Functional analysis

Lecture 14: CONVOLUTION AND FOURIER TRANSFORM; FOURIER INVERSION FORMULA AND THE UNIQUENESS THEOREM FOR FOURIER TRANSFORM

9 An application: Fourier transform

The Fourier transform is one of the most important notions in mathematics. It has numerous applications in wave analysis, electrical engineering, image processing etc. and also provides an extremely useful tool in many mathematical problems coming from differential equations and probability theory (where it appears as the characteristic function of a random variable). In order to provide a motivation for the definition of the Fourier transform, we need to understand that it is a continuous version of the Fourier series or, in other words, it extends periodic phenomenons to a non-periodic setting when we let the period tend to infinity.

To be more precise, suppose we are given a function $f \in L_1(\mathbb{R})$, not necessarily periodic. In order to apply the theory of Fourier series, pick any large number $T > 0$, consider the restriction of f to the interval $[-\frac{T}{2}, \frac{T}{2}]$ and extend it periodically to the whole of $\mathbb{R}$. In this way we obtain a T-periodic function f_T such that $f_T(x) = f(x)$ for $x \in [-\frac{T}{2}, \frac{T}{2}]$. Instead of the ordinary trigonometric system $(u_n)_{n\in\mathbb{Z}}$ we now consider exponents of the form

$$u_{n,T}(x) = e^{2\pi \mathrm{i} \frac{n}{T} x} \quad (n \in \mathbb{Z}).$$

It is easy to verify that $(u_{n,T})_{n\in\mathbb{Z}}$ forms an orthonormal set in the space $L_2[-\frac{T}{2}, \frac{T}{2}]$ equipped with the normalized Lebesgue measure $\frac{1}{T}\mathrm{d}x$. This set is also complete which follows by repeating the proof of Fejér's theorem with obvious modifications, or by deriving it directly from the 2π-periodic case after rescaling the domain $[-\pi, \pi]$ to $[-\frac{T}{2}, \frac{T}{2}]$. Hence, the Fourier coefficients of the T-periodic function f_T are given by the formula

$$\widehat{f_T}(n) = (f_T, u_{n,T}) = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-2\pi \mathrm{i} \frac{n}{T} t}\, \mathrm{d}t \quad (n \in \mathbb{Z}), \tag{9.1}$$

and the corresponding Fourier series is

$$f_T(x) \sim \sum_{n=-\infty}^{\infty} \widehat{f_T}(n) e^{2\pi \mathrm{i} \frac{n}{T} x}.$$

Observe that f_T in his expansion has exponents $e^{\mathrm{i}\xi x}$ associated with every number of the form $\xi = 2\pi n/T$ ($n \in \mathbb{Z}$), whereas in the classical 2π-periodic case we had only exponents associated with integers. In other words, the larger the period T is, the more narrowly distributed are the arguments of the exponential functions involved in the Fourier series. So, if we want to draw a graph of f_T in the frequency scale, we should plot the values of $\widehat{f_T}(n)$ (or their magnitudes) at each argument of the form $2\pi n/T$ ($n \in \mathbb{Z}$), which are $\frac{1}{T}$ apart, whereas in the case of 2π-periodic functions we plot the values of Fourier coefficients at integers which are just 1 apart. We thus suspect that in the limiting case, when $T \to \infty$, we arrive at a continuous spectrum, that is, a certain function of a continuous real variable which describes the asymptotic behavior of Fourier coefficients of f_T's.

However, in order to define an appropriate limit version of Fourier coefficients, it would not make much sense just to pass to the limit in formula (9.1) as $T \to \infty$, because we simply have $\widehat{f_T}(n) \to 0$ for every $n \in \mathbb{Z}$. This is because the integrals in (9.1) are bounded as $T \to \infty$ and hence, the coefficients $\widehat{f_T}(n)$ converge to zero like $\frac{1}{T}$. Therefore, we scale up by T and consider new coefficients

$$c_{T,n} = \int_{-T/2}^{T/2} f(t) e^{-2\pi \mathrm{i} \frac{n}{T} t} \,\mathrm{d}t \quad (n \in \mathbb{Z})$$

which on the frequence scale should be plotted at the points $2\pi n/T$ $(n \in \mathbb{Z})$.

As an example, consider the function $f = \mathbb{1}_{[-1,1]}$. Take any $T > 1$ and consider the periodized function f_T, i.e. $f_T(x) = 1$ if $x \in [nT-1, nT+1]$ for some $n \in \mathbb{Z}$ and $f(x) = 0$ otherwise. Calculate

$$c_{T,n} = \int_{-T/2}^{T/2} f(t) e^{-2\pi \mathrm{i} \frac{n}{T} t} \,\mathrm{d}t = \int_{-1}^{1} e^{-2\pi \mathrm{i} \frac{n}{T} t} \,\mathrm{d}t = \frac{\mathrm{i}T}{2\pi n} e^{-2\pi \mathrm{i} \frac{n}{T} t} \,\mathrm{d}t \Big|_{-1}^{1} = \frac{N}{\pi n} \sin \frac{2\pi n}{T}.$$

After plotting these values we obtain a discrete graph of the function $\frac{2 \sin x}{x}$, for the arguments $x = 2\pi n/T$ $(n \in \mathbb{Z})$. For $T = 4\pi$ we get the following picture.

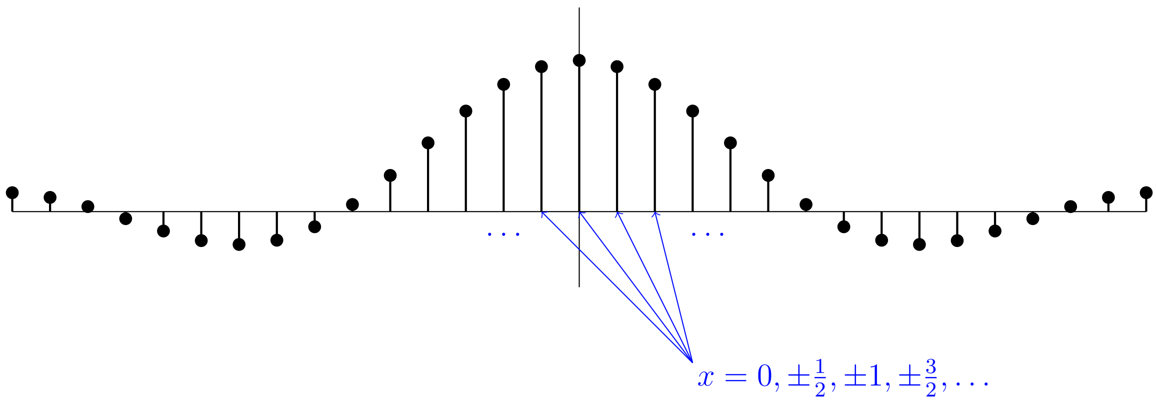

Fig. 1. The plot of the Fourier coefficients of $\mathbb{1}_{[-1,1]}$ periodized over $[-2\pi, 2\pi]$, i.e. $T = 4\pi$

Definition 9.1. Let $f, g \in L_1(\mathbb{R})$. We define the *convolution* $f * g$ by the formula

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-y) g(y) \,\mathrm{d}y, \tag{9.2}$$

and the *Fourier transform* $\widehat{f}$ of f by

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-\mathrm{i}tx} \,\mathrm{d}t. \tag{9.3}$$

The map $L_1(\mathbb{R}) \ni f \mapsto \widehat{f}$ is called the *Fourier transform.*

The choice of the factor $(2\pi)^{-1/2}$ is common and makes many formulas more elegant, as well as the symmetry between f and $\widehat{f}$ in the L_2-case, which we observe in the Plancherel theorem. So, under this definition we can easily calculate e.g. the Fourier transform of the function $f = \mathbb{1}_{[-1,1]}$:

$$\widehat{\mathbb{1}_{[-1,1]}}(x) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\mathrm{i}tx} \,\mathrm{d}t = \frac{\mathrm{i}}{\sqrt{2\pi} x} e^{-\mathrm{i}tx} \Big|_{-1}^{1} = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x}.$$

Notice that formula (9.2) makes sense, i.e. the map $y \mapsto f(x-y)g(y)$ is measurable and integrable on $\mathbb{R}$ for a.e. $x \in \mathbb{R}$. Hence, $f * g$ is defined by (9.2) a.e., moreover, $f*g \in L_1(\mathbb{R})$. This follows by a simple application of Fubini's theorem. Namely, replacing (if necessary) f and g by Borel functions coinciding with them a.e. on $\mathbb{R}$, we verify that $F(x,y) \coloneqq f(x-y)g(y)$ is also Borel on $\mathbb{R}^2$. By Fubini's theorem, we have

$$\int_{\mathbb{R}^2} |F(x,y)|\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}} |F(x,y)|\,\mathrm{d}x = \int_{\mathbb{R}} |g(y)|\,\mathrm{d}y \int_{\mathbb{R}} |f(x-y)|\,\mathrm{d}x = \|f\|_1 \|g\|_1.$$

Therefore, $F \in L_1(\mathbb{R}^2)$ which implies that the integral defining $f * g(x)$ exists for a.e. $x \in \mathbb{R}$ and $f * g \in L_1(\mathbb{R})$. Appealing once again to Fubini's theorem, we obtain

$$\|f * g\|_1 \leq \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}} |F(x,y)|\,\mathrm{d}x = \|f\|_1 \|g\|_1.$$

Lemma 9.2. *Let $f \in L_1(\mathbb{R})$ and $\alpha, \lambda \in \mathbb{R}$. Then:*

(a) *for $g(t) = e^{\mathrm{i}\alpha t} f(t)$ we have $\widehat{g}(x) = \widehat{f}(x-\alpha)$;*
(b) *for $g(t) = f(t-\alpha)$ we have $\widehat{g}(x) = \widehat{f}(x) e^{-\mathrm{i}\alpha x}$;*
(c) *if $g \in L_1(\mathbb{R})$, then $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$;*
(d) *for $g(t) = \overline{f(-t)}$ we have $\widehat{g}(x) = \overline{\widehat{f}(x)}$;*
(e) *for $g(t) = f\left(\frac{t}{\lambda}\right)$ $(\lambda \neq 0)$ we have $\widehat{g}(x) = |\lambda| \widehat{f}(\lambda x)$;*
(f) *for $g(t) = -\mathrm{i}t f(t)$, if $g \in L_1(\mathbb{R})$, then $\widehat{f}$ is differentiable and $(\widehat{f})' = \widehat{g}$;*
(g) *if $f \in C^1(\mathbb{R})$ and $f' \in L_1(\mathbb{R})$, then $\widehat{(f')}(x) = \mathrm{i}x\widehat{f}(x)$.*

Proof. Assertions (a), (b), (d) and (e) follow automatically from formula (9.3). For proving (c) we use the Fubini theorem:

$$\begin{aligned}
\widehat{f * g}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\mathrm{i}tx}\,\mathrm{d}t \int_{\mathbb{R}} f(t-s) g(s)\,\mathrm{d}s \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} g(s) e^{-\mathrm{i}sx}\,\mathrm{d}s \int_{\mathbb{R}} f(t-s) e^{-\mathrm{i}x(t-s)}\,\mathrm{d}t \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} g(s) e^{-\mathrm{i}sx}\,\mathrm{d}s \int_{\mathbb{R}} f(t) e^{-\mathrm{i}xt}\,\mathrm{d}t = \widehat{f}(x)\widehat{g}(x).
\end{aligned}$$

Assertion (f) is left as an exercise (see **Problem 6.17**). For (g), integration by parts gives

$$\begin{aligned}
\widehat{(f')}(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f'(t) e^{-\mathrm{i}tx}\,\mathrm{d}t \\
&= f(t) e^{-\mathrm{i}tx}\Big|_{-\infty}^{+\infty} + \frac{\mathrm{i}x}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-\mathrm{i}tx}\,\mathrm{d}t = \mathrm{i}x\widehat{f}(x).
\end{aligned}$$ □

Now, our goal is to find a way to 'invert' a Fourier transform, that is, to reconstruct the original function f from its transform $\widehat{f}$. Since $\widehat{f}$, as we explained before, is a continuous analogue of the sequence $(\widehat{f}(n))_{n\in\mathbb{Z}}$ of Fourier coefficients, let us first see how to reconstruct an integrable, 2π-periodic function from its Fourier series. So, let $f \in L_1(\mathbb{T})$ and consider the corresponding Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{\mathrm{i}nx}, \quad \text{where} \quad \widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-\mathrm{i}nt}\,\mathrm{d}t.$$

We know that in general it is not possible to represent $f(x)$ as the value of the Fourier series at x, and this series can be divergent even for continuous functions. However, assume additionally that

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty. \tag{9.4}$$

Then the formula

$$g(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{\mathrm{i}nx} \tag{9.5}$$

defines a continuous function g, because (9.4) assures that the series is uniformly convergent by the Weierstrass M-test. Moreover, for each $n \in \mathbb{Z}$ we have

$$\begin{aligned}\widehat{g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-\mathrm{i}nx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big\{\sum_{k=-\infty}^{\infty}\widehat{f}(k)e^{\mathrm{i}kx}\Big\}e^{-\mathrm{i}nx}\,\mathrm{d}x \\ &= \sum_{k=-\infty}^{\infty}\widehat{f}(k)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}(k-n)x}\,\mathrm{d}x = \widehat{f}(n).\end{aligned} \tag{9.6}$$

Hence, $f(x) = g(x)$ a.e. because the operator $L_1(\mathbb{T}) \ni f \mapsto (\widehat{f}(n))_{n\in\mathbb{Z}}$ is injective (recall Theorem 8.11). In other words, under condition (9.4) the Fourier series of f converges to $f(x)$ a.e. on $\mathbb{R}$.

The natural conjecture is thus that if both f and $\widehat{f}$ belong to $L_1(\mathbb{R})$ (the latter being an analogue to assumption (9.4)), we have the formula

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{f}(x)e^{\mathrm{i}tx}\,\mathrm{d}x, \tag{9.7}$$

which corresponds to (9.5). In fact, we shall prove it is true, but it is much more delicate matter than in the case of Fourier series. Observe that we cannot simply repeat the same argument as above, replacing everywhere the formula for Fourier coefficients by the formula for Fourier transform. By doing so, in computation (9.6) we would arrive at an integral $\int_{-\infty}^{\infty} e^{\mathrm{i}(x-y)t}\,\mathrm{d}t$ which does not make sense.

From now on, we consider the L_p spaces (basically for $p = 1, 2$) on $\mathbb{R}$ equipped with the measure $\mathrm{d}\mu = (2\pi)^{-1/2}\,\mathrm{d}x$, so that

$$\|f\|_p = \Big\{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |f(x)|^p\,\mathrm{d}x\Big\}^{1/p} \quad \text{for } f \in L_p(\mathbb{R}).$$

Proposition 9.3. *For every $f \in L_1(\mathbb{R})$, $\widehat{f} \in C_0(\mathbb{R})$ and $\|\widehat{f}\|_\infty \le \|f\|_1$.*

Proof. The above inequality is obvious from formula (9.3). To see that $\widehat{f}$ is continuous, fix any sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ converging to some $x \in \mathbb{R}$. Then

$$|\widehat{f}(x_n) - \widehat{f}(x)| \le \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |f(t)||e^{-\mathrm{i}x_nt} - e^{-\mathrm{i}xt}|\,\mathrm{d}t$$

and since the function under the integral is majorized by $2|f(t)|$, we obtain $\widehat{f}(x_n) \to \widehat{f}(x)$ by Lebesgue's theorem.

Now, we prove that $\widehat{f}$ vanishes at infinity. For every $x \in \mathbb{R}$ we have

$$\widehat{f}(x) = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-\mathrm{i}x(t+\frac{\pi}{x})} \,\mathrm{d}t = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f\big(s - \frac{\pi}{x}\big) e^{-\mathrm{i}xs} \,\mathrm{d}s,$$

hence

$$2\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Big(f(t) - f\big(t - \frac{\pi}{x}\big)\Big) e^{-\mathrm{i}xt} \,\mathrm{d}t.$$

Therefore, $2|\widehat{f}(x)| \leq \|f - f_{\pi/x}\|_1$, where f_y denotes the shifted function, $f_y(x) = f(x-y)$. It is easy to show that the map $\mathbb{R} \ni y \mapsto f_y \in L_1(\mathbb{R})$ is uniformly continuous (**classes**), hence we obtain $|\widehat{f}(x)| \to 0$ as $|x| \to \infty$. $\square$

As we explained before, trying to prove the announced assertion (9.7) by simply plugging into it the formula (9.3) leads to a divergent integral. However, if instead of $\widehat{f}$ we had $\widehat{f}$ multiplied by some integrable function of variable t, then using the Fubini theorem and changing the order of integration would lead to a convergent integral. So, our strategy is to convolve f with some nicely integrable functions in such a way that in the limit (in some sense) we obtain f itself. An appropriate sequence of such functions is called an *approximate identity* of $L_1(\mathbb{R})$ under the convolution operation, and there are many possible choices of them. We choose the following: define

$$H(t) = e^{-|t|} \qquad (t \in \mathbb{R})$$

and

$$h_\lambda(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H(\lambda t) e^{\mathrm{i}tx} \,\mathrm{d}t \qquad (\lambda > 0). \tag{9.8}$$

We can observe that h_1 is the Fourier transform of the function $H(-t)$, but the point is that all h_λ's are positive and their integrals are easy to calculate. Indeed, we have

$$h_\lambda(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\lambda^2 + x^2} \qquad (\lambda > 0), \tag{9.9}$$

whence

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h_\lambda(x) \,\mathrm{d}x = 1 \qquad (\lambda > 0). \tag{9.10}$$

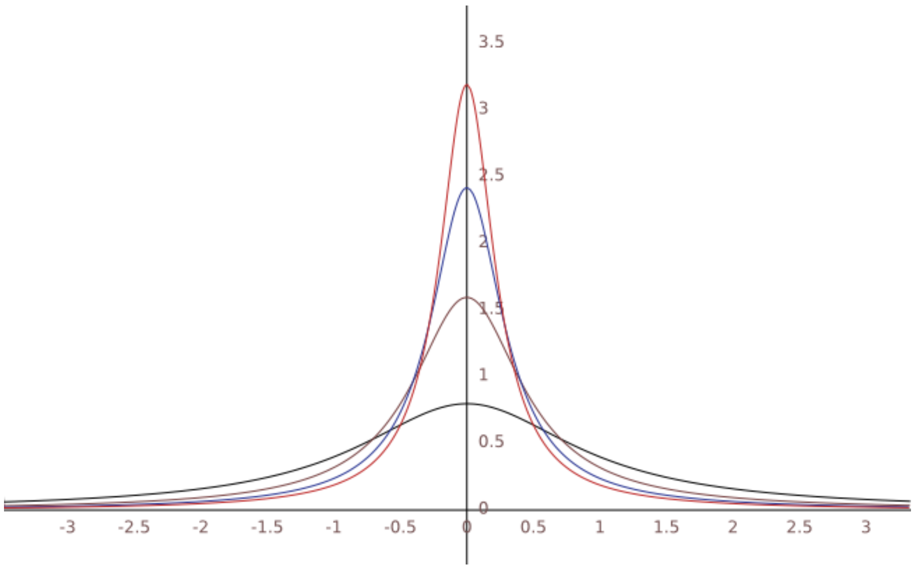

Fig. 2. The graphs of h_λ for $\lambda = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

The following result collects the most important features of the functions H and h_λ.

Proposition 9.4. *Let $f \in L_1(\mathbb{R})$, $g \in L_\infty(\mathbb{R})$ and assume that g is continuous at some point $x \in \mathbb{R}$. Then:*

(a) $(f * h_\lambda)(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}} H(\lambda t)\widehat{f}(t)e^{\mathrm{i}xt}\,\mathrm{d}t$;

(b) $\lim_{\lambda\to 0+}(g * h_\lambda)(x) = g(x)$;

(c) $\lim_{\lambda\to 0+} \|f * h_\lambda - f\|_1 = 0$;

(d) $\lim_{\lambda\to 0+} \|f * h_\lambda - f\|_p = 0$ *if* $1 \le p < \infty$ *and* $f \in L_p(\mathbb{R})$;

Proof. Assertion (a) follows immediately from (9.8) by applying Fubini's theorem.

For (b), observe that using (9.10) we obtain

$$\begin{aligned}
(g * h_\lambda)(x) - g(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \big(g(x-y) - g(x)\big)h_\lambda(y)\,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \big(g(x-y) - g(x)\big)\lambda^{-1}h_1(\lambda^{-1}y)\,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \big(g(x-\lambda y) - g(x)\big)h_1(y)\,\mathrm{d}y.
\end{aligned}$$

Since the integrated function is bounded by $2\|g\|_\infty h_1(y)$, we get that $(g * h_\lambda)(x) \to g(x)$ as $\lambda \to 0^+$ by Lebesgue's dominated convergence theorem.

For assertion (c), observe first that $f * h_\lambda$ is well-defined and continuous. In general, using Hölder's inequality and the fact that the map $\mathbb{R} \ni z \mapsto G_z \in L_q(\mathbb{R})$ is uniformly continuous for any $G \in L_q(\mathbb{R})$, we may infer that for all $f \in L_p(\mathbb{R})$ and $G \in L_q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$ the convolution $F * G$ is uniformly continuous.

Using (9.10) we may write

$$(f * h_\lambda)(x) - f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \big(f(x-y) - f(x)\big)h_\lambda(y)\,\mathrm{d}y.$$

Integrating over $x \in \mathbb{R}$ and using Fubini's theorem we obtain

$$\|f * h_\lambda - f\|_1 \le \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \|f_y - f\|_1 \cdot h_\lambda(y)\,\mathrm{d}y.$$

Hence, applying assertion (b) to the continuous function $g(y) := \|f_y - f\|_1$ with $g(0) = 0$, we infer that the right-hand side tends to 0 as $\lambda \to 0^+$.

Assertion (d) is proved in the same way as above with one difference that in order to estimate $\|f * h_\lambda - f\|_p$ one should use the Jensen inequality for integrals, applied to the convex function $t \mapsto t^p$. $\square$

Theorem 9.5 (Fourier inversion formula). *If $f, \widehat{f} \in L_1(\mathbb{R})$, then the function g defined by*

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(x)e^{\mathrm{i}tx}\,\mathrm{d}x$$

belongs to $C_0(\mathbb{R})$ and $f(t) = g(t)$ a.e. on $\mathbb{R}$.

Proof. Since we assume that $\widehat{f} \in L_1(\mathbb{R})$, the above integral makes sense and it defines $g \in C_0(\mathbb{R})$ according to Proposition 9.3. Note that

$$|H(\lambda t)\widehat{f}(t)e^{itx}| \leq |\widehat{f}(t)| \quad \text{for every } t \in \mathbb{R}$$

and $H(\lambda t) \to 1$ as $\lambda \to 0^+$. Hence, by Proposition 9.4(a) and Lebesgue's theorem, we obtain

$$\lim_{\lambda \to 0+} (f * h_\lambda)(x) = g(x) \quad \text{for every } x \in \mathbb{R}.$$

On theother hand, Proposition 9.4(c) says that

$$\lim_{\lambda \to 0+} \|f * h_\lambda - f\|_1 = 0.$$

Hence, there exists a sequence $(\lambda_n)_{n=1}^{\infty}$ of positive numbers converging to zero such that

$$\lim_{n \to \infty} (f * h_{\lambda_n})(f) = f(x) \quad \text{a.e. on } \mathbb{R}.$$

(see the proof of Theorem 1.12). Consequently, $f(x) = g(x)$ a.e. □

Corollary 9.6. *If $f \in L_1(\mathbb{R})$ and $\widehat{f}(x) \equiv 0$, then $f = 0$ a.e.*