Combinatorics in Banach space theory

Lecture 9

9 A quantitative version of Krein's theorem

The classical Krein theorem says that the convex hull of a weakly compact subset of a Banach space is relatively weakly compact. The aim of this section is to derive a certain quantitative version of this fact which was proved in [FHM05] and which asserts that the property of being weakly compact is stable with resect to the convex hull operation. To formulate this result we need a notion of 'almost' weak compactness, or ε-weak compactness. Hereinafter, we will often identify an underlying Banach space X with its canonical image in X^{**}. In particular, for any $A \subset X$ the symbol $\overline{A}^{w*}$ stands for the weak* closure of A in X^{**}.

Definition 9.1. Let A be a subset of a Banach space X and let $\varepsilon \geqslant 0$. We say that A is *ε-relatively weakly compact* (ε-RWC for short) if it is bounded and $\overline{A}^{w*} \subset X + \varepsilon B_{X^{**}}$.

Observe that this definition harmonises perfectly with the standard fact that $A \subset X$ is relatively weakly compact if and only $\overline{A}^{w*} \subset X$. In other words, A is relatively weakly compact if and only if it is 0-RWC (see Problem 1.3). Our goal is to prove the following remarkable improvement of Krein's theorem:

Theorem 9.2 (Fabian, Hájek, Montesinos, Zizler, 2005)**.** *Let X be a Banach space and $\varepsilon \geqslant 0$. If $A \subset X$ is ε-RWC, then* $\mathrm{conv}(A)$ *is* 2ε-RWC.

Using double limits is the key technique of the proof. The following notion defines a perturbation of Grothendieck's double-limit condition characterising relatively weakly compact sets.

Definition 9.3. Let X be a Banach space, $\varepsilon \geqslant 0$ and let $A \subset X$ and $B \subset X^*$ be bounded sets. We say that A *ε-interchanges limits with* B, and we write A-ε-B, if for any sequences $(x_m)_{m=1}^\infty \subset A$ and $(x_n^*)_{n=1}^\infty \subset B$ we have

$$\Big| \lim_{m\to\infty} \lim_{n\to\infty} \langle x_m, x_n^* \rangle - \lim_{n\to\infty} \lim_{m\to\infty} \langle x_m, x_n^* \rangle \Big| \leqslant \varepsilon, \tag{9.1}$$

provided that both of the two iterated limits above exist.

The whole proof of Theorem 9.2 is contained in three steps formulated as the assertions (i)-(iii) in Theorem 9.4 below. The first assertion is elementary and rather technical. The second one requires a piece of topological machinery, that is, the upper (weak*) semicontinuous envelope which will be applied to functionals defined on B_{X^*}, in particular, to those that are not weak* continuous. Finally, the third assertion is based on a crucial combinatorial lemma due to Pták [Ptá63]. It is quite surprising that Pták's lemma, seemingly elementary and purely combinatorial, may be proved with the aid of Banach space theoretic methods. However, it fully pays back, as it is the key part of the proof of Theorem 9.2 which clearly lies in the heart of the Banach space theory.

Theorem 9.4. *Let X be a Banach space, $A \subset X$ be a bounded set and $\varepsilon \geqslant 0$. Then, the following assertions hold true:*

(i) *If A is ε-*RWC*, then A-2ε-B_{X^*}.*
(ii) *If A-ε-B_{X^*}, then A is ε-*RWC.
(iii) *If A-ε-B_{X^*}, then* $\operatorname{conv}(A)$*-ε-B_{X^*}*

Proof of assertion (i). Let $x_0^* \in X^*$ be a w^*-cluster point of the sequence $(x_n^*)_{n=1}^\infty$. Then, for every $y \in X$ and $\delta > 0$ all but finitely many x_n^*'s belong to the weak* open basis neighbourhood $\{y^* \in X^* \colon |\langle y, y^* - x_0^*\rangle| < \delta\}$ of x_0^*. In particular, we have

$$\lim_{n\to\infty} \langle x_m, x_n^*\rangle = \langle x_m, x_0^*\rangle \quad \text{for each } m \in \mathbb{N}. \tag{9.2}$$

Similarly, let $x_0^{**} \in \overline{A}^{w*}$ be a w^*-cluster point of $(x_m)_{m=1}^\infty$. Then, we have

$$\lim_{m\to\infty} \langle x_m, x_n^*\rangle = \langle x_n^*, x_0^{**}\rangle \quad \text{for each } n \in \mathbb{N}. \tag{9.3}$$

In view of (9.2) and (9.3), the two double limits appearing in condition (9.1) equal:

$$\lim_{m\to\infty}\lim_{n\to\infty} \langle x_m, x_n^*\rangle = \lim_{m\to\infty} \langle x_m, x_0^*\rangle = \langle x_0^*, x_0^{**}\rangle$$

and

$$\lim_{n\to\infty}\lim_{m\to\infty} \langle x_m, x_n^*\rangle = \lim_{n\to\infty} \langle x_n^*, x_0^{**}\rangle.$$

In order to estimate the absolute value of the difference between these double limits, we use our assumption $\overline{A}^{w*} \subset X + \varepsilon B_{X^{**}}$ to pick $x_0 \in X$ such that $\|x_0^{**} - x_0\| \leqslant \varepsilon$. Then, for each $n \in \mathbb{N}$ we have $\|x_0^{**} - x_0\| \cdot \|x_n^* - x_0^*\| \leqslant 2\varepsilon$ (recall that $x_n^*, x_0^* \in B_{X^*}$), whence

$$\Big|\lim_{n\to\infty} \langle x_n^*, x_0^{**}\rangle - \langle x_0^*, x_0^{**}\rangle\Big| = \Big|\lim_{n\to\infty} \langle x_n^* - x_0^*, x_0^{**}\rangle\Big| \leqslant \Big|\lim_{n\to\infty} \langle x_0, x_n^* - x_0^*\rangle\Big| + 2\varepsilon = 2\varepsilon. \quad \square$$

For any $x^{**} \in X^{**}$ let $\hat{x}^{**} \colon B_{X^*} \to \mathbb{R}$ be the upper w^*-semicontinuous envelope (regularisation) of x^{**} on the unit ball of X^*, that is,

$$\hat{x}^{**}(y^*) = \inf\big\{ f(y^*) \;\big|\; f \colon B_{X^*} \to \mathbb{R} \text{ is a } w^*\text{-continuous function such that } x^{**} \leqslant f \big\}. \tag{9.4}$$

It is a standard fact from general topology that the so-defined map $\hat{x}^{**}$ is the least upper w^*-semicontinuous function that majorises x^{**} (of course, for $x^{**} \in X$ we have $\hat{x}^{**} = x^{**}|_{B_{X^*}}$, since in this case x^{**} itself is w^*-continuous). It is also well-known that $\hat{x}^{**}$ may be equivalently defined by the formula

$$\hat{x}^{**}(y^*) = \lim_{U \in \mathcal{N}(y^*)} \sup x^{**}(U), \tag{9.5}$$

where $\mathcal{N}(y^*)$ is the collection of all weak* open neighbourhoods of y^* contained in B_{X^*}, directed by reversed inclusion. The equivalence of formulas (9.4) and (9.5) holds true for general topological spaces and for upper semicontinuous envelopes of arbitrary functions. We omit the routine proof of this fact. Now, before showing assertion (ii) of Theorem 9.4 we need to derive the following simple lemma:

Lemma 9.5. *For every $x^{**} \in X^{**}$ we have $\hat{x}^{**}(0) = \operatorname{dist}(x^{**}, X)$.*

Proof. Denote $d = \operatorname{dist}(x^{**}, X)$. To show our claim we will use formula (9.5).

For showing the inequality '$\leqslant$' choose a sequence $(x_n)_{n=1}^{\infty} \subset X$ satisfying $\|x^{**} - x_n\| < d + 1/n$, for $n \in \mathbb{N}$. Consider the weak* open basis neighbourhoods $U_n \in \mathcal{N}(0)$ given by

$$U_n = \big\{y^* \in B_{X^*} \colon |\langle x_j, y^* \rangle| < n^{-1} \text{ for } 1 \leqslant j \leqslant n\big\}.$$

If $y^* \in U_n$, then

$$\langle y^*, x^{**} \rangle \leqslant |\langle y^*, x^{**} - x_n \rangle| + |\langle x_n, y^* \rangle| < \|x^{**} - x_n\| + \frac{1}{n} < d + \frac{2}{n}.$$

Therefore, $\sup x^{**}(U_n) \leqslant d + 2/n$ and formula (9.5) yields $\hat{x}^{**}(0) \leqslant d$.

In order to show the inequality '$\geqslant$' it is enough to prove that for every basis neighbourhood $U \in \mathcal{N}(0)$ the supremum of x^{**} over U is at least d. So, suppose U has the form

$$U = \big\{y^* \in B_{X^*} \colon |\langle x_j, y^* \rangle| < \varepsilon \text{ for } 1 \leqslant j \leqslant k\big\}$$

for some $\varepsilon > 0$ and $x_1, \ldots, x_k \in X$. The classical corollary from the Hahn–Banach extension theorem (see, e.g., Corollary 1.9.7 in [Meg98]) implies that there exists a functional $\psi \in X^{***}$ such that $\|\psi\| = 1$, $\psi(x^{**}) = d$ and ψ vanishes on X. Moreover, Goldstine's theorem says that the canonical image of the unit ball B_{X^*} is w^*-dense in the unit ball $B_{X^{***}}$. Consequently, the values of ψ at the finite number of points: $x^{**}, x_1, \ldots, x_k$ may be approximated to within any positive number δ by a functional $y^* \in B_{X^*}$ ($\hookrightarrow B_{X^{***}}$). Taking $\delta < \varepsilon$ we get in this way a functional $y^* \in U$ satisfying $\langle y^*, x^{**} \rangle > d - \delta$. Since $\delta > 0$ may be arbitrarily small, our claim follows. □

The following corollary results immediately from Lemma 9.5 and formula (9.5).

Corollary 9.6. *Let $x^{**} \in X^{**}$ and $d = \operatorname{dist}(x^{**}, X)$. Then, for every $U \in \mathcal{N}(0)$ we have $\sup x^{**}(U) \geqslant d$ and for every $\delta > 0$ there exists $U_\delta \in \mathcal{N}(0)$ such that $\sup x^{**}(V) < d + \delta$ for every $V \in \mathcal{N}(0)$ with $V \subset U_\delta$.*

Proof of assertion (ii). Assume that A-ε-B_{X^*}. Fix any $x^{**} \in \overline{A}^{w*}$ and let $d = \operatorname{dist}(x^{**}, X)$. We are to prove that $d \leqslant \varepsilon$. To this end we will construct two sequences, $(x_m)_{m=1}^{\infty} \subset A$ and $(x_n^*)_{n=1}^{\infty} \subset B_{X^*}$ such that

$$\big|\lim_{m\to\infty}\lim_{n\to\infty} \langle x_m, x_n^* \rangle - \lim_{n\to\infty}\lim_{m\to\infty} \langle x_m, x_n^* \rangle\big| = d, \tag{9.6}$$

whereas both of the two double limits above exist. Once this is done, the proof is completed, in view of the assumption.

Pick any $x_1 \in A$ and let

$$U(x_1; 1) = \{y^* \in B_{X^*} \colon |\langle x_1, y^* \rangle| < 1\} \in \mathcal{N}(0).$$

By Corollary 9.6, we may find $x_1^* \in U(x_1; 1)$ such that $d - 1 \leqslant \langle x_1^*, x^{**} \rangle \leqslant d + 1$. Now, pick $x_2 \in A$ satisfying $|\langle x_1^*, x^{**} - x_2 \rangle| < 1/2$ (this may be done, since $x^{**} \in \overline{A}^{w*}$) and let

$$U\Big(x_1, x_2; \frac{1}{2}\Big) = \Big\{y^* \in B_{X^*} \colon |\langle x_j, y^* \rangle| < \frac{1}{2} \text{ for } j = 1, 2\Big\} \in \mathcal{N}(0).$$

Then, again by Corollary 9.6, there is $x_2^* \in U(x_1, x_2; 1/2)$ satisfying $d - 1/2 \leqslant \langle x_2^*, x^{**} \rangle \leqslant d + 1/2$. Continuing in this way we obtain sequences $(x_m)_{m=1}^{\infty} \subset A$ and $(x_n^*)_{n=1}^{\infty} \subset B_{X^*}$ which satisfy the following conditions:

(a) $|\langle x_j^*, x^{**} - x_m\rangle| < \frac{1}{m}$ for each $m \in \mathbb{N}$ and $j = 1, 2, \dots, m-1$;
(b) $x_n^* \in U\big(x_1, \dots, x_n, \frac{1}{n}\big)$, thus $|\langle x_j, x_n^*\rangle| < \frac{1}{n}$ for each $n \in \mathbb{N}$ and $j = 1, 2, \dots, n$;
(c) $d - \frac{1}{n} \leqslant \langle x_n^*, x^{**}\rangle \leqslant d + \frac{1}{n}$ for each $n \in \mathbb{N}$.

Consequently, condition (b) implies that $\lim_{m\to\infty} \lim_{n\to\infty} \langle x_m, x_n^*\rangle = 0$, whereas conditions (a) and (c) give $\lim_{n\to\infty} \lim_{m\to\infty} \langle x_m, x_n^*\rangle = d$. Therefore, equality (9.6) is valid and the proof is completed. □

We now proceed to the proof of assertion (iii) of Theorem 9.4 which involves the convex hull operation and requires a combinatorial insight. A family $\mathcal{G} \subset \mathcal{F}\mathbb{N}$ is called *hereditary*, provided that for every $G \in \mathcal{G}$ each subset of G belongs to $\mathcal{G}$.

Lemma 9.7 (Pták's lemma, 1963)**.** *Let $\mathcal{G} \subset \mathcal{F}\mathbb{N}$ be a hereditary family with the following property: There exists $\delta > 0$ such that for every finite sequence $a_1, \dots, a_n$ of non-negative numbers with $\sum_{j=1}^n a_j = 1$ there is a set $G \in \mathcal{G}$ for which $\sum_{j\in G} a_j \geqslant \delta$. Then, there exists an infinite set $M \subset \mathbb{N}$ such that $\mathcal{F}M \subset \mathcal{G}$.*

Proof. Define a non-negative function $\|\cdot\|_{\mathcal{G}}$ on c_{00}, the space of all finitely supported real sequences, by the formula

$$\Big\|\sum_{j=1}^{\infty} a_j e_j\Big\|_{\mathcal{G}} = \sup\Big\{\Big|\sum_{j\in G} a_j\Big| \colon G \in \mathcal{G}\Big\}.$$

Plainly, $\|\cdot\|_{\mathcal{G}}$ is homogeneous and satisfies the triangle inequality. Moreover, it easily follows from the assumption that $\mathcal{G}$ covers the whole of $\mathbb{N}$ and, being hereditary, contains all singletons. Therefore $\|x\|_{\mathcal{G}}$ is positive whenever $x \in c_{00}$ is non-zero, whence $\|\cdot\|_{\mathcal{G}}$ is a norm. Let X be the completion of $(c_{00}, \|\cdot\|_{\mathcal{G}})$.

Observe that for any $n \in \mathbb{N}$ and any scalars $a_1, \dots, a_n$ we have

$$\frac{\delta}{2}\sum_{j=1}^{n} |a_j| \leqslant \Big\|\sum_{j=1}^{n} a_j e_j\Big\|_{\mathcal{G}} \leqslant \sum_{j=1}^{n} |a_j|.$$

The right inequality is obvious, whereas the left one follows easily from the assumption. This means that the sequence $(e_n)_{n=1}^{\infty} \subset X$ is equivalent to the canonical basis of ℓ_1 and there exists an isomorphism from X onto ℓ_1 that maps each of the vectors e_n's from X to the corresponding canonical unit vector in ℓ_1.

Notice that every set $G \in \mathcal{G}$ gives rise to a functional $\varphi_G \in X^*$ defined on the dense subspace c_{00} of X by the formula

$$\varphi_G\bigg(\sum_{j=1}^{\infty} a_j e_j\bigg) = \sum_{j\in G} a_j.$$

Of course, for each $G \in \mathcal{G}$ we have $\|\varphi_G\| \leqslant 1$ and the collection $\{\varphi_G \colon G \in \mathcal{G}\}$ is norming, that is, for every $x \in X$ we have $\|x\| = \sup\{|\varphi_G(x)| \colon G \in \mathcal{G}\}$. Let K be the w^*-closure of the set of all φ_G's, for $G \in \mathcal{G}$. Then, K is a compact Hausdorff space and the correspondence $X \ni x \mapsto f_x \in C(K)$, given by $f_x(\varphi_G) = \langle x, \varphi_G\rangle$, is an isometric embedding of X into $C(K)$.

It is evident from the Hahn–Banach theorem that any subspace of a Banach space with a separable dual has also a separable dual. Hence, since $X \simeq \ell_1$ and X embeds into

$C(K)$, the dual of $C(K)$ is non-separable. If K was countable, then it would follows easily from the Riesz Representation Theorem that $C(K)^*$ is separable*.

Let ψ be any functional from K; we may write it formally as $\psi = \sum_{n=1}^{\infty} c_n e_n^*$, where $c_n = \psi(e_n)$ for each $n \in \mathbb{N}$. Since ψ lies in the weak* closure of the set $\{\varphi_G \colon G \in \mathcal{G}\}$, for every $\varepsilon > 0$ and $n \in \mathbb{N}$ there are infinitely many G's in $\mathcal{G}$ satisfying $|c_j - \varphi_G(e_j)| < \varepsilon$ for each $1 \leqslant j \leqslant n$. This, in particular, implies that for every $n \in \mathbb{N}$ we have $c_n \in \{0,1\}$ and the functional ψ may be identified with a subset H of $\mathbb{N}$ (consisting of all those $n \in \mathbb{N}$ for which $c_n = 1$). Moreover, the set H is the limit in the product (pointwise) topology of $2^{\mathbb{N}}$ of a sequence of some sets $G \in \mathcal{G}$. In other words, K may be identified with the closure of $\mathcal{G}$ in the pointwise topology on $2^{\mathbb{N}}$. Since K is uncountable, there exists an infinite set $M \subset \mathbb{N}$ which is a pointwise limit of elements from $\mathcal{G}$ and since $\mathcal{G}$ is hereditary, every finite subset of M must belong to $\mathcal{G}$. $\square$

The assertion of Pták's lemma may be also formulated without assuming that the underlying family is hereditary. In fact, this is how we will use Pták's lemma in the proof of assertion (iii) of Theorem 9.4.

Lemma 9.8 ('Non-hereditary' version of Pták's lemma)**.** *Assume $\mathcal{G} \subset \mathcal{F}\mathbb{N}$ has the following property: There exists $\delta > 0$ such that for every finite sequence $a_1, \ldots, a_n$ of non-negative numbers with $\sum_{j=1}^{n} a_j = 1$ there is a set $G \in \mathcal{G}$ for which $\sum_{j \in G} a_j \geqslant \delta$. Then, there exists a strictly increasing sequence $H_1 \subset H_2 \subset \ldots$ of finite subsets of $\mathbb{N}$ and a sequence $(G_n)_{n=1}^{\infty} \subset \mathcal{G}$ such that $H_n \subset G_n$ for each $n \in \mathbb{N}$.*

Proof. Define $\widetilde{\mathcal{G}} = \{H \in \mathcal{F}\mathbb{N} \colon H \subset G \text{ for some } G \in \mathcal{G}\}$ and apply Pták's Lemma 9.7 to the (hereditary) family $\widetilde{\mathcal{G}}$ instead of $\mathcal{G}$. $\square$

Proof of assertion (iii)*.* Pick a positive number M such that $\|x\| \leqslant M$ for every $x \in A$. Let $\varepsilon > 0$ and suppose that there exist two sequences: $(x_m)_{m=1}^{\infty} \subset \mathrm{conv}(A)$ and $(x_n^*)_{n=1}^{\infty} \subset B_{X^*}$ satisfying

$$\Big| \lim_{m\to\infty} \lim_{n\to\infty} \langle x_m, x_n^* \rangle - \lim_{n\to\infty} \lim_{m\to\infty} \langle x_m, x_n^* \rangle \Big| = \varepsilon. \tag{9.7}$$

Pick any $\beta \in (0, \varepsilon)$. We are going to construct a sequence $(t_j)_{j=1}^{\infty} \subset A$ and a subsequence $(x_{n_k}^*)_{k=1}^{\infty}$ of $(x_n^*)_{n=1}^{\infty}$ such that

$$\Big| \lim_{j\to\infty} \lim_{k\to\infty} \langle t_j, x_{n_k}^* \rangle - \lim_{k\to\infty} \lim_{j\to\infty} \langle t_j, x_{n_k}^* \rangle \Big| \geqslant \beta \tag{9.8}$$

and then we will be done.

---

*Suppose K is a countable compact Hausdorff space. Then, for every $(x,y) \in K \times K$ with $x \neq y$ the set $H_{x,y} = \{f \in C(K) \colon f(x) = f(y)\}$ is the kernel of the non-zero functional $C(K) \ni f \mapsto f(x) - f(y)$, so it is a closed, one-codimensional hyperplane. By the Baire Category Theorem, the union $\bigcup_{(x,y) \in K \times K, x \neq y} H_{x,y}$ is a meagre subset of $C(K)$, thus there is some function $f \in C(K)$ that belongs to none of $H_{x,y}$'s. In other words, f is a one-to-one continuous map from K into $\mathbb{R}$ and therefore it gives rise to a metric on K, consistent with the original topology. We have thus proved that every countable compact Hausdorff space is metrisable. With no much extra effort it may be also shown that every such space is *scattered*, that is, every its non-empty closed subset has an isolated point (see [FHH10, Lemma 14.21]). Moreover, according to Rudin's theorem [FHH10, Theorem 14.24], for every scattered compact Hausdorff space K the dual of $C(K)$ is isometrically isomorphic to $\ell_1(\Gamma)$ for some index set Γ and $C(K)^* \simeq \ell_1$ whenever K is countable.

Considering the set $\{\delta_t \colon t \in K\} \subset C(K)^*$ of all Dirac's measures, it is easy to see that $C(K)^*$ is non-separable whenever K is uncountable. Consequently, the dual of $C(K)$ is separable if, and only if, K is countable and in every such case (except finite-dimensional spaces) $C(K)^*$ is isometric to ℓ_1.

Let $x_0^* \in B_{X^*}$ be a w^*-cluster point of $(x_n^*)_{n=1}^\infty$ and let $A_0 \subset A$ be a countable set such that $(x_m)_{m=1}^\infty \subset \mathrm{conv}(A_0)$. For any point $x \in X$ there is a subsequence of $(x_n^*)_{n=1}^\infty$ which converges at x to $x_0^* x$. Using the diagonal procedure we may extract a common subsequence for all the elements of the countable set $A_0 \cup \{x_m \colon m \in \mathbb{N}\}$. For simplicity, let us denote that subsequence again by $(x_n^*)_{n=1}^\infty$. Then, we have $x_n^* \to x_0^*$ pointwise on the set $A_0 \cup \{x_m \colon m \in \mathbb{N}\}$.

Since $\lim_n \langle x_m, x_n^* \rangle = \langle x_m, x_0^* \rangle$, we may rewrite (9.7) as

$$\sigma\bigl(\lim_{m\to\infty} \langle x_m, x_0^* \rangle - \lim_{n\to\infty} \lim_{m\to\infty} \langle x_m, x_n^* \rangle\bigr) = \varepsilon,$$

with a suitable $\sigma \in \{-1, 1\}$. Pick any $\delta > 0$ (we shall impose an additional condition upon δ later); by deleting a finite number of x_n^*'s we may assume that

$$\sigma\bigl(\lim_{m\to\infty} \langle x_m, x_0^* \rangle - \lim_{m\to\infty} \langle x_m, x_n^* \rangle\bigr) > \varepsilon - \delta \quad \text{for each } n \in \mathbb{N},$$

that is,

$$\sigma \lim_{m\to\infty} \langle x_m, x_0^* - x_n^* \rangle > \varepsilon - \delta \quad \text{for each } n \in \mathbb{N}. \tag{9.9}$$

Now, let us give a heuristic argument in order to see in what way Pták's lemma might be used. Suppose we have found a sequence $(t_j)_{j=1}^\infty$ satisfying (9.8) and suppose that it is contained in A_0, just to make our life easier. Then for each $j \in \mathbb{N}$ we have $\lim_k \langle t_j, x_{n_k}^* \rangle = \langle t_j, x_0^* \rangle$, so (9.8) really means that $\lim_k \lim_j |\langle t_j, x_0^* - x_{n_k}^* \rangle| \geqslant \beta$. That would be true, if we had $|\langle t_j, x_0^* - x_{n_k}^* \rangle| \geqslant \beta$, for each $j \in \mathbb{N}$ and each $k = 1, \ldots, j$. This observation suggests to define sets

$$\Gamma(t) = \bigl\{n \in \mathbb{N} \colon |\langle t, x_0^* - x_n^* \rangle| \geqslant \beta\bigr\} \quad (t \in A_0).$$

At once, we may note that every $\Gamma(t)$ is a finite subset of $\mathbb{N}$, as $x_n^* \to x_0^*$ pointwise on A_0. If we could prove that the family $\mathcal{G} = \{\Gamma(t) \colon t \in A_0\}$ satisfies the assumption of Pták's Lemma 9.8, then for some sequence $(t_j)_{j=1}^\infty \subset A_0$ and some natural numbers $n_1 < n_2 < \ldots$ we would have

$$\{n_1, \ldots, n_j\} \subset \Gamma(t_j) \quad \text{for each } j \in \mathbb{N}. \tag{9.10}$$

By extracting appropriate subsequences from $(t_j)_{j=1}^\infty$ and $(x_{n_k}^*)_{k=1}^\infty$ we could also assume that:

- $\lim_j \langle t_j, x_0^* - x_{n_k}^* \rangle$ exists for every $k \in \mathbb{N}$,
- $\lim_k \lim_j \langle t_j, x_0^* - x_{n_k}^* \rangle$ exists,
- $\lim_j \langle t_j, x_0^* \rangle$ exists,

and this would not influence the validity of (9.10). Having all these conditions, both of the two double limits in (9.8) exist and, as it was explained above, equality (9.8) holds true. So, what remains to be proved is that the family $\mathcal{G}$ satisfies the assumption of Pták's lemma with some number $\gamma > 0$.

Suppose that no choice of $\gamma > 0$ is possible. Then, for every $\gamma > 0$ there is a sequence $a_1, \ldots, a_N$ of non-negative numbers with $\sum_{j=1}^N a_j = 1$ and such that $\sum_{j \in G} a_j < \gamma$ for every $G \in \mathcal{G}$. Define

$$x^* = \sum_{j=1}^N a_j (x_0^* - x_j^*) \in 2B_{X^*}.$$

For every $t \in A_0$ we have

$$\begin{aligned} |\langle t, x^* \rangle| &= \left| \sum_{j=1}^{N} a_j \langle t, x_0^* - x_j^* \rangle \right| \\ &\leqslant \sum_{\substack{1 \leqslant j \leqslant N \\ j \in \Gamma(t)}} a_j \underbrace{|\langle t, x_0^* - x_j^* \rangle|}_{\leqslant 2\|t\| \leqslant 2M} + \sum_{\substack{1 \leqslant j \leqslant N \\ j \notin \Gamma(t)}} a_j |\langle t, x_0^* - x_j^* \rangle| \leqslant 2M\gamma + \beta. \end{aligned}$$

Since $(x_m)_{m=1}^{\infty} \subset \operatorname{conv}(A_0)$, we have also $|\langle x_m, x^* \rangle| \leqslant 2M\gamma + \beta$ for each $m \in \mathbb{N}$. Therefore, in view of (9.9), we have

$$\begin{aligned} 2M\gamma + \beta \geqslant \lim_{m\to\infty} |\langle x_m, x^* \rangle| &= \left| \sum_{j=1}^{N} a_j \lim_{m\to\infty} \langle x_m, x_0^* - x_j^* \rangle \right| \\ &= \sigma \sum_{j=1}^{N} a_j \lim_{m\to\infty} \langle x_m, x_0^* - x_j^* \rangle > \varepsilon - \delta. \end{aligned}$$

Consequently, the supposition that $\mathcal{G}$ does not satisfy the assumption of Pták's lemma leads to the conclusion $\delta + 2M\gamma > \varepsilon - \beta$. However, for any $\beta < \varepsilon$ we may choose $\delta > 0$ and $\gamma > 0$ so small that $\delta + 2M\gamma < \varepsilon - \beta$. $\square$