Functional analysis

Lecture 1: Normed space; examples; Riesz' lemma; equivalence of norms on a finite-dimensional space

1 Normed spaces and Banach spaces

Throughout this lecture we will be considering vector spaces over a field $\mathbb{K}$ of either real or complex numbers, i.e. $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Some of the presented results remain valid in the same form when the scalar field is $\mathbb{R}$ and $\mathbb{C}$. Sometimes, however, the transition from the real case to the complex one requires a nontrivial argument (like in the Hahn–Banach theorem), but it also happens that the situation is essentially different in both these cases (as in the spectral theory). Usually, it should be clear from the context over which field are the vector (normed, Banach) spaces in question. If not, it should be explicitely stated. If we do not mention the scalar field, it means that the result works the same way over either $\mathbb{R}$ and $\mathbb{C}$.

Definition 1.1. Let X be a vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A function $\|\cdot\|\colon X \to [0, \infty)$ is called a *norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$,
(ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{K}$, $x \in X$,
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (the *triangle inequality*).

The vector space X equipped with a norm $\|\cdot\|$, that is, the pair $(X, \|\cdot\|)$ we shall call a *normed space*.

Each norm is naturally associated with a metric given by $\rho(x, y) = \|x - y\|$, so every normed space is automatically a metric space, hence also a topological space with the topology generated by the basis $\{D(x, r)\colon x \in X,\ r > 0\}$, where $D(x, r)$ stands for the open ball:

- $D(x, r) = \{y \in X\colon \|x - y\| < r\}$.

Using any topological notions (like convergence, closedness, density etc.) in reference to a normed space, we shall always have in mind the topolgy given by the underlying norm, unless otherwise stated. In particular, the norm as a function defined on X and values in $[0, \infty)$ is continuous with respect to itself **(classes)**.

The (closed) unit ball and the unit sphere of a normed space $(X, \|\cdot\|)$ are denoted, respectively, by:

- $B_X = \{x \in X\colon \|x\| \leq 1\}$,
- $S_X = \{x \in X\colon \|x\| = 1\}$.

For simplifying notation, instead of mentioning the pair $(X, \|\cdot\|)$, we shall often speak of X alone as of a normed space. In most cases it will be clear from the context which norm we consider. Below, we list the most classical examples of normed spaces—these objects appear all over mathematics.

Example 1.2. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. The following vector spaces over $\mathbb{K}$, supplied with norms described below, are normed spaces.

(1) The n-dimensional space $\mathbb{K}^n$ supplied with one of the norms:

- $\|\mathbf{x}\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$,
- $\|\mathbf{x}\|_p = (|x_1|^p + \ldots + |x_n|^p)^{1/p}$ for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n$,

where p is any parameter from $[1, \infty)$. (Notice that $\lim_{p\to\infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$, so the 'maximum norm' is the limit case of the 'p-norms' as p grows to infinity.)

(2) The sequence spaces:

- $c_0 = \big\{(x_n)_{n=1}^\infty \in \mathbb{K}^\mathbb{N}\colon\ \lim_n x_n = 0\big\}$,
- $c = \big\{(x_n)_{n=1}^\infty \in \mathbb{K}^\mathbb{N}\colon\ \lim_n x_n \text{ exists in } \mathbb{K}\big\}$,
- $\ell_\infty = \big\{(x_n)_{n=1}^\infty \in \mathbb{K}^\mathbb{N}\colon\ \sup_n |x_n| < \infty\big\}$,
- $\ell_p = \big\{(x_n)_{n=1}^\infty \in \mathbb{K}^\mathbb{N}\colon\ \sum_n |x_n|^p < \infty\big\}$ $(1 \leq p < \infty)$,

where each of the spaces: c_0, c and ℓ_∞ is supplied with the 'supremum norm':

$$\|(x_n)_{n=1}^\infty\|_\infty = \sup_n |x_n|,$$

whereas ℓ_p is supplied with a norm defined by

$$\|(x_n)_{n=1}^\infty\|_p = \Big(\sum_{n=1}^\infty |x_n|^p\Big)^{1/p}.$$

(Of course, the linear structure in all the sequence spaces above is given by the standard coordinatewise operations.)

All the spaces $(\mathbb{K}^n, \|\cdot\|_p)$ and $(\ell_p, \|\cdot\|_p)$ for $1 \leq p \leq \infty$ are particular cases of the following, more general construction:

(3) Let $(X, \mathfrak{M}, \mu)$ be a measure space with a positive (not necessarily finite) measure μ. On the set of all measurable scalar-valued functions on X we introduce an equivalence relation $\sim$ by saying that $f \sim g$ if and only if $f(x) = g(x)$ holds true μ-a.e. on X, that is, $\mu(\{x\colon f(x) \neq g(x)\}) = 0$. Denote by $[f]_\sim$ the class of abstraction corresponding to a function f. For $1 \leq p < \infty$, we define

$$L_p(\mu) = \Big\{[f]_\sim \mid\ f\colon X \to \mathbb{K} \text{ is } \mathfrak{M}\text{-measurable and } \textstyle\int_X |f|^p \,\mathrm{d}\mu < \infty\Big\},$$

and for $p = \infty$,

$$L_\infty(\mu) = \Big\{[f]_\sim \mid\ f\colon X \to \mathbb{K} \text{ is } \mathfrak{M}\text{-measurable and essentially bounded, i.e. } \exists_{A\in\mathfrak{M},\, \mu(A)=0}\ \sup_{x\in X\setminus A} |f(x)| < \infty\Big\}.$$

These sets form vector spaces over $\mathbb{K}$ with the canonical operations $[f]_\sim + [g]_\sim = [f + g]_\sim$ and $\lambda[f]_\sim = [\lambda f]_\sim$. They are obviously well-defined, as well as the following functions which are actually norms:

- $\|[f]_\sim\|_p = \Big\{ \int\limits_X |f|^p \, \mathrm{d}\mu \Big\}^{1/p}$ for $[f]_\sim \in L_p(\mu)$, $1 \leq p < \infty$,
- $$\begin{aligned} \|[f]_\sim\|_\infty &= \operatorname{ess\,sup} f \\ &\overset{\text{df.}}{=} \inf_{A\in\mathfrak{M},\, \mu(A)=0} \sup_{x\in X\setminus A} |f(x)| \\ &= \inf\big\{a \in \mathbb{R} \colon \mu(|f|^{-1}(a,\infty)) = 0\big\} \quad \text{for } [f]_\sim \in L_\infty(\mu). \end{aligned}$$

Remark. (a) Even though the elements of $L_p(\mu)$ are classes of abstraction, we will always treat them as genuine functions. This shall not cause any confusion as long as we remember to identify functions equal almost everywhere.
(b) The normed spaces $(\mathbb{K}^n, \|\cdot\|_p)$ and $(\ell_p, \|\cdot\|_p)$ $(1 \leq p \leq \infty)$ are particular cases of the $L_p(\mu)$-spaces. This can be seen by considering the counting measure μ on an n-element set and on the set $\mathbb{N}$ of natural numbers, respectively (i.e. $\mu(A)$ is the cardinality of A if A is finite and is ∞ if A is infinite). Similarly, considering any set of indices Γ (of arbitrarily large cardinality) and taking μ to be the counting measure on Γ we obtain 'long' versions of the ℓ_p-spaces $(1 \leq p \leq \infty)$, and also of the space c_0:

- $c_0(\Gamma) = \Big\{x \colon \Gamma \to \mathbb{K} \,\big|\, \forall_{\varepsilon>0}\ \{\gamma \in \Gamma \colon |x(\gamma)| \geq \varepsilon\} \text{ is finite}\Big\}$,
- $\ell_\infty(\Gamma) = \big\{x \colon \Gamma \to \mathbb{K} \,\big|\, \|x\|_\infty \coloneqq \sup_{\gamma\in\Gamma} |x(\gamma)| < \infty\big\}$,
- $\ell_p(\Gamma) = \big\{x \colon \Gamma \to \mathbb{K} \,\big|\, \|x\|_p \coloneqq \big(\sum_{\gamma\in\Gamma} |x(\gamma)|^p\big)^{1/p} < \infty\big\}$ $(1 \leq p < \infty)$.

Notice that each element x of $c_0(\Gamma)$ or $\ell_p(\Gamma)$ for $1 \leq p < \infty$ must have countable support, i.e. $x(\gamma) = 0$ for all but countably many γ's. Nevertheless, each of these space is nonseparable whenever Γ is uncountable **(classes)**. At this point, let us introduce a common notation:

- $\ell_p^n = (\mathbb{K}^n, \|\cdot\|_p)$, for $1 \leq p \leq \infty$;
- c_0^n is sometimes used instead of ℓ_∞^n (for reasons that should be clear).

(4) Let K be a compact Hausdorff space, L be a locally compact (i.e. every point has a neighborhood of compact closure) Hausdorff space, and let X be any Hausdorff space. Then, all the following three spaces:

  - $C(K) = \{f \colon K \to \mathbb{K} \,|\, f \text{ is continuous}\}$,
  - $C_0(L) = \Big\{f \colon L \to \mathbb{K} \,\big|\, \forall_{\varepsilon>0}\ \{x \in L \colon |f(x)| \geq \varepsilon\} \text{ is compact}\Big\}$,
  - $C_b(X) = \{f \colon K \to \mathbb{K} \,|\, f \text{ is continuous and bounded}\}$,

  are normed spaces when supplied with the supremum norm $\|f\|_\infty = \sup_x |f(x)|$. (Functions belonging to $C_0(L)$ are called *vanishing at infinity*.)

Proof (that examples (1)–(4) yield normed spaces). Definitely, all the spaces above are vector spaces over $\mathbb{K}$ and the conditions (i) and (ii) from the definition of norm are trivially satisfied. The triangle inequality is very easy to verify in the cases of: c_0^n, c_0, ℓ_∞, $L_\infty(\mu)$, $C(K)$, $C_0(L)$ and $C_b(X)$. We just need to show the triangle inequality for the

$L_p(\mu)$-spaces, for $1 \le p < \infty$ (as mentioned, ℓ_p^n's and ℓ_p are particular cases), but this is a direct consequence of the famous Minkowski inequality:

$$\left\{ \int_X |f+g|^p \,\mathrm{d}\mu \right\}^{1/p} \le \left\{ \int_X |f|^p \,\mathrm{d}\mu \right\}^{1/p} + \left\{ \int_X |g|^p \,\mathrm{d}\mu \right\}^{1/p},$$

which can be proven with the aid of Hölder's inequality:

$$\int_X |fg| \,\mathrm{d}\mu \le \left\{ \int_X |f|^p \,\mathrm{d}\mu \right\}^{1/p} \left\{ \int_X |g|^{p'} \,\mathrm{d}\mu \right\}^{1/p'};$$

they both hold true for all measurable functions f, g, and each $p \in (1, \infty)$ (for $p = 1$ the Minkowski inequality is trivially fulfilled as well), where $p' \in (1, \infty)$ is the *conjugate exponent* to p, i.e. the one that satisfies $\frac{1}{p} + \frac{1}{p'} = 1$ **(classes)**. □

If Y is any subset (in particular, a subspace) of a normed space X, then the distance from x to Y is denoted by

- $\mathrm{dist}(x, Y) = \inf\{\|x - y\| : y \in Y\}$.

The first nontrivial information on the geometry of general normed spaces we get from the following lemma sometimes called the lemma about an *almost orthogonal element.*

Lemma 1.3 (Riesz' lemma). *Let X be a normed space and $Y \subsetneq X$ be its proper closed subspace. Then, for every $\varepsilon > 0$ there exists a vector $x \in S_X$ such that* $\mathrm{dist}(x, Y) \ge 1 - \varepsilon$.

Proof. Pick any $u \in X \setminus Y$ and let $\delta = \mathrm{dist}(u, Y)$. Since Y is closed, we have $\delta > 0$. For any $\eta > 0$ we can find $v \in Y$ such that $\delta \le \|u - v\| \le \delta + \eta$. Define

$$x = \frac{u - v}{\|u - v\|}$$

and observe that for each $y \in Y$ we have

$$\|x - y\| = \left\| \frac{u - v}{\|u - v\|} - y \right\| = \frac{1}{\|u - v\|} \big\| u - \underbrace{(v + \|u - v\| y)}_{\in Y} \big\| \ge \frac{\delta}{\delta + \eta}.$$

Hence, it is enough to take η so that $\frac{\delta}{\delta+\eta} \ge 1 - \varepsilon$, and then $\mathrm{dist}(x, Y) \ge 1 - \varepsilon$. □

Corollary 1.4. *If X is an infinite-dimensional normed space, then the unit sphere S_X contains a sequence $(x_n)_{n=1}^\infty$ such that $\|x_m - x_n\| \ge \frac{1}{2}$ for all $m, n \in \mathbb{N}$, $m \ne n$, and therefore S_X (and B_X) fails to be compact.*

Definition 1.5. Let X be a vector space over either $\mathbb{R}$ or $\mathbb{C}$ and let $\|\cdot\|_1$, $\|\cdot\|_2$ be two norms on X. We say that these norms are *equivalent* provided there exist constants $c, C > 0$ such that

$$c\|x\|_1 \le \|x\|_2 \le C\|x\|_1 \quad \text{for every } x \in X.$$

Remark. The motivation for this definition comes from the fact that two norms are equivalent if and only if they generate the same topology **(classes)**.

Theorem 1.6. *If X is a finite-dimensional normed space, then all norms on X are equivalent to each other.*

Proof. Let $n = \dim X < \infty$ and $(\mathrm{e}_1, \ldots, \mathrm{e}_n)$ be a Hamel (algebraic) basis of X. Any $x \in X$ can be then uniquely written in the form $x = \sum_{j=1}^{n} \alpha_j \mathrm{e}_j$ for some scalars α_j. For such an x we set $\|x\|_\infty = \max_{1\le j\le n} |\alpha_j|$. Plainly, this formula defines a norm $\|\cdot\|_\infty$ on X.

Let $\|\cdot\|$ be any other norm on X; it is enough to show that $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent. First, observe that

$$\|x\| \le \sum_{j=1}^{n} |\alpha_j| \|\mathrm{e}_j\| \le C\|x\|_\infty, \tag{1.1}$$

where $C := \sum_{j=1}^{n} \|\mathrm{e}_j\|$.

Let $\mathcal{T}$ and $\mathcal{U}$ stand for the topologies on X generated by the norms $\|\cdot\|_\infty$ and $\|\cdot\|$, respectively. Let $B = \{x \in X \colon \|x\|_\infty \le 1\}$. In view of (1.1), we have $\mathcal{U} \subseteq \mathcal{T}$. Since B is $\mathcal{T}$-compact, it is also $\mathcal{U}$-compact and since any compact Hausdorff topology is *rigid* (cannot be weakened without losing Hausdorffness and cannot be enriched without spoiling compactness—**(classes)**), we infer that $\mathcal{T}$ and $\mathcal{U}$ agree on the set B.

Define $A = \{x \in X \colon \|x\|_\infty < 1\}$. Since it is a $\mathcal{T}$-open subset of B, it is also $\mathcal{U}$-open. Hence, there is $U \in \mathcal{U}$ such that $U \cap B = A$. Obviously, $0 \in A$, so U must contain a $\mathcal{U}$-neighborhood of zero which means that there is $r > 0$ with $\{x \in X \colon \|x\| < r\} \subseteq U$. Therefore,

$$\big(\|x\| < r \ \text{ and } \ \|x\|_\infty \le 1\big) \implies \|x\|_\infty \le 1. \tag{1.2}$$

Claim. $\|x\| < r \implies \|x\|_\infty < 1$.

Assume that $\|x\| < r$. Obviously, $\frac{x}{\|x\|_\infty} \in B$ and now if $\|x\|_\infty \ge 1$, then we would have

$$\left\| \frac{x}{\|x\|_\infty} \right\| < \frac{r}{\|x\|_\infty} \le r.$$

Hence, condition (1.2) would imply that $1 = \frac{x}{\|x\|_\infty} < 1$; a contradiciton which establishes our claim. By homogeneity, it follows easily that $\|x\|_\infty < r^{-1}\|x\|$ for every $x \in X$. □

Corollary 1.7. *If M is a finite-dimensional subspace of a normed space, then M is closed.*

Proof. **(classes)** □

Corollary 1.8. *A normed space X is locally compact if and only if* $\dim X < \infty$.

Proof. One direction is contained in Corollary 1.4. Namely, if $\dim X = \infty$, then X is not locally compact. On the other hand, if $n := \dim X < \infty$, then Theorem 1.6 guarantees that the norm on X is equivalent, for example, to the Euclidean norm in ℓ_2^n (we identify X with $\mathbb{K}^n$ in an obvious manner). This means that B_X is a closed set contained in a Euclidean ball of certain radius. Hence, B_X is compact due to the Heine–Borel theorem. Consequently, every closed ball in X is compact and thus every point has a neighborhood with compact closure. □