Functional analysis

Lecture 13: Fejér's theorem on Cesàro summability of Fourier series; Dirichlet's and Fejér's kernels; an application of the Banach–Steinhaus theorem to the problem of pointwise convergence of Fourier series

Now, our goal is to show that the orthonormal trigonometric system $(u_n)_{n\in\mathbb{Z}}$ is linearly dense in the Hilbert space $L_2[0,2\pi]$, i.e. it is an orthonormal basis. First, let us reduce this problem to the problem of density of trigonometric polynomials with respect to the supremum norm in $C(\mathbb{T})$.

Remark 8.2. Recall that $C(\mathbb{T})$ is dense in $L_2(\mathbb{T})$ (identified with $L_2[0,2\pi]$), which follows easily from Lusin's theorem (see the proof of Theorem 6.10). We claim that once we prove that trigonometric polynomials form a dense subspace of $C(T)$ under the supremum norm, it follows that they form a dense subspace of $L_2(\mathbb{T})$ under the L_2-norm. Indeed, fix any $f\in L_2(\mathbb{T})$ and $\varepsilon>0$ and pick $g\in C(\mathbb{T})$ such that $\|f-g\|_2<\varepsilon$. Let also P be a trigonometric polynomial satisfying $\|f-P\|_\infty<\varepsilon$ (if it exists). Then

$$\|f-P\|_2\leq\|f-g\|_2+\|P-g\|_2<\varepsilon+\Big\{\frac{1}{2\pi}\int_0^{2\pi}|P(t)-g(t)|^2\,\mathrm{d}t\Big\}^{1/2}\leq\varepsilon+\|P-g\|_\infty<2\varepsilon.$$

Therefore, we have reduced our problem to the question whether trigonometric polynomials are dense in $C(\mathbb{T})$ under the supremum norm, i.e. if every continuous function on $\mathbb{T}$ can be uniformly approximated by trigonometric polynomials. This can be proved by appealing to the abstract Stone–Weierstrass theorem. The required assumptions are satisfied: $\mathbb{T}$ is compact and trigonometric polynomials form a self-adjoint (i.e. closed under complex conjugation) subalgebra $\mathcal{P}$ of $C(\mathbb{T})$ which separates points of $\mathbb{T}$ and contains all constant functions. Hence, $\mathcal{P}$ is dense in $C(T)$ which is what we want to prove. However, our intention is to give a direct proof, not appealing to the Stone–Weierstrass theorem and, moreover, provide a concrete description of trigonometric polynomials approximating any given function $f\in C(\mathbb{T})$.

For any $f\in L_2(\mathbb{T})$ and $N=0,1,2,\dots$ we use the following notation:

- $s_N(f;x)$ the N^{th} symmetric partial sum of the Fourier series corresponding to f at a point $x\in\mathbb{R}$,
- $\sigma_N(f;x)$ the arithmetic mean of the first $N+1$ partial sums $s_j(f;x)$.

More precisely,

$$s_N(f;x)=\sum_{n=-N}^{N}\widehat{f}(n)e^{\mathrm{i}nx}=\frac{1}{2\pi}\sum_{n=-N}^{N}\int_0^{2\pi}f(t)e^{\mathrm{i}n(x-t)}\,\mathrm{d}t$$

and

$$\sigma_N(f;x)=\frac{s_0(f;x)+s_1(f;x)+\ldots+s_N(f;x)}{N+1}.$$

Despite of the fact that the Fourier series of a continuous functions $f\in C(\mathbb{T})$ does not in general converge pointwise to f on $\mathbb{T}$ (as we will see later), it is *Cesàro summable* to f, i.e. the sequence of arithmetic means of partial sums converge to f on $\mathbb{T}$, and the convergence is also uniform.

Theorem 8.3 (Fejér's theorem). *For every $f \in C(\mathbb{T})$ we have $\sigma_N(f;x) \rightrightarrows f(x)$ on $\mathbb{R}$. In other words, $\|\sigma_N(f) - f\|_\infty \to 0$ as $N \to \infty$.*

Before proving Fejér theorem, we need some preparations. We introduce two important *kernels*, that is sequences of continuous 2π-periodic functions on $\mathbb{R}$. The first one corresponds to the sequence $(s_n)_{n=0}^\infty$ of partial sums of a Fourier series, whereas the second one corresponds to the sequence of arithmetic means $(\sigma_n)_{n=0}^\infty$.

Definition 8.4. The sequence $(D_N)_{N=0}^\infty$ of trigonometric polynomials defined by the formula

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = 1 + 2\sum_{n=1}^{N} \cos nx$$

is called the *Dirichlet kernel*.

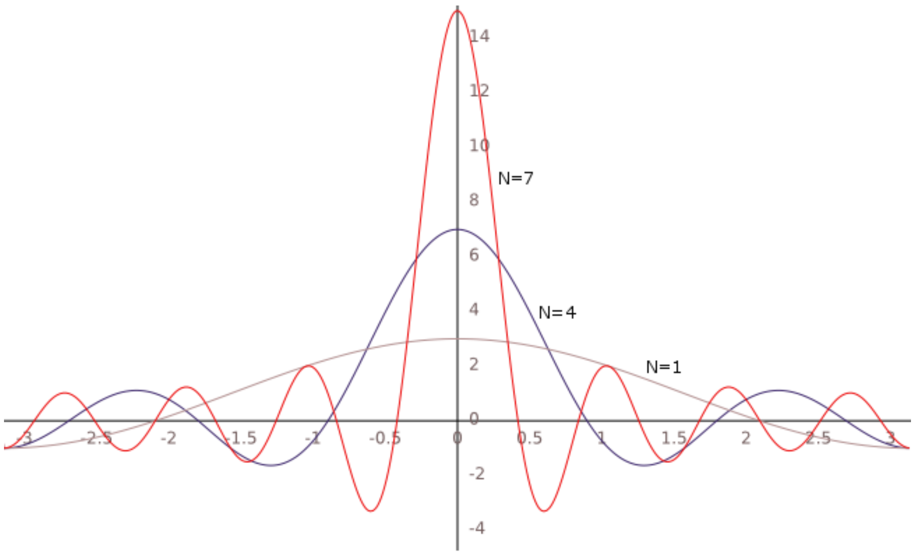

Fig. 1. The graphs of $D_N(x)$ for $N = 1, 4, 7$ on the interval $[-\pi, \pi]$

Lemma 8.5. *For every $N = 0, 1, 2, \ldots$ the Dirichlet function $D_N(x)$ has the following properties:*

(a) *D_N is even and 2π-periodic,*

(b) *$D_N(0) = 2N + 1$,*

(c) *$|D_N(x)| \le 2N + 1$ for every $x \in \mathbb{R}$,*

(d) $\displaystyle \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,\mathrm{d}x = 1$,

(e) $\displaystyle D_N(x) = \frac{\sin\left(N + \frac{1}{2}\right)x}{\sin\frac{x}{2}}$ *for every $x \in \mathbb{R}$, $x \neq 2k\pi$ $(k \in \mathbb{Z})$.*

Proof. Assertions (a)–(d) follow easily just from the definition. To derive the compact formula (e) for $D_N(x)$ we use a simple telescoping sum trick:

$$D_N(x) = 1 + 2\sum_{n=1}^{N} \cos nx = \sum_{n=-N}^{N} \cos nx + \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} \sum_{n=-N}^{N} \sin nx$$
$$= \frac{1}{\sin\frac{x}{2}} \sum_{n=-N}^{N} \Big(\sin\frac{x}{2}\cos nx + \cos\frac{x}{2}\sin nx\Big)$$
$$= \frac{1}{\sin\frac{x}{2}} \sum_{n=-N}^{N} \sin\Big(k + \frac{1}{2}\Big)x = \frac{\sin\big(N+\frac{1}{2}\big)x}{\sin\frac{x}{2}}. \qquad \square$$

Definition 8.6. The sequence $(K_N)_{N=0}^{\infty}$ of trigonometric polynomials defined by the formula

$$K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x)$$

is called the *Fejér kernel.*

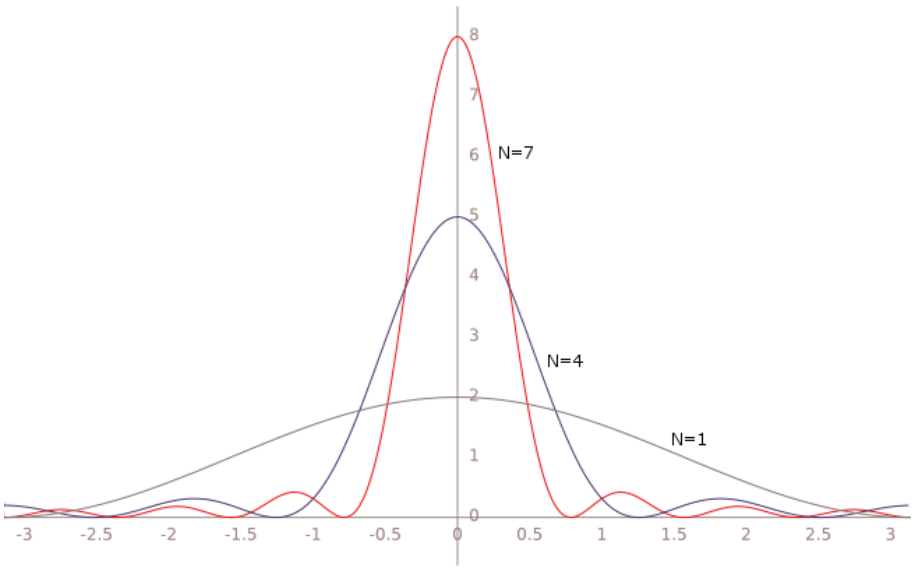

Fig. 2. The graphs of $K_N(x)$ for $N = 1, 4, 7$ on the interval $[-\pi, \pi]$

The crucial property of the Fejér kernel is that it is always nonnegative, as it is suggested by the above picture. This is by no means clear from the definition, as $K_N(x)$ is just an average of some sign-changing functions.

Lemma 8.7. *For every $N = 0, 1, 2, \ldots$ the Fejér function $K_N(x)$ has the following properties:*

(a) *K_N is even and 2π-periodic,*

(b) *$K_N(0) = N + 1$,*

(c) $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,\mathrm{d}x = 1$,

(d) *$K_N(x) \geq 0$ for every $x \in \mathbb{R}$,*

(e) $K_N(x) \leq \dfrac{1}{N+1} \cdot \dfrac{2}{1-\cos\delta}$ *for $0 < \delta \leq |x| \leq \pi$,*

(f) *for every* $x \in \mathbb{R}$, $x \neq 2k\pi$ $(k \in \mathbb{Z})$,

$$K_N(x) = \frac{1}{N+1}\left(\frac{\sin\frac{N+1}{2}x}{\sin\frac{x}{2}}\right)^2 = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x}.$$

Proof. Assertions (a)–(c) are obvious, while (d) and (e) follow easily from formulas in assertion (f). To prove them, we use the identity $2\sin\alpha\sin\beta = \cos(\alpha-\beta)-\cos(\alpha+\beta)$. In view of Lemma 8.5(e) we obtain

$$\begin{aligned}
K_N(x) &= \frac{1}{N+1}\sum_{n=0}^{N}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}} = \frac{1}{(N+1)\sin^2\frac{x}{2}}\sum_{n=0}^{N}\sin\frac{x}{2}\sin\left(n+\frac{1}{2}\right)x \\
&= \frac{1}{2(N+1)\sin^2\frac{x}{2}}\sum_{n=0}^{N}\big(\cos nx-\cos(n+1)x\big) \\
&= \frac{1-\cos(N+1)x}{2(N+1)\sin^2\frac{x}{2}} = \frac{1}{N+1}\left(\frac{\sin\frac{N+1}{2}x}{\sin\frac{x}{2}}\right)^2. \qquad \square
\end{aligned}$$

Note that assertion (e) above implies that $K_N(x) \rightrightarrows 0$ on $[-\pi,-\delta]\cup[\delta,\pi]$, for every $\delta>0$.

Proof of Theorem 8.3. Fix any $f \in C(\mathbb{T})$. We start with an important observation that the Fourier partial sums $s_N(f)$ are given by a convolution with the Dirichlet kernel. Namely,

$$\begin{aligned}
s_N(f;x) &= \sum_{n=-N}^{N}\widehat{f}(n)e^{\mathrm{i}nx} = \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{-\pi}^{\pi}f(t)e^{\mathrm{i}n(x-t)}\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-s)\sum_{n=-N}^{N}e^{\mathrm{i}ns}\,\mathrm{d}s \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-s)D_N(s)\,\mathrm{d}s \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)D_N(x-t)\,\mathrm{d}t.
\end{aligned} \tag{8.1}$$

Similarly, for the artithmetic means $\sigma_N(f)$ we have

$$\begin{aligned}
\sigma_N(f;x) &= \frac{1}{N+1}\sum_{n=0}^{N}s_n(f;x) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-s)K_N(s)\,\mathrm{d}s \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)K_N(x-t)\,\mathrm{d}t.
\end{aligned} \tag{8.2}$$

Let $M = \|f\|_\infty < \infty$. Fix any $\varepsilon>0$ and use the uniform continuity of f to pick $\delta>0$ so that $|f(x-t)-f(x)|<\varepsilon/3$ whenever $|t|\leq\delta$. In view of Lemma 8.7(e) there is $N\in\mathbb{N}$ so large that for each $n\geq N$ we have

$$\int_{-\pi}^{\delta}K_n(t)\,\mathrm{d}t\,,\ \int_{\delta}^{\pi}K_n(t)\,\mathrm{d}t < \frac{\pi\varepsilon}{3M}.$$

Then, for $n \geq N$, using successively formula (8.2) and Lemma 8.7(c), (d), we obtain

$$\begin{aligned}
|\sigma_n(f;x) - f(x)| &= \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} \big(f(x-t) - f(x)\big) K_n(t)\,\mathrm{d}t\Big| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)| K_n(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\Big(\int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi}\Big) \\
&\leq \frac{M}{\pi}\int_{-\pi}^{-\delta} K_n(t)\,\mathrm{d}t + \frac{1}{2\pi}\cdot\frac{\varepsilon}{3}\int_{-\delta}^{\delta} K_n(t)\,\mathrm{d}t + \frac{M}{\pi}\int_{\delta}^{\pi} K_n(t)\,\mathrm{d}t \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

which shows that $\sigma_n(f;x) \rightrightarrows f(x)$. □

Obviously, all averages $\sigma_N(f)$ are trigonometric polynomials. Hence, according to Remark 8.2 we can conclude from Fejér's theorem that the trigonometric system $(u_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of $L_2[-\pi,\pi]$ (or $L_2[0,2\pi]$). The general theory of Hilbert spaces (see Proposition 5.11) thus says that every function $f \in L_2[-\pi,\pi]$ can be expressed as its Fourier series, where convergence in understood in the L_2-norm. The issue of pointwise convergence is however a bit more subtle and without additional information on regularity of f at a given point (see, for example, **Problem 6.13**) one cannot state that $f(x)$ is the limit of $(s_n(f;x))_{n=1}^{\infty}$. As we shall see below, it is even not true that continuous functions on $\mathbb{T}$ are pointwise limits of their Fourier series. The first counterexample was given by P. du Bois-Reymond in 1873 who was able to construct, for any given point $x_0 \in \mathbb{R}$, a concrete function $f \in C(\mathbb{T})$ whose Fourier series diverges at x_0. In fact, a simple application of the Banach–Steinhaus theorem shows that pointwise convergence of Fourier series of continuous functions fails quite drastically.

Theorem 8.8. *For every $x_0 \in \mathbb{R}$ there exists a dense G_δ subset $E_{x_0} \subset C(\mathbb{T})$ such that for every function $f \in E_{x_0}$ the Fourier series $(s_n(f;x_0))_{n=1}^{\infty}$ is divergent.*

Lemma 8.9. *For every $n \in \mathbb{N}$ we have*

$$\|D_n\|_1 > \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k} \xrightarrow[n\to\infty]{} \infty,$$

where $\|\cdot\|_1$ stands for the L_1-norm on $[-\pi,\pi]$ with respect to the normalized Lebesgue measure $(2\pi)^{-1}\,\mathrm{d}x$.

Proof. Using the inequality $\left|\sin\frac{x}{2}\right| \leq \left|\frac{x}{2}\right|$ and Lemma 8.5(e), we obtain

$$\begin{aligned}
\|D_n\|_1 &= \frac{1}{\pi}\int_0^{\pi}\Big|\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\Big|\,\mathrm{d}x \geq \frac{2}{\pi}\int_0^{\pi}\Big|\sin\Big(n+\frac{1}{2}\Big)x\Big|\,\frac{\mathrm{d}x}{x} \\
&= \frac{2}{\pi}\int_0^{(n+\frac{1}{2})\pi}|\sin t|\,\frac{\mathrm{d}t}{t} > \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi}|\sin t|\,\mathrm{d}t = \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k}.
\end{aligned}$$

□

Proof of Theorem 8.8. Fix any $x_0 \in \mathbb{R}$ and define

$$\Lambda_n f = s_n(f; x_0) \quad \text{for } f \in C(\mathbb{T}) \text{ and } n \in \mathbb{N}.$$

In view of formula (8.1), we have $\Lambda_n \in C(\mathbb{T})^*$ and $\|\Lambda_n\| \leq \|D_n\|_1$ for each $n \in \mathbb{N}$. In fact, we have $\|\Lambda_n\| = \|D_n\|_1$ which can be seen by considering the function $g(t) = \operatorname{sgn}(D_n(x_0 - t))$ and picking a sequence $(f_j)_{j=1}^\infty \subset C(\mathbb{T})$ such that $-1 \leq f_j \leq 1$ for $j \in \mathbb{N}$ and $f_j \to g$ pointwise. Then, by Lebesgue's theorem,

$$\lim_{j\to\infty} \Lambda_n(f_j) = \lim_{j\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f_j(t) D_n(x_0 - t)\, \mathrm{d}t = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) D_n(x_0 - t)\, \mathrm{d}t = \|D_n\|_1.$$

Hence, Lemma 8.9 shows that $\|\Lambda_n\| \to \infty$ which means that the sequence $(\Lambda_n)_{n=1}^\infty$ is not uniformly bounded. By the Banach–Steinhaus theorem, there exists a dense G_δ set $E_{x_0} \subset C(\mathbb{T})$ such that

$$\sup_n |\Lambda_n f| = \sup_n |s_n(f; x_0)| = \infty \quad \text{for every } f \in E_{x_0}. \qquad \square$$

Now, we will see how the open mapping theorem can quickly answer another important question in the theory of Fourier series. Notice that the formula for Fourier coefficients, which for functions in $L_2(\mathbb{T})$ stems from the general theory of Hilbert spaces, makes perfect sense also for all integrable functions. So, we define

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-\mathrm{i}nt}\, \mathrm{d}t \quad \text{for } f \in L_1(\mathbb{T}). \tag{8.3}$$

(Note that since the measure of $\mathbb{T}$ is finite, we have $L_2(\mathbb{T}) \subset L_1(\mathbb{T})$.)

The aforementioned question is motivated by the following important result.

Lemma 8.10 (**Riemann–Lebesgue lemma**). *For every $f \in L_1(\mathbb{T})$ we have*

$$\lim_{|n|\to\infty} \widehat{f}(n) = 0. \tag{8.4}$$

In other words,

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t) \cos nt\, \mathrm{d}t = \lim_{n\to\infty} \int_{-\pi}^{\pi} f(t) \sin nt\, \mathrm{d}t = 0.$$

Proof. As we have already noted, $C(\mathbb{T})$ is dense in $L_1(\mathbb{T})$ and hence Fejér's theorem implies that trigonometric polynomials are dense in $L_1(\mathbb{T})$. Observe that formula (8.4) holds true trivially for f being a trigonometric polynomial, as for $|n|$ large enough we simply have $\widehat{f}(n) = 0$.

So, let $f \in C(\mathbb{T})$, $\varepsilon > 0$ and pick a trigonometric polynomial P such that $\|f - P\|_1 < \varepsilon$. For $n \in \mathbb{Z}$ of sufficiently large modulus (larger than the degree of P), we have

$$|\widehat{f}(n)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(t) - P(t))\, \mathrm{d}t \right| \leq \|f - P\|_1 < \varepsilon. \qquad \square$$

Therefore, the Riemann–Lebesgue lemma says that the natural operator $L_1(\mathbb{T}) \ni f \mapsto (\widehat{f}(n))_{n\in\mathbb{Z}}$ takes values in the Banach space $c_0(\mathbb{Z})$. We can thus ask whether an effect similar to the Riesz–Fischer theorem (Theorem 5.14) holds true, that is, whether for every sequence $(\xi_n)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z})$ one can find a function $f \in L_1(\mathbb{T})$ satisfying $\widehat{f}(n) = \xi_n$ for each $n \in \mathbb{Z}$. The following result provides a negative answer.

Theorem 8.11. *The map* $L_1(\mathbb{T}) \ni f \mapsto \Lambda f := (\widehat{f}(n))_{n\in\mathbb{Z}}$ *is an injective bounded linear operator into* $c_0(\mathbb{Z})$*. However,* Λ *is not surjective.*

Proof. Linearity of Λ is obvious. That $\Lambda(L_1(\mathbb{T})) \subseteq c_0(\mathbb{Z})$ follows from Lemma 8.10. Boundedness is also easy, as by formula (8.3) we have $|\widehat{f}(n)| \leq \|f\|_1$ for each $n \in \mathbb{Z}$, thus $\|\Lambda\| \leq 1$ and in fact $\|\Lambda\| = 1$ (consider e.g. the constant 1 function).

For injectivity of Λ, suppose $f \in L_1(\mathbb{T})$ and $\Lambda f = 0$. Then, $\int_{-\pi}^{\pi} f(t)P(t)\,\mathrm{d}t = 0$ for every trigonometric polynomial P. Hence, Lebesgue's theorem and Theorem 8.3 imply that $\int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t = 0$ for every $g \in C(\mathbb{T})$. Using Lusin's theorem, we infer that $\int_A f(t)\,\mathrm{d}t = 0$ for every measurable set $A \subseteq [-\pi, \pi]$ which implies that $f = 0$ a.e.

Finally, notice that Λ is not surjective. If it was, then by the open mapping theorem, there would exist $\delta > 0$ such that $\|\Lambda f\|_\infty \geq \delta\|f\|_1$ for every $f \in L_1(\mathbb{T})$. This is, however, not the case because for $f = D_n$ ($n \in \mathbb{N}$) we have $\Lambda D_n = \mathbb{1}_{\{0,\pm1,\ldots,\pm n\}}$, hence $\|\Lambda D_n\|_\infty = 1$, but $\|D_n\|_1 \to \infty$ by Lemma 8.9. $\square$