Functional analysis

Lecture 10: Boundedness of variation of complex measures; The Radon–Nikodym theorem and duality $L_p(\mu)^* \cong L_q(\mu)$

Proof of Theorem 6.1(b). The main observation is the following

Claim. For every $E \in \mathfrak{M}$ with $|\mu|(E) = \infty$, there exists a decomposition $E = A \cup B$ such that $A, B \in \mathfrak{M}$, $A \cap B = \varnothing$, $|\mu(A)| > 1$ and $|\mu|(B) = \infty$.

Indeed, take any $t > 6(1 + |\mu(E)|)$ and choose a partition $(E_i)_{i=1}^n \in \Pi(E)$ such that $\sum_{i=1}^n |\mu(E_i)| > t$. By Lemma 6.2 applied to the complex numbers $z_i = \mu(E_i)$ $(i = 1, \dots, n)$ there is a set $S \subseteq \{1, \dots, n\}$ for which $|\mu(A)| \geq 1$, where $A = \bigcup_{i \in S} E_i$. By the definition of t and our choice of S, we also have

$$|\mu(E \setminus A)| \geq |\mu(A)| - |\mu(E)| \geq 1.$$

Since the variation of μ is additive, at least one of the values $|\mu|(A)$ and $|\mu|(E \setminus A)$ must be infinite. Interchanging A with $E \setminus A$ if necessary, we obtain the desired decomposition.

Now, suppose that $|\mu|(X) = \infty$. Using the above Claim, by an easy induction, we construct a descending sequence of measurable sets $(B_n)_{n=0}^\infty$ with $B_0 = X$ such that for each $n \in \mathbb{N}_0$ we have:

- $B_n = A_{n+1} \cup B_{n+1}$,
- $A_{n+1} \cap B_{n+1} = \varnothing$,
- $|\mu(A_{n+1})| \geq 1$ and $|\mu|(B_{n+1}) = \infty$.

Then, the sets $(A_n)_{n=1}^\infty$ are pairwise disjoint and the series $\sum_{n=1}^\infty \mu(A_n)$ is plainly divergent. This contradicts the fact that $|\mu|$ is σ-additive as the set $C = \bigcup_{n=1}^\infty A_n$ belongs to $\mathfrak{M}$. $\square$

Definition 6.3. Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ and let λ be either a positive (not necessarily finite) measure or a complex-valued measure on $\mathfrak{M}$. We say that λ is *absolutely continuous* with respect to μ and write

$$\lambda \ll \mu$$

provided that $\mu(E) = 0$ implies $\lambda(E) = 0$ for every $E \in \mathfrak{M}$. We say that λ is *supported* on a set $E \in \mathfrak{M}$ if for every $A \in \mathfrak{M}$ we have $\lambda(A) = \lambda(A \cap E)$. If λ_1, λ_2 are two positive (not necessarily finite) measures, or two complex measures on $\mathfrak{M}$, then we call them *singular* and write

$$\lambda_1 \perp \lambda_2$$

if they are supported on disjoint sets.

Let μ be a positive measure and $h \in L_1(\mu)$. It is easy to show that the formula $\lambda(E) = \int_E h \,\mathrm{d}\mu$ defines a (signed or complex) measure which is absolutely continuous with respect to μ. The fundamental theorem of Radon and Nikodym says that this is basically the only method of producing measures absolutely continuous with respect to the given one. Below we present a proof due to von Neumann which actually gives two results simultaneously: the classical Lebesgue decomposition theorem which says that any complex measure has a uniquely determined singular part and absolutely continuous part (assertion (a)) and the mentioned Radon–Nikodym theorem which represents the absolutely continuous part as an intergal.

Theorem 6.4 (**Lebesgue–Radon–Nikodym theorem**). *Let $\mathfrak{M}$ be a σ-algebra of subsets of X. Let μ be a σ-finite positive measure on $\mathfrak{M}$ and λ a complex measure on $\mathfrak{M}$.*

(a) *There is a uniquely determined pair of complex measures $(\lambda_{\mathrm{a}}, \lambda_{\mathrm{s}})$ on $\mathfrak{M}$ such that*

$$\lambda = \lambda_{\mathrm{a}} + \lambda_{\mathrm{s}}, \quad \lambda_{\mathrm{a}} \ll \mu, \quad \lambda_{\mathrm{s}} \perp \mu. \tag{6.1}$$

Moreover, $\lambda_{\mathrm{a}} \perp \lambda_{\mathrm{s}}$ and if λ is positive, then λ_{a} and λ_{s} are also positive.

(b) *There is a unique function $h \in L_1(\mu)$ such that*

$$\lambda_{\mathrm{a}}(E) = \int_E h \, \mathrm{d}\mu \qquad (E \in \mathfrak{M}). \tag{6.2}$$

Proof. First, we remark that we can reduce the proof to the case where both μ and λ are positive and finite. So, assume that the result has been proved in this particular case. In the first step, if μ is positive σ-finite and λ is positive finite, pick a disjoint family of measurable sets $(X_n)_{n=1}^{\infty}$ such that $\mu(X_n) < \infty$ for each $n \in \mathbb{N}$. Consider the σ-algebra $\mathfrak{M}_n = \{E \cap X_n \colon E \in \mathfrak{M}\}$ of subsets of X_n and let μ_n and λ_n be the restrictions of μ and λ to $\mathfrak{M}_n$. We get a sequence of Lebesgue decompositions:

$$\lambda_n = \lambda_{\mathrm{a}}^{(n)} + \lambda_{\mathrm{s}}^{(n)}, \quad \lambda_{\mathrm{s}}^{(n)} \ll \mu_n, \quad \lambda_{\mathrm{s}}^{(n)} \perp \mu_n,$$

as well as a sequence of functions $h_n \in L_1(\mu_n)$ such that $\lambda_n(E) = \int_E h_n \, \mathrm{d}\mu_n$ for all $n \in \mathbb{N}$ and $E \in \mathfrak{M}_n$. Then, the formulas

$$\lambda_{\mathrm{a}}(E) = \sum_{n=1}^{\infty} \lambda_{\mathrm{a}}^{(n)}(E \cap X_n), \quad \lambda_{\mathrm{s}}(E) = \sum_{n=1}^{\infty} \lambda_{\mathrm{s}}^{(n)}(E \cap X_n)$$

give the absolutely continuous and singular parts in the Lebesgue decomposition of λ. On the other hand, the function $h \colon X \to \mathbb{R}$ defined by $h(x) = h_n(x)$, for $x \in X_n$, represents the measure λ as the integral with respect to μ. Notice that $h \in L_1(\mu)$ because $\lambda(X) < \infty$.

Now, assume that μ is still positive and σ-finite, but λ is a complex measure. Then $\lambda = \lambda_1 + \mathrm{i}\lambda_2$, where λ_1, λ_2 are signed measures, the real and imaginary parts of λ. By the Hahn decomposition theorem (see Theorem 3.22), we can write both these measures as differences of positive finite measures (positive and negative parts):

$$\lambda_1 = \lambda_1^+ - \lambda_1^-, \quad \lambda_2 = \lambda_2^+ - \lambda_2^-$$

Applying the previous part to μ and each of the measures $\lambda_i^{\pm}$ $(i = 1, 2)$ we obtain the Lebesgue decomposition of λ in an obvious way. If $h_i^{\pm} \in L_1(\mu)$ yield the integral representation of $\lambda_i^{\pm}$, then plainly $h \in L_1(\mu)$ defined by $h = (h_1^+ - h_1^-) + \mathrm{i}(h_2^+ - h_2^-)$ gives the integral representation of λ.

For proving the uniqueness of λ_{a} and λ_{s}, suppose we have two pairs $(\lambda_{\mathrm{a}}, \lambda_{\mathrm{s}})$ and $(\lambda'_{\mathrm{a}}, \lambda'_{\mathrm{s}})$ which give Lebesgue's decomosition of λ with respect to μ, i.e. pairs of measures satisfying all the properties listed in (6.1). Then $\lambda'_{\mathrm{a}} - \lambda_{\mathrm{a}} = \lambda_{\mathrm{s}} - \lambda'_{\mathrm{s}}$ and both sides represent a measure which is simultaneously absolutely continuous and singular with respect to μ. From this it follows easily that they are both the zero measure, that is, $\lambda_{\mathrm{a}} = \lambda'_{\mathrm{a}}$ and $\lambda_{\mathrm{s}} = \lambda'_{\mathrm{s}}$.

From now on, we assume that μ and λ are positive finite measures.

Define $\varphi = \lambda + \mu$. An elementary verification, by the definition of Lebesgue integral, shows that $\int_X f \,\mathrm{d}\varphi = \int_X f \,\mathrm{d}\lambda + \int_X f \,\mathrm{d}\mu$ for every $f \in L_1(\varphi)$. Hence, by the standard Cauchy–Schwarz inequality for integrals,

$$\Big|\int_X f \,\mathrm{d}\lambda\Big| \leq \int_X |f| \,\mathrm{d}\lambda \leq \int_X |f| \,\mathrm{d}\varphi \leq (\varphi(X))^{1/2}\Big(\int_X |f|^2 \,\mathrm{d}\varphi\Big)^{1/2}.$$

This estimate shows that the map

$$L_1(\varphi) \ni f \longmapsto \int_X f \,\mathrm{d}\lambda$$

is a continuous linear functional (with norm at most $(\varphi(X))^{1/2} < \infty$), that is, an element of $(L_2(\varphi))^*$. By the Riesz representation theorem (Theorem 5.7), there exists a function $g \in L_2(\varphi)$ such that

$$\int_X f \,\mathrm{d}\lambda = \int_X fg \,\mathrm{d}\varphi \quad \text{for every } f \in L_2(\varphi). \tag{6.3}$$

(Observe that we essentially used the fact that $L_2(\varphi)$ is complete, and hence a Hilbert space, which is the content of Theorem 1.12.)

Let $E \in \mathfrak{M}$, $\varphi(E) > 0$ and put $f = \mathbb{1}_E$ in equation (6.3). Since $0 \leq \lambda \leq \varphi$, we obtain

$$0 \leq \frac{1}{\varphi(E)} \int_E g \,\mathrm{d}\varphi \leq 1.$$

As it holds true for every set E of positive measure, we have $0 \leq g(x) \leq 1$ a.e. Here, we have just used the following simple observation: If ν is any positive measure, $f \in L_1(\nu)$ and $S \subseteq \mathbb{C}$ is a closed set such that the integral mean $(\nu(E))^{-1} \int_X f \,\mathrm{d}\nu \in S$ for every measurable set E with $\nu(E) > 0$, then $f(x) \in S$ for a.e. To see this, note that $\mathbb{C} \setminus S$ is the union of countably many open balls disjoint from S. Hence, it is enough to prove that for every open ball $D \subset \mathbb{C} \setminus S$ we have $\nu(f^{-1}(D)) = 0$. Let D be centered at x_0 and of radius $r > 0$, and suppose that $E \coloneqq f^{-1}(D)$ is of positive measure. Then

$$\Big|\frac{1}{\nu(E)} \int_E f \,\mathrm{d}\nu - x_0\Big| \leq \frac{1}{\nu(E)} \int_E |f(x) - x_0| \,\mathrm{d}\nu(x) < r$$

which means that the integral mean over E belongs to the ball D, thus it is disjoint from S. This, however, contradicts our assumption. We are going to use the just proved proposition a couple of times in the sequel.

By modifying g on a measure zero set, we can assume that $0 \leq g(x) \leq 1$ holds true for every $x \in X$. Let us rewrite formula (6.3) in the form

$$\int_X (1 - g) f \,\mathrm{d}\lambda = \int_X fg \,\mathrm{d}\mu \quad \text{for every } f \in L_2(\varphi). \tag{6.4}$$

Define

$$A = \{x \in X \colon 0 \leq g(x) < 1\} \quad \text{and} \quad B = \{x \in X \colon g(x) = 1\},$$

and the corresponding restricted measures:

$$\lambda_{\mathrm{a}}(E) = \lambda(A \cap E) \quad \text{and} \quad \lambda_{\mathrm{s}}(E) = \lambda(B \cap E) \qquad (E \in \mathfrak{M}).$$

Putting $f = \mathbb{1}_B$ in equation (6.4) we see that $\mu(B) = 0$ and hence $\lambda_s \perp \mu$. Similarly, if $\mu(E) = 0$, then (6.4) shows that

$$\int_{A\cap E} (1-g)\, \mathrm{d}\lambda = 0$$

and since $1 - g$ is positive on A, we obtain $\lambda_a(E) = 0$. This shows that $\lambda_a \ll \mu$ (which, of course, will also follow once we prove the Radon–Nikodym representation). Obviously, $\lambda = \lambda_a + \lambda_s$ and hence we have proved assertion (a).

Since g is bounded, we can put $f := (1 + g + g^2 + \ldots + g^n)\mathbb{1}_E$ in equation (6.4), with any $n \in \mathbb{N}$ and $E \in \mathfrak{M}$. Thus, we obtain

$$\int_E (1 - g^{n+1})\, \mathrm{d}\lambda = \int_E g(1 + g + g^2 + \ldots + g^n)\, \mathrm{d}\mu.$$

Notice that the left-hand side converges to $\lambda(A \cap E) = \lambda_a(E)$ as $n \to \infty$. On the other hand, the sequence $(g(1 + g + g^2 + \ldots + g^n))_{n=1}^{\infty}$ is pointwise monotone increasing and converges to some nonnegative measurable function h. By Lebesgue's theorem on monotone convergence, the right-hand side converges to $\int_E h\, \mathrm{d}\mu$. Consequently,

$$\lambda_a(E) = \int_E h\, \mathrm{d}\mu. \qquad \square$$

The Radon–Nikodym theorem, which we have just proved, says that if μ is a σ-finite positive measure, then every complex measure λ_a which is absolutely continuous with respect to μ admits a representation (6.2) with some function $h \in L_1(\mu)$, uniquely determined as an element of $L_1(\mu)$. Thus, it makes sense to call this h the *Radon–Nikodym derivative* of λ_a with respect to μ and to use the notation

- $h = \dfrac{\mathrm{d}\lambda_a}{\mathrm{d}\mu}$ if $h \in L_1(\mu)$ represents λ_a by formula (6.2).

Remark. The integral represenation (6.2) of the Radon–Nikodym theorem is still valid under the assumption that λ_a is positive and σ-finite, however, we cannot guarantee that h is then μ-integrable. In this case, h can be claimed to be just 'locally' integrable (see **Problem 5.14**).

The following important corollary is a measure-theoretic analogue of the usual polar decomposition of complex numbers.

Corollary 6.5. *Let μ be a complex measure on a σ-algebra $\mathfrak{M}$ of subsets of X. Then there exists a measurable function h such that $|h(x)| = 1$ for every $x \in X$ and*

$$\mathrm{d}\mu = h\, \mathrm{d}|\mu|. \tag{6.5}$$

Proof. Obviously, $\mu \ll |\mu|$ and hence the Radon–Nikodym theorem produces a function $h \in L_1(|\mu|)$ satisfying (6.5). For any $r > 0$ consider the set $A_r = \{x \in X\colon |h(x)| < r\}$. For any partition $(E_j)_{j=1}^n \in \Pi(A_r)$ we have

$$\sum_{j=1}^n |\mu(E_j)| = \sum_{j=1}^n \Big| \int_{E_j} h\, \mathrm{d}|\mu| \Big| \le \sum_{j=1}^n \int_{E_j} |h|\, \mathrm{d}|\mu| = \int_{A_r} |h|\, \mathrm{d}|\mu| \le r|\mu|(A_r).$$

Passing to supremum we get $|\mu|(A_r) \le r|\mu|(A_r)$. Therefore, $|\mu|(A_r) = 0$ for $r < 1$, which means that $|h(x)| \ge 1$ a.e. on X.

On the other hand, for every $E \in \mathfrak{M}$ with $|\mu|(E) > 0$ we have

$$\Big|\frac{1}{|\mu|(E)}\int_E h\,\mathrm{d}|\mu|\Big| = \frac{|\mu(E)|}{|\mu|(E)} \le 1.$$

Hence, our observation concerning integral means (see the proof of Theorem 6.4) yields that $|h(x)| \le 1$ a.e. on X. By modifying the function h on a measure zero set, we can thus guarantee that $|h(x)| = 1$ for every $x \in X$. □

Now, we present another important application of the Radon–Nikodym theorem. Namely, we extend the duality between $L_p(\mu)$- and $L_q(\mu)$-spaces to arbitrary positive (σ-finite for $p = 1$) measures. Recall that we proved it earlier only for the Lebesgue measure on an interval (or, with no essential changes in the proof, on a half line or the whole real line); see Theorem 3.6.

Theorem 6.6. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and $p \in [1, \infty)$. Let $q \in (1, \infty]$ be the conjugate exponent, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ and:*

- *if $p > 1$, then μ can be an arbitrary positive measure;*
- *if $p = 1$ and $q = \infty$ we assume that μ is σ-finite.*

For every functional $\Lambda \in L_p(\mu)^*$ there exists a unique $g \in L_q(\mu)$ such that

$$\Lambda f = \int_X fg\,\mathrm{d}\mu \quad \textit{for every } f \in L_p(\mu) \tag{6.6}$$

and, moreover, $\|\Lambda\| = \|g\|_q$. On the other hand, any $g \in L_q(\mu)$ defines a continuous linear functional on $L_p(\mu)$ via formula (6.6)*. Consequently, the map $\Lambda \mapsto g$ yields an isometric isomorphism*

$$L_p(\mu)^* \cong L_q(\mu).$$

Proof. For now, we deal with the σ-finite case and later we will explain how to reduce the general cases to this one, for $1 < p, q < \infty$.

Fix any $\Lambda \in L_p(\mu)^*$. Note that the uniqueness of g which represents Λ by formula (6.6) is obvious. Also, by Hölder's inequality, we easily get $\|\Lambda\| \le \|g\|_q$. We have to prove the existence of g and that the reverse inequality holds true. For $\Lambda = 0$ we take $g = 0$, so we assume that $\Lambda \ne 0$.

First, assume that $\mu(X) < \infty$. Define $\lambda(E) = \Lambda \mathbb{1}_E$ for $E \in \mathfrak{M}$ and observe it is a σ-additive set function. For, let $E = \bigcup_{n=1}^\infty E_n$, where each $E_n \in \mathfrak{M}$, and define $S_N = \bigcup_{n=1}^N E_n$. Then

$$\|\mathbb{1}_E - \mathbb{1}_{S_N}\|_p = (\mu(E \setminus S_N))^{1/p} \xrightarrow[n\to\infty]{} 0$$

and by the continuity of Λ, we have $\lambda(S_N) \to \lambda(E)$ (notice that it is not true for $p = \infty$). Hence, λ is a complex measure on $\mathfrak{M}$.

Plainly, $\mu(E) = 0$ implies that $\lambda(E) = 0$, that is, $\lambda \ll \mu$. By the Radon–Nikodym theorem, there exists $g \in L_1(\mu)$ such that

$$\Lambda \mathbb{1}_E = \int_E g\,\mathrm{d}\mu = \int_X \mathbb{1}_E \cdot g\,\mathrm{d}\mu \quad \text{for every } E \in \mathfrak{M}.$$

By linearity, the above formula holds true for any simple function in the place of $\mathbb{1}_E$. Since every bounded measurable function is the pointwise limit of a uniformly bounded sequence of simple functions, we also have

$$\Lambda f = \int_X fg \, \mathrm{d}\mu \quad \text{for every } f \in L_\infty(\mu). \tag{6.7}$$

By standard measure-theoretic arguments, simple functions are also dense in $L_p(\mu)$. This can be seen by first considering any nonnegative $f \in L_p(\mu)$, approximating it by simple functions $(s_n)_{n=1}^\infty$ and then using Lebesgue's theorem on dominated convergence to conclude that $\|s_n - f\|_p \to 0$. The case of a general complex-valued function $f \in L_p(\mu)$ follows from then easily. Therefore, we have proved formula (6.6).

It remains to prove that $g \in L_q(\mu)$ and $\|\Lambda\| \geq \|g\|_q$. For $p = 1$, we observe that for each $E \in \mathfrak{M}$ we have

$$\Big| \int_E g \, \mathrm{d}\mu \Big| \leq \|\Lambda\| \cdot \|\mathbb{1}_E\|_1 = \|\Lambda\| \mu(E).$$

Hence, in view of our observation on integral means (see the proof of Theorem 6.4), we have $\|g\|_\infty \leq \|\Lambda\|$.

If $p > 1$, we take a measurable function $\alpha\colon X \to \mathbb{C}$ such that $|\alpha(x)| = 1$ for every $x \in X$ and $\alpha g = |g|$. For any $n \in \mathbb{N}$, define

$$E_n = \{x \in X \colon |g(x)| \leq n\} \quad \text{and} \quad f = \mathbb{1}_{E_n} |g|^{q-1} \alpha.$$

Then $|f|^p = |g|^q$ on E_n, hence $f \in L_\infty(\mu)$ According to (6.7), we have

$$\int_{E_n} |g|^q \, \mathrm{d}\mu = \int_X fg \, \mathrm{d}\mu = \Lambda f \leq \|\Lambda\| \cdot \Big(\int_{E_n} |g|^q \, \mathrm{d}\mu \Big)^{1/p}.$$

It follows that

$$\int_X \mathbb{1}_{E_n} |g|^q \, \mathrm{d}\mu \leq \|\Lambda\|^q$$

which means that $\|g\|_q \leq \|\Lambda\|$, as n was arbitrary. `TBC`