} + B_p\Big(\sum_{j=1}^n b_j^2\Big)^{1/2} \leqslant \sqrt{2} B_p\Big(\sum_{j=1}^n a_j^2 + \sum_{j=1}^n b_j^2\Big)^{1/2}$$

(as $\sqrt{t} + \sqrt{u} \leqslant \sqrt{2(t+u)}$ for $t, u \geqslant 0$), whence $B_p^{\mathbb{C}} \leqslant \sqrt{2} B_p$. For the converse estimate, observe that for all $t, u \geqslant 0$ and $q > 0$ we have

$$(t+u)^q \geqslant \begin{cases} t^q + u^q & \text{if } q \geqslant 1 \\ 2^{q-1}(t^q + u^q) & \text{if } q < 1. \end{cases}$$

Indeed, the first inequality is obvious whereas the second follows from the power-mean inequality $((t^q + u^q)/2)^{1/q} \leqslant (t+u)/2$ (for $q < 1$). Similarly,

$$(t+u)^q \leqslant \begin{cases} t^q + u^q & \text{if } q \leqslant 1 \\ 2^{q-1}(t^q + u^q) & \text{if } q > 1. \end{cases}$$

Therefore, setting

$$c_p = \begin{cases} 1 & \text{if } p \geqslant 2 \\ 2^{p/2-1} & \text{if } p < 2 \end{cases} \quad \text{and} \quad d_p = \begin{cases} 1 & \text{if } p \leqslant 2 \\ 2^{p/2-1} & \text{if } p > 2 \end{cases}$$

and using the above inequalities for $q = p/2$ we obtain

$$\Big\|\sum_{j=1}^n (a_j + \mathrm{i}b_j) r_j\Big\|_{L_p}^p = \int_0^1 \Big|\sum_{j=1}^n (a_j + \mathrm{i}b_j) r_j(t)\Big|^p \,\mathrm{d}t$$

$$= \int_0^1 \Big\{\Big(\sum_{j=1}^n a_j r_j(t)\Big)^2 + \Big(\sum_{j=1}^n b_j r_j(t)\Big)^2\Big\}^{p/2} \,\mathrm{d}t$$

$$\geqslant c_p\bigg(\int_0^1 \Big(\sum_{j=1}^n a_j r_j(t)\Big)^p \,\mathrm{d}t + \int_0^1 \Big(\sum_{j=1}^n b_j r_j(t)\Big)^p \,\mathrm{d}t\bigg)$$

$$\geqslant c_p A_p^p\bigg(\Big(\sum_{j=1}^n a_j^2\Big)^{p/2} + \Big(\sum_{j=1}^n b_j^2\Big)^{p/2}\bigg) \geqslant c_p d_p^{-1} A_p^p\Big(\sum_{j=1}^n a_j^2 + \sum_{j=1}^n b_j^2\Big).$$

Consequently, $A_p^{\mathbb{C}} \geqslant (c_p d_p^{-1})^{1/p} A_p = 2^{-|1/p-1/2|} A_p$.

Remark 7.5. The sharp constants A_p and B_p in inequality (7.2) were determined by Haagerup [Haa82]. His result reads as follows:

$$A_p = \begin{cases} 2^{1/2-1/p} & \text{if } 0 < p \leqslant p_0 \\ 2^{1/2}\big(\Gamma((p+1)/2)/\sqrt{\pi}\big)^{1/p} & \text{if } p_0 < p < 2 \\ 1 & \text{if } 2 \leqslant p < \infty \end{cases}$$

and

$$B_p = \begin{cases} 1 & \text{if } 1 < p \leqslant 2 \\ 2^{1/2}\big(\Gamma((p+1)/2)/\sqrt{\pi}\big)^{1/p} & \text{if } 2 < p < \infty, \end{cases}$$

where p_0 is the unique solution of the equation $\Gamma((p+1)/2) = \sqrt{\pi}/2$ in the interval $(1, 2)$, $p_0 \approx 1.84742$.

Proof of Theorem 7.1. First, observe that it is enough to prove that $\ell_\infty(\Gamma)^*$ contains an isomorphic copy of $\ell_2(2^\Gamma)$. Indeed, suppose there exists an operator $T\colon \ell_2(2^\Gamma) \to \ell_\infty(\Gamma)^*$ which is an embedding, that is a one-to-one operator with a closed range. Since T is w^*-to-w continuous (the weak* and weak topologies on $\ell_2(2^\Gamma)$ coincide), it is also w^*-to-w^* continuous and therefore it is an adjoint operator, $T = S^*$ for some $S\colon \ell_\infty(\Gamma) \to \ell_2(2^\Gamma)$. Now, S has a dense range because T is injective and S has a closed range because so does T. Consequently, S would be a quotient operator.

In order to find a copy of $\ell_2(2^\Gamma)$ inside $\ell_\infty(\Gamma)^*$, we appeal to the Fichtenholz–Kantorovich–Hausdorff theorem (see Problem 2.9) which produces an independent family $\mathcal{F} \subset \mathcal{P}\Gamma$ with cardinality 2^Γ. Let $V \subset \ell_\infty(\Gamma)$ be defined as

$$V = \Big\{\prod_{i=1}^m \mathbb{1}_{A_i} \cdot \prod_{j=1}^n \mathbb{1}_{\Gamma\setminus B_j}\colon A_1,\ldots,A_m,B_1,\ldots,B_n \text{ are distinct members of } \mathcal{F},\ m,n\in\mathbb{N}_0\Big\}$$

and let Y be the linear span of V. Since $\mathcal{F}$ is independent, the set V is linearly independent, so by putting

$$\varphi\Big(\prod_{i=1}^m \mathbb{1}_{A_i} \cdot \prod_{j=1}^n \mathbb{1}_{\Gamma\setminus B_j}\Big) = 2^{-m-n} \tag{7.4}$$

we define a linear functional on Y. Moreover, the fact that $\mathcal{F}$ is independent implies that φ has norm 1 on Y, whence the Hahn–Banach theorem produces a norm-1 extension (still denoted φ) of φ to the whole of $\ell_\infty(\Gamma)$. Since $\ell_\infty(\Gamma) \simeq C(\beta\Gamma)$ (see Problem 4.5; notice that $\ell_\infty(\Gamma)$ is the same as $B(\mathcal{P}\Gamma)$), we have $\ell_\infty(\Gamma)^* \simeq \mathcal{M}(\beta\Gamma)$ which is the Banach space of all scalar-valued, σ-additive, regular Borel measures on $\beta\Gamma$, equipped with the total variation norm. So, regarding φ as one of those measures we may replace φ by its variation $|\varphi|$ (which is still a member of $\mathcal{M}(\beta\Gamma)$) and observe that $|\varphi|$ corresponds to a functional which still satisfies the formula analogous to (7.4). Consequently, we may assume that φ is a probabilistic measure on $\beta\Gamma$.

Now, for any $A \in \mathcal{F}$ define a functional $\psi_A \in \ell_\infty(\Gamma)^*$ by $\psi_a(x) = \varphi((\mathbb{1}_A - \mathbb{1}_{\Gamma\setminus A})x)$. We claim that $\{\psi_A\colon A\in\mathcal{F}\}$ is equivalent to the standard basis $\{e_\gamma\colon \gamma\in\Gamma\}$ of $\ell_2(\Gamma)$ in the following sense: there is a one-to-one correspondence $\Gamma \ni \gamma \mapsto A_\gamma \in \mathcal{F}$ such that there exists an isomorphism $T\colon \ell_2(\Gamma) \to \overline{\mathrm{span}}\{\psi_A\colon A\in\mathcal{F}\}$ satisfying $T(e_\gamma) = \psi_{A_\gamma}$ for every $\gamma\in\Gamma$. To this end it suffices to show that for some constants $0 < A, B < \infty$ we have

$$A\Big\|\sum_{j=1}^n a_j e_{\gamma_j}\Big\|_{\ell_2(\Gamma)} \leqslant \Big\|\sum_{j=1}^n a_j \psi_{A_{\gamma_j}}\Big\|_{\ell_\infty(\Gamma)^*} \leqslant B\Big\|\sum_{j=1}^n a_j e_{\gamma_j}\Big\|_{\ell_2(\Gamma)} \tag{7.5}$$

for all distinct $\gamma_1,\ldots,\gamma_n \in \Gamma$ and all scalars $a_1,\ldots,a_n$ (see [AK06, Theorem 1.3.2]).

For any distinct $A_1,\ldots,A_n \in \mathcal{F}$ and any scalars $a_1,\ldots,a_n$ we have

$$\Big\|\sum_{j=1}^n a_j\psi_{A_j}\Big\|_{\ell_\infty(\Gamma)^*} = \sup_{x\in B_{\ell_\infty(\Gamma)}} \sum_{j=1}^n a_j\big(\varphi(\mathbb{1}_{A_j}x) - \varphi(\mathbb{1}_{\Gamma\setminus A_j}x)\big).$$

Let us use the notation $A^1 = A$ and $A^{-1} = \Gamma\setminus A$. Observe that each of the summands $\varphi(\mathbb{1}_{A_j^\varepsilon}x)$, for $1 \leqslant j \leqslant n$ and $\varepsilon = \pm 1$, may be decomposed as

$$\varphi(\mathbb{1}_{A_j^\varepsilon}x) = \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_n)\in\{-1,1\}^n\\ \varepsilon_j=\varepsilon}} \varphi\Big(x\cdot\prod_{i=1}^n \mathbb{1}_{A_i^{\varepsilon_i}}\Big).$$

In this way we obtain

$$\Big\|\sum_{j=1}^n a_j\psi_{A_j}\Big\|_{\ell_\infty(\Gamma)^*} = \sup_{x\in B_{\ell_\infty(\Gamma)}} \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\{-1,1\}^n} \Big(\sum_{j=1}^n \varepsilon_j a_j\Big)\varphi\Big(x\cdot\prod_{i=1}^n \mathbb{1}_{A_i^{\varepsilon_i}}\Big)$$

$$= \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\{-1,1\}^n} \Big|\sum_{j=1}^n \varepsilon_j a_j\Big|\cdot 2^{-n} = \int_0^1\Big|\sum_{j=1}^n a_j r_j(t)\Big|\,\mathrm{d}t,$$

where $(r_j)_{j=1}^\infty$ is the Rademacher system on $[0,1]$. By appealing to Khintchine's inequality we conclude that condition (7.5) is valid with $A=A_1$ and $B=B_1$. □