# Combinatorics in Banach space theory

## Lecture 8

## 8 The Johnson–Lindenstrauss space

In this section we will present a construction, due to Johnson and Lindenstrauss [JL74], which gives an interesting counterexample in the theory of the 'three-space problem'. The key role in the construction is played by an almost disjoint family. To motivate this topic, we shall start with some basic facts concerning the notion of weakly compactly generated spaces which yields a natural generalisation of both separable and reflexive spaces.

**Definition 8.1.** We say that a Banach space $X$ is *weakly compactly generated* (WCG for short) if there exists a weakly compact set $K\subset X$ such that $X=\overline{\mathrm{span}}(K)$. Any set $K$ with such a property will be called *fundamental*.

**Example 8.2.** Every separable Banach space is WCG.

To see this, consider any countable and dense subset $\{x_n\colon n\in\mathbb{N}\}$ of the unit ball of $X$ and let $K=\{\frac{1}{n}x_n\colon n\in\mathbb{N}\}\cup\{0\}$. Plainly, $K$ is (weakly) compact and fundamental. Observe that we have just shown that every separable Banach space is in fact generated by a compact set. Conversely, every Banach space that is $\|\cdot\|$-compactly generated must be separable, since compact metric spaces are separable and every separable subset generates only separable subspace.

**Example 8.3.** Every reflexive Banach space is WCG.

This follows immediately from the fact that the unit dual ball of any reflexive space is weakly compact and it is, of course, fundamental.

**Example 8.4.** Every $L_1(\mu)$-space, with $\mu$ being a $\sigma$-finite, non-negative measure, is WCG.

First, notice that every $L_1(\mu)$-space as above is isometrically isomorphic to another $L_1(\nu)$-space with $\nu$ being a finite measure. Indeed, there exists a $\mu$-integrable positive function $\varphi$ with $\int\varphi\,\mathrm{d}\mu=1$, thus the measure $\nu$ given by $\mathrm{d}\nu=\varphi\mathrm{d}\mu$ is a probabilistic measure and the map $U\colon L_1(\mu)\to L_1(\nu)$ given by $Uf=f/\varphi$ is an isometry.

Now, if $\mu$ is finite, then Hölder's inequality implies that the inclusion embedding $j\colon L_2(\mu)\hookrightarrow L_1(\mu)$ is a bounded linear operator and, by a standard result from measure theory, its range is dense in $L_1(\mu)$. Hence, $j(B_{L_2(\mu)})$ is a fundamental subset of $L_1(\mu)$, but it is also weakly compact as $B_{L_2(\mu)}$ is weakly compact and $j$ is weak-to-weak continuous.

**Example 8.5.** For every non-empty index set $\Gamma$ the space $c_0(\Gamma)$ is WCG.

Indeed, the inclusion embedding $j\colon\ell_2(\Gamma)\to c_0(\Gamma)$ is bounded and has a dense range, hence $j(B_{\ell_2(\Gamma)})$ is a weakly compact, fundamental subset of $c_0(\Gamma)$. Another way of showing our claim is to observe that the set $\{e_\gamma\colon\gamma\in\Gamma\}\cup\{0\}\subset c_0(\Gamma)$ is weakly compact (and is, of course, fundamental). Indeed, let $\mathcal{U}$ be any weakly open covering of this set and let

$U \in \mathcal{U}$ be such that $0 \in U$. There is a finite set $F \subset \ell_1(\Gamma) \simeq c_0(\Gamma)^*$ and some $\varepsilon > 0$ such that
$$\big\{x \in c_0(\Gamma)\colon |x^* x| < \varepsilon \text{ for } x^* \in F\big\} \subset U.$$
For any $x^* \in F$ the inequality $|x^* e_\gamma| < \varepsilon$ is valid for all but finitely many $\gamma$'s from $\Gamma$, thus all but finitely many vectors $e_\gamma$ ($\gamma \in \Gamma$) belong to $U$, which proves that $\mathcal{U}$ has a finite subcovering.

On the other hand, there are plenty of negative examples. For instance, the space $\ell_\infty$ is a non-WCG Banach space as every weakly compact subset of $\ell_\infty$ is norm separable. More generally, this property is shared by all Banach spaces that are duals of separable spaces (see Problem 1.4), hence every non-separable Banach space that is a dual of a separable one is not WCG. Another class of non-WCG spaces is formed by $\ell_1(\Gamma)$-spaces, for $\Gamma$ being any uncountable set. In this case, the reason for every weakly compact subset of $\ell_1(\Gamma)$ being norm separable is Schur's property which even implies that every weakly compact subset of $\ell_1(\Gamma)$ is compact (see Problem 4.9).

We may now proceed to the general framework for the 'three-space problem'. Later we will discuss some more advanced machinery, developed mainly by Kalton and Peck ([Kal78], [KP79]), which gives deep insight in this problem. For now, it is enough just to understand what the problem is about.

Suppose we are interested in a certain property P (such as reflexivity, separability, being isomorphic to a Hilbert space etc.) and we are given Banach spaces $X$, $Y$ and $Z$ such that $Z$ contains a subspace $Y_1$ isomorphic to $Y$ with the quotient $Z/Y_1$ isomorphic to $X$. The question is: Assuming that both $X$ and $Y$ satisfy P, must $Z$ also satisfy P?

Another way of formulating this problem is to use the language of exact sequences. Given Banach spaces $X$, $Y$ and $Z$, a *short exact sequence* is a diagram
$$0 \longrightarrow Y \xrightarrow{\;i\;} Z \xrightarrow{\;q\;} X \longrightarrow 0 \tag{8.1}$$
where each arrow represents an operator and the image of each arrow coincides with the kernel of the one that follows. Hence, $i$ is injective and $q$ is surjective. The exact sequence (8.1) corresponds exactly to the situation desribed in the previous paragraph. Indeed, $Y_1 = i(Y) = \ker(q)$ is a (closed) subspace of $Z$ and the Open Mapping Theorem implies that the (unique) algebraic isomorphism $T\colon Z/Y_1 \to X$ satisfying $T \circ \pi = q$ (where $\pi\colon Z \to Z/Y_1$ is the canonical projection) is in fact an isomorphism in the sense of the Banach space theory. We say that P is a *three-space property* (3SP *property* for short), provided that for all Banach spaces $X$, $Y$ satisfying P and every exact sequence of the form (8.1), with $Z$ being a Banach space, $Z$ also satisfies P. We say that P is a 3SP in some prescribed class $\mathcal{C}$ of linear topological spaces, provided that the above condition holds true, when instead of assuming that $X$, $Y$ and $Z$ are Banach spaces we assume that $X$, $Y$ and $Z$ belong to $\mathcal{C}$.

It is a well-known fact that if $X$ is a Banach space and $X^*$ is separable, then $X$ is separable as well. The same is true if we replace the word 'separable' by 'reflexive'. So, since WCG Banach spaces generalise both separable and reflexive spaces, the following question arises:

**Question 1.** Let $X$ be a Banach space such that $X^*$ is WCG. Does it imply that $X$ is also WCG?

Similarly, since both separabilty and reflexivity are 3SP properties (see Corollaries 1.12.10

and 1.11.19 in [Meg98]), we may also ask the following question:

**Question 2.** Is being WCG a 3SP property?

These two questions were formulated by Lindenstrauss in 1967. Seven years later Johnson and Lindenstrauss [JL74] constructed a Banach space, which we will denote as $\mathrm{JL}_2$ and call the *Johnson–Lindenstrauss space*, such that substituting $X = \mathrm{JL}_2$ gives negative answers to both of them. Let us now proceed to the definition of $\mathrm{JL}_2$.

Let $\{N_\gamma\}_{\gamma\in\Gamma}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$ with $|\Gamma| = \mathfrak{c}$ and for each $\gamma \in \Gamma$ let $\varphi_\gamma = \mathbb{1}_{N_\gamma}$ be the characteristic function of $N_\gamma$, defined on $\mathbb{N}$. Let also

$$V = \mathrm{span}\big(c_0 \cup \{\varphi_\gamma \colon \gamma \in \Gamma\}\big)$$

which is a linear subspace of the linear space $\ell_\infty$. Observe that every $x \in V$ may be written in the form

$$x = y + \sum_{j=1}^{k} a_{\gamma_j}\varphi_{\gamma_j}, \quad \text{where } y \in c_0,\ a_{\gamma_j} \in \mathbb{R},\ \gamma_j \in \Gamma \text{ and } \gamma_i \neq \gamma_j \text{ for } i \neq j. \tag{8.2}$$

Moreover, for any such $x$ the scalars $a_\gamma$ (for $\gamma \in \Gamma$) are uniquely determined (almost all of them are zero) and they may be calculated by

$$a_\gamma = \lim_{\substack{n\to\infty \\ n\in N_\gamma}} x(n).$$

Therefore, the formula

$$\|x\|_{\mathrm{JL}_2} = \max\left\{ \|x\|_\infty, \Big(\sum_{j=1}^{k} |a_{\gamma_j}|^2\Big)^{1/2} \right\} \tag{8.3}$$

is well-posed and, as is easily seen, defines a norm on the linear space $V$. The Johnson–Lindenstrauss space $\mathrm{JL}_2$ is defined as the completion of the normed space $(V, \|\cdot\|_{\mathrm{JL}_2})$. Now, we need two key properties of $\mathrm{JL}_2$.

**Proposition 8.6.** *There exists an exact sequence*

$$0 \to c_0 \to \mathrm{JL}_2 \to \ell_2(\Gamma) \to 0.$$

*Proof.* In view of formula (8.3), the identity embedding $c_0 \hookrightarrow V$ is an isometry, thus $\mathrm{JL}_2$ contains an isometric copy of $c_0$. We shall only prove that the quotient $\mathrm{JL}_2/c_0$ is isomorphic to $\ell_2(\Gamma)$.

Obviously, for any $x \in V$ of the form (8.2) we have

$$\|x\|_{\mathrm{JL}_2} \geqslant \Big(\sum_{j=1}^{k} |a_{\gamma_j}|^2\Big)^{1/2} = \|(a_\gamma)_{\gamma\in\Gamma}\|_{\ell_2(\Gamma)}.$$

On the other hand, since the sets $N_{\gamma_j}$, for $1 \leqslant j \leqslant k$, may meet only at finitely many places, there exists a finitely supported sequence $z$ (hence $z \in c_0$) such that

$$\Big\| z + \sum_{j=1}^{k} a_{\gamma_j}\varphi_{\gamma_j} \Big\|_\infty = \max_{1\leqslant j\leqslant k} |a_{\gamma_j}|,$$

whence the $\mathrm{JL}_2$-norm of the vector at the left-hand side is equal to the $\ell_2(\Gamma)$-norm of the sequence $(a_\gamma)_{\gamma\in\Gamma}$. We have thus proved that the distance between any $x \in V$ and the subspace $c_0 \hookrightarrow V$ is equal to $\|(a_\gamma)_{\gamma\in\Gamma}\|_{\ell_2(\Gamma)}$. Consequently, $V/c_0$ is isometrically isomorphic to the (non-complete) space of all finitely supported sequences in $\ell_2(\Gamma)$. By passing to completions, we get the result. □

Since $c_0$ and $\ell_2(\Gamma)$ are both WCG, if the answer to Question 2 was positive, then $\mathrm{JL}_2$ would also be WCG. But this is not the case. In fact, the collection of all evaluation functionals $\mathrm{JL}_2 \ni x \mapsto x(n)$ (for $n \in \mathbb{N}$) is a countable and total (i.e. separating points) subset of $\mathrm{JL}_2^*$. Hence, every weakly compact subset $K$ of $\mathrm{JL}_2$ is metrisable in the weak topology, therefore $K$ is weakly separable and, since weak and strong separability are equivalent, $K$ is norm separable. Consequently, $\overline{\mathrm{span}}(K)$ is also separable, thus it cannot be equal to the non-separable space $\mathrm{JL}_2$.

**Proposition 8.7.** $\mathrm{JL}_2^* \simeq \ell_2(\Gamma) \oplus \ell_1$.

*Proof.* Passing to adjoint operators in the exact sequence

$$0 \longrightarrow c_0 \xrightarrow{S} \mathrm{JL}_2 \xrightarrow{T} \ell_2(\Gamma) \longrightarrow 0$$

we obtain another (dual) exact sequence

$$0 \longrightarrow \ell_2(\Gamma) \xrightarrow{T^*} \mathrm{JL}_2^* \xrightarrow{S^*} \ell_1 \longrightarrow 0,$$

where $S^*$ is a surjective operator from $\mathrm{JL}_2^*$ onto $\ell_1$. By the projectivity of $\ell_1$ (see Problem 1.9), the operator $S^*$ admits a lifting, that is, an operator $L\colon \ell_1 \to \mathrm{JL}_2^*$ such that $S^*L = I_{\ell_1}$. This means that the exact sequence *splits* and the space $\mathrm{JL}_2^*$ is isomorphic to the direct sum of $\ell_2(\Gamma)$ and $\ell_1$. Indeed, for every $z \in \mathrm{JL}_2^*$ we have $z - LS^*z \in \ker(S^*) = \mathrm{im}(T^*) \simeq \ell_2(\Gamma)$. Therefore, the formula $z = (z - LS^*z) + LS^*z$ gives the decomposition $\mathrm{JL}_2^* = \mathrm{im}(T^*) + \mathrm{im}(L)$. Since $\mathrm{im}(T^*) \cap \mathrm{im}(L) = \{0\}$, this sum is direct. Finally, $\mathrm{im}(L)$ is a closed subspace of $\mathrm{JL}_2^*$ and the Open Mapping Theorem implies that it is isomorphic to $\ell_1$ which completes the proof. □

Consequently, the answer to Question 1 is also negative, since the dual $\mathrm{JL}_1^*$ is WCG being a direct sum of two WCG Banach spaces.

The construction of the Johnson–Lindenstrauss space presented above may be carried on with no essential changes for any $p \in (1, \infty)$ instead of 2. In this way we would get a space which is called the $p$th *Johnson–Lindenstrauss space* and is denoted as $\mathrm{JL}_p$. For any $p \in (1, \infty)$ the so-defined spaces $\mathrm{JL}_p$ enjoy properties analogous to those formulated in Propositions 8.6 and 8.7.