Functional analysis

Lecture 11: Complex version of the Riesz–Markov–Kakutani theorem; The Baire category theorem; the uniform boundedness principle

Proof of Theorem 6.6 (cont.) Now, we assume that $\mu(X)=\infty$ and μ is σ-finite. Let $(X_n)_{n=1}^\infty$ be a sequence of pairwise disjoint measurable sets such that $X=\bigcup_{n=1}^\infty X_n$ and $0<\mu(X_n)<\infty$ for $n\in\mathbb{N}$. Define a measurable map $h\colon X\to(0,\infty)$ by

$$h(x)=\frac{1}{n^2\mu(X_n)}\quad\text{if } x\in X_n\ (n\in\mathbb{N}).$$

Then $h\in L_1(\mu)$ and the formula $\widetilde{\mu}(E)=\int_E h\,\mathrm{d}\mu$ defines a finite measure on $\mathfrak{M}$. Moreover, the map $f\mapsto h^{1/p}f$ is a linear isometry from $L_p(\widetilde{\mu})$ onto $L_p(\mu)$.

Thus, given any $\Lambda\in L_p(\mu)^*$, we can define $\Psi\in L_p(\widetilde{\mu})^*$ by $\Psi(f)=\Lambda(h^{1/p}f)$. From the first part of the proof we infer that there exists $G\in L_q(\widetilde{\mu})$ such that

$$\Psi(f)=\int_X fG\,\mathrm{d}\widetilde{\mu}\quad\text{for every } f\in L_p(\widetilde{\mu}).$$

Define $g=h^{1/q}G$ (for $p=1$ we take $g=G$) and observe that

$$\int_X |g|^q\,\mathrm{d}\mu=\int_X |G|^q\,\mathrm{d}\widetilde{\mu}=\|\Psi\|^q=\|\Lambda\|^q$$

if $p>1$, whereas for $p=1$ we have $\|g\|_\infty=\|G\|_\infty=\|\Psi\|=\|\Lambda\|$. Hence, we have proved that $\|\Lambda\|=\|g\|_q$ and, finally, for every $f\in L_p(\mu)$ we have

$$\begin{aligned}\Lambda f=\Psi(h^{-1/p}f)&=\int_X h^{-1/p}fG\,\mathrm{d}\widetilde{\mu}\\&=\int_X h^{-1/p-1/q}fh^{1/q}G\,\mathrm{d}\widetilde{\mu}=\int_X h^{-1}fg\,\mathrm{d}\widetilde{\mu}=\int_X fg\,\mathrm{d}\mu.\end{aligned}$$ □

Remark. For $p=2$, the space $L_p(\mu)$ is a Hilbert space, so the above duality result, for an arbitrary positive measure μ, we already knew from the Riesz representation theorem (Theorem 5.7). In fact, we used this particular case $L_2(\mu)^*\cong L_2(\mu)$ in the above proof, because it was an essential part of the Radon–Nikodym theorem.

Remark 6.7. If $1<p<\infty$ and $\frac1p+\frac1q=1$, then the duality $L_p(\mu)^*\cong L_q(\mu)$ described in Theorem 6.6 holds true for an arbitrary positive measure μ, not necessarily σ-finite (see **Problem 5.23**). However, for $p=1$ we have the following example.

Example 6.8. Define $X=\{a,b\}$ and a measure μ on X by $\mu\{a\}=1$ and $\mu\{b\}=\mu(X)=\infty$. Then, obviously, $L_1(\mu)^*\ncong L_\infty(\mu)$ as the first space is one-dimensional, while the second one is two-dimensional.

Now, our goal is to complete the proof of the Riesz–Markov–Kakutani theorem (Theorem 3.23) in the complex case. It will be another application of the Radon–Nikodym theorem. First, we need the following simple lemma.

Lemma 6.9. *Let $(X,\mathfrak{M},\mu)$ be a measure space with a positive measure μ. Suppose that $g\in L_1(\mu)$ and define a complex measure λ on $\mathfrak{M}$ by the formula $\lambda(E)=\int_E g\,\mathrm{d}\mu$. Then $|\lambda|(E)=\int_E|g|\,\mathrm{d}\mu$ for every $E\in\mathfrak{M}$.*

Proof. By Corollary 6.5 (the polar decomposition of measures), there exists a function $h \in L_1(\mu)$ such that $|h(x)| = 1$ for every $x \in X$ and $\mathrm{d}\lambda = h\,\mathrm{d}|\lambda|$. Hence, $h\,\mathrm{d}|\lambda| = g\,\mathrm{d}\mu$ and, by an easy verification, we have $\mathrm{d}|\lambda| = \overline{h}g\,\mathrm{d}\mu$. Since $|\lambda|$ and μ are positive measures, we have $\overline{h}g \geq 0$ a.e. which means that $\overline{h}g = |g|$ a.e. Thus, $\mathrm{d}|\lambda| = |g|\,\mathrm{d}\mu$ as desired. □

Theorem 6.10 (Riesz–Markov–Kakutani for $C_0(X)^*$ over complex numbers). *Let X be a locally compact Hausdorff space and $\Lambda \in C_0(X)^*$ be a continuous linear functional on the complex Banach space of complex continuous functions on X vanishing at infinity. Then, there exists a unique regular Borel complex measure μ on X such that*

$$\Lambda f = \int_X f\,\mathrm{d}\mu \quad \text{for every } f \in C_0(X). \tag{6.1}$$

Moreover, we have $\|\Lambda\| = |\mu|(X)$. On the other hand, every $\mu \in \mathcal{M}(X)$ gives rise to an element Λ of $C_0(X)^*$ via formula (6.1). Consequently, the map $\Lambda \mapsto \mu$ is an isometric isomorphism

$$C_0(X)^* \cong \mathcal{M}(X).$$

(The symbol $\mathcal{M}(X)$ stands here for the space of all regular Borel complex measures on X, equipped with the total variation norm.)

Proof. To prove the uniqueness, assume that $\mu \in \mathcal{M}(X)$ satisfies $\int_X f\,\mathrm{d}\mu = 0$ for every $f \in C_0(X)$. In view of Corollary 6.5, there is $h \in L_1(\mu)$ such that $|h| = 1$ and $\mathrm{d}\mu = h\,\mathrm{d}|mu|$. For any sequence $(f_n)_{n=1}^\infty \subset C_0(X)$ we thus have

$$|\mu|(X) = \int_X (\overline{h} - f_n)h\,\mathrm{d}|\mu| \leq \int_X |\overline{h} - f_n|\,\mathrm{d}|\mu|. \tag{6.2}$$

Now, observe that $C_c(X)$ is dense in $L_1(|\mu|)$. Indeed, by Lusin's theorem for any measurable complex function s on X and any $\varepsilon > 0$ there exists a function $g \in C_c(X)$ such that

$$|\mu|\big(\{x \in X \colon g(x) \neq s(x)\big) < \varepsilon \quad \text{ and } \quad \|g\|_\infty \leq \|s\|_\infty.$$

Hence, $\|g - s\|_1 \leq 2\varepsilon\|s\|_\infty$ and since every function from $L_1(\mu)$ can be approximated by a sequence of simple functions, we obtain the announced claim. Therefore, we can arrange the sequence $(f_n)_{n=1}^\infty$ so that the right-hand side in (6.2) converges to zero. Thus, $|\mu|(X) = 0$ which shows the uniqueness part. (Notice that we have silently used the fact that $\mathcal{M}(X)$ is a linear space, in particular, that the difference of two regular Borel complex measures is also regular.)

Fix any $\Lambda \in C_0(X)^*$ with $\|\Lambda\| = 1$.

Claim. There exists a positive linear functional Ψ on $C_0(X)$ such that

$$|\Lambda f| \leq \Psi(|f|) \leq \|f\|_\infty \quad \text{for every } f \in C_0(X). \tag{6.3}$$

Indeed, consider Λ restricted to the real part of $C_0(X)$, that is, real-valued continuous functions vanishing at infinity. By Lemma 3.18, $C_0(X)^*$ is a Banach lattice, hence it makes sense to consider the modulus of Λ, $|\Lambda| = \Lambda \vee (-\Lambda)$ which is defined according to the lattice operations on $C_0(X)^*$ (see the proof of Lemma 3.18 and the remark following it). Hence, on the real part of $C_0(X)$ we define

$$\Psi = |\Lambda|, \quad \text{that is,} \quad \Psi f = \sup\big\{|\Lambda g| \colon g \in C_0(X),\, g(X) \subseteq \mathbb{R},\, |g| \leq f\big\}.$$

By basic properties of Banach lattices, the norm of Ψ is the same as that of Λ, which means that inequality (6.3) holds true for each real function $f \in C_0(X)$. Now, if $f \in C_0(X)$ is complex-valued, we define $\Psi f = \Psi(\mathrm{Re}\, f) + \mathrm{i}\Psi(\mathrm{Im}\, f)$. Linearity and positivity of Ψ is obvious, as well as the fact that inequality (6.3) holds true for every complex-valued function $f \in C_0(X)$.

Having established the Claim, we apply the 'positive version' of the Riesz–Markov–Kakutani theorem (Theorem 3.16). There exists a positive Borel measure λ on X such that $\Psi f = \int_X f\, \mathrm{d}\lambda$. Moreover, since $|\Psi f| \le 1$ for $\|f\|_\infty \le 1$, we have

$$\lambda(X) = \sup\{\Psi f\colon f \in C_c(X),\ 0 \le f \le 1\} \le 1.$$

Therefore, λ being a finite measure, is regular (see assertions (b)–(d) of Theorem 3.16).

Regarding any $f \in C_c(X)$ as an element of $L_1(\lambda)$, we have

$$|\Lambda f| \le \Psi(|f|) = \int_X |f|\, \mathrm{d}\lambda = \|f\|_1.$$

This means that Λ is of norm at most 1 on the dense linear subspace $L_1(\lambda) \cap C_c(X)$ of $L_1(\lambda)$. Hence, we can extend it to a linear functional of norm at most 1 on the whole of $L_1(\lambda)$. By virtue of Theorem 6.6, there exists a Borel function $g\colon X \to \mathbb{C}$ with essential supremum at most 1 (thus we can assume that $|g| \le 1$ on X) and such that

$$\Lambda f = \int_X fg\, \mathrm{d}\lambda \quad \text{for every } f \in C_c(X). \tag{6.4}$$

Note that both sides of this equation express a continuous linear functional on $C_0(X)$. Since $C_c(X)$ is dense $C_0(X)$, we obtain the desired representation with

$$\mathrm{d}\mu = g\, \mathrm{d}\lambda.$$

Notice that since $\|\Lambda\| = 1$, equation (6.4) yields

$$\int_X |g|\, \mathrm{d}\lambda \ge \sup\big\{|\Lambda f|\colon f \in C_0(X),\ \|f\|_\infty \le 1\big\} = 1.$$

But since also $\lambda(X) \le 1$ and $|g| \le 1$, the preceding inequality is only possible when $\lambda(X) = 1$ and $|g| = 1$ a.e. In view of Lemma 6.9, we have $\mathrm{d}|\lambda| = |g|\, \mathrm{d}\lambda = \mathrm{d}\lambda$ and hence

$$|\mu|(X) = \lambda(X) = 1 = \|\Lambda\|. \qquad \square$$

7 Three fundamental results based on a category argument

In this section, we present the 'big three'—three classical Banach space theoretic results, inherently connected with completeness. These are: the uniform boundedness principle, the open mapping theorem and the closed graph theorem. The first one was published by S. Banach and H. Steinhaus in *Fundamenta Mathematicae* (1927) and, together with the remaining two results, appeared in Banach's fundamental monograph "*Théorie des opérations linéaires*" published in Warsaw, 1932.

At the core of all these three results is applying a category argument based on the fact that every complete metric space X is of second category in itself. In other words, a countable union of nowhere dense set cannot exhaust the whole of X. This is the content of the following fundamental theorem.

Theorem 7.1 (**Baire category theorem**). *Let (X, ρ) be a complete metric space. If $(V_n)_{n=1}^{\infty}$ is a sequence of open dense subsets of X, then the intersection $\bigcap_{n=1}^{\infty} V_n$ is dense in X.*

Proof. Fix any nonempty open set $U \subseteq X$. We shall show that $\bigcap_{n=1}^{\infty} V_n$ contains at least one point from U. We denote by $D(x, r)$ the open ball centered at x and with radius r, i.e. the set of those points $z \in X$ for which $\rho(x, z) < r$. By $\overline{D}(x, r)$ we denote the closure of $D(x, r)$. (Nota that it can happen that $\overline{D}(x, r) \neq \{x \in X : \rho(x, z) \leq r\}$.)

By induction, we construct a sequence of points $(x_n)_{n=1}^{\infty} \subset X$ and positive numbers $(r_n)_{n=1}^{\infty}$ such that:

(i) $\overline{D}(x_1, r_1) \subseteq U \cap V_1$,

(ii) $\overline{D}(x_n, r_n) \subseteq V_n \cap D(x_{n-1}, r_{n-1})$ for each $n \in \mathbb{N}$, $n \geq 2$,

(iii) $r_n < \frac{1}{n}$ for each $n \in \mathbb{N}$.

First, since V_1 is dense and open, we find $x_1 \in X_1$ and $r_1 \in (0, 1)$ so that condition (i) is satisfied. If $n \geq 2$ and $x_1, \ldots, x_{n-1}$ were already chosen, we note that $V_n \cap D(x_{n-1}, r_{n-1}) \neq \varnothing$ is open, hence there exist $x_n \in X$ and $r_n \in (0, \frac{1}{n})$ such that condition (ii) is valid.

Observe that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Indeed, for any $m, n, N \in \mathbb{N}$ with $m, n > N$ we have $x_m, n_n \in D(x_N, r_N)$, thus $\rho(x_m, x_n) < 2r_N < 2/N$. Let $x = \lim_{n \to infty} x_n$. Then $x \in \bigcap_{n=1}^{\infty} \overline{D}(x_n, r_n)$ because for each $n \in \mathbb{N}$ we have $x_j \in \overline{D}(x_n, r_n)$ whenever $j > n$ and all these sets are closed. Hence $x \in \bigcap_{n=1}^{\infty} V_n$ and also $x \in U$ which follows, respectively, from (ii) and (i). □

Remark. Another way of stating Baire's theorem is that a countable union of nowhere dense sets in a complete metric space X must have nonempty complement. For, let $A_n \subset X$ be nowhere dense ($n \in \mathbb{N}$). Then $\operatorname{int} \overline{A_n} = \varnothing$ which means that the complement $X \setminus \overline{A_n}$ is an open dense subset of X. By Theorem 7.1, we have $\bigcap_{n=1}^{\infty} (X \setminus \overline{A_n}) \neq \varnothing$ (it is even dense, but quite typically we use Baire's theorem just to derive nonemptiness). Hence, $\bigcup_{n=1}^{\infty} A_n \subsetneq X$.

A subset of X which is a countable union of nowhere dense sets is called a set *of first category* (in X). All other sets are called *of second category* (in X). Therefore, Baire's category theorem says that every complete metric space if of second category in itself.

The next result is usually called the *uniform boundedness principle*, for an obvious reason: it implies that every collection of bounded linear operators on a Banach space which is pointwise bounded must be actually uniformly bounded, that is, bounded in norm.

Theorem 7.2 (**Banach–Steinhaus theorem**). *Let X be a Banach space and Y be a normed space. Let also $\{T_\alpha : \alpha \in A\} \subset \mathscr{L}(X, Y)$ be a collection of operators which is not uniformly bounded, that is,*

$$\sup_{\alpha \in A} \|T_\alpha\| = \infty.$$

Then, there exists a dense G_δ-subset B of X such that

$$\sup_{\alpha \in A} \|T_\alpha x\| = \infty \quad \textit{for every } x \in B.$$

Proof. Define a function $\varphi\colon X \to [0,\infty]$ by $\varphi(x) = \sup_{\alpha\in A} \|T_\alpha x\|$ and sets

$$V_n = \{x \in X \colon \varphi(x) > n\} \quad (n \in \mathbb{N}).$$

Since all the functions $X \ni x \mapsto \|T_\alpha x\|$ are continuous, it is easily seen that each V_n is open (in fact, φ is lower semicontinuous as the supremum of continuous functions). Consider two cases.

Case 1. V_N is not dense in X, for some $N \in \mathbb{N}$.

Then, the complement of V_N contains an open ball, so there exist $x_0 \in X$ and $r > 0$ such that $x_0 + x \notin V_N$ whenever $\|x\| \le r$. For such x's we have $\varphi(x_0 + x) \le N$, hence

$$\|T_\alpha x\| \le \|T_\alpha x_0\| + \|T_\alpha(x_0 + x)\| \le 2N \quad \text{for each } \alpha \in A.$$

It follows that for $\|x\| \le 1$ we have $\|T_\alpha x\| \le 2N/r$, i.e. $\|T_\alpha\| \le 2N/r$ which contradicts the assumption that $\{T_\alpha \colon \alpha \in A\}$ is not uniformly bounded.

Case 2. V_n is dense in X, for every $n \in \mathbb{N}$.

Then, by the Baire category theorem, $B := \bigcap_{n=1}^\infty V_n$ is a dense G_δ-subset of X. Plainly, $\sup_{\alpha\in A} \|T_\alpha x\| = \infty$ for every $x \in B$. □