Functional analysis

Lecture 15: Fourier transform for L_2-functions; Plancherel's theorem; generalized Fourier inversion formula

As we have seen, formula (9.3) originally defines the Fourier transform which maps the space $L_1(\mathbb{R})$ into $C_0(\mathbb{R})$, but not in a surjective way (see **Problem 6.30**). So, there is no Fourier-like correspondence between all integrable functions and all continuous functions vanishing at infinity. Nonetheless, the machinery of Fourier transform can be adapted to functions from $L_2(\mathbb{R})$ and in that space the resulting theory manifests full symmetry between f and $\widehat{f}$. This is the content of the following celebrated theorem whose most important part is the formula in assertion (b).

Theorem 9.7 (**Plancherel theorem**). *There exists a map $L_2(\mathbb{R}) \ni f \mapsto \widehat{f}$ from $L_2(\mathbb{R})$ into itself having the following properties:*

(a) *if $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, then $\widehat{f}$ is the Fourier transform of f in the sense of Definition 9.1;*
(b) $\|f\|_2 = \|\widehat{f}\|_2$ *for every $f \in L_2(\mathbb{R})$;*
(c) *the map $L_2(\mathbb{R}) \ni f \mapsto \widehat{f}$ is an isometric isomorphism of $L_2(\mathbb{R})$ onto itself;*
(d) *if for every $A > 0$ we define*

$$\varphi_A(x) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(t) e^{-\mathrm{i}tx} \, \mathrm{d}t \quad \text{and} \quad \psi_A(t) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \widehat{f}(x) e^{\mathrm{i}tx} \, \mathrm{d}x,$$

then $\|\varphi_A - \widehat{f}\|_2 \to 0$ and $\|\psi_A - f\|_2 \to 0$ as $A \to \infty$.

Proof. For transparency, we divide the proof into several parts. We start with proving equality (b) for functions from $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and then we extend the Fourier transform to $L_2(\mathbb{R})$ and complete the proof of (b) for all L_2-functions.

<u>Part 1.</u> $\|f\|_2 = \|f\|_2$ for every $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.

For any fixed $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ define

$$\widetilde{f}(x) = \overline{f(-x)} \quad \text{and} \quad g = f * \widetilde{f}.$$

We have

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-y)\overline{f(-y)} \, \mathrm{d}y = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x+y)\overline{f(y)} \, \mathrm{d}y,$$

hence we can write g in a form of the inner product in $L_2(\mathbb{R})$, namely, $g(x) = (f_{-x}, f)$. This means that $g(x)$ is the evaluation of the continuous linear functional on $L_2(\mathbb{R})$ generated by f (like in the Riesz representation theorem) on the shifted function f_{-x}. Since the map $x \mapsto f_{-x}$ is also continuous, we infer that g is continuous.

By the Cauchy–Schwarz inequality,

$$|g(x)| \le \|f_{-x}\|_2 \|f\|_2 = \|f\|_2^2$$

and hence g is bounded. Notice also that since $f, \widetilde{f} \in L_1(\mathbb{R})$, the function g as their convolution belongs to $L_1(\mathbb{R})$ as well. By Proposition 9.4(a), we have

$$(g * h_\lambda)(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H(\lambda t) \widehat{g}(t) \, \mathrm{d}t.$$

In view of Lemma 9.2(c), (d), we have $\widehat{g} = |\widehat{f}|^2 \geq 0$. Since also $H(\lambda t) \nearrow 1$ as $\lambda \to 0$, the Lebesgue theorem yields

$$\lim_{\lambda \to 0^+} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H(\lambda t)\widehat{g}(t)\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\widehat{f}(t)|^2\,\mathrm{d}t = \|\widehat{f}\|_2^2.$$

On the other hand, by Proposition 9.4(b), we have

$$\lim_{\lambda \to 0^+} (g * h_\lambda)(0) = g(0) = \|f\|_2^2$$

which proves that $\|f\|_2 = \|\widehat{f}\|_2$.

Part 2. An extension of the Fourier transform to $L_2(\mathbb{R})$.

Fix any $f \in L_2(\mathbb{R})$ and define $f_A = f \cdot \mathbb{1}_{[-A,A]}$ for $A > 0$. Plainly, $f_A \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and Lebesgue's theorem implies that $\|f_A - f\|_2 \to 0$ as $A \to \infty$. If we define φ_A as in assertion (d), i.e.

$$\varphi_A(x) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(t)e^{-\mathrm{i}tx}\,\mathrm{d}t,$$

then, obviously, $\varphi_A = \widehat{f_A}$.

By Part 1, we have $\|\varphi_A\|_2 = \|f_A\|_2$ and since $(f_A)_{A\in\mathbb{N}}$ is a Cauchy sequence in $L_2(\mathbb{R})$, so is $(\varphi_A)_{A\in\mathbb{N}}$. Therefore, we can define

$$\widehat{f} = \lim_{A \to \infty} \varphi_A \quad \text{(the limit in } L_2(\mathbb{R})\text{)}.$$

Notice that in the case where $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, the sequence $(\varphi_A)_{A\in\mathbb{N}}$ converges pointwise to $\widehat{f}$. Therefore, the just defined map $f \mapsto \widehat{f}$ is indeed an extension of the Fourier transform.

Observe also that for every $f \in L_2(\mathbb{R})$ we have

$$\|\widehat{f}\|_2 = \lim_{A \to \infty} \|\varphi_A\|_2 = \lim_{A \to \infty} \|f_A\|_2 = \|f\|_2.$$

Consequently, we have proved assertions (a), (b) and the first part of (d).

Part 3. Generalizing the Fourier inversion formula (the second part of assertion (d)).

For clarity, we use here the symbols Φ and Ψ for the linear operators on $L_2(\mathbb{R})$ given by $\Phi(f) = \widehat{f}$ and

$$\Psi(g) = \lim_{A \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} g(x)e^{\mathrm{i}tx}\,\mathrm{d}x \quad \text{(the limit in } L_2(\mathbb{R})\text{)}.$$

(Note that the latter definition makes sense because, as we have already seen, the sequence under the limit sign is Cauchy.)

We *claim* that

$$\Psi\Phi(f) = f \quad \text{for every } f \in L_2(\mathbb{R}). \tag{9.1}$$

Since $\Psi g(x) = \Phi g(-x)$, assertion (b) implies that both Φ and Ψ are isometries of L_2 into itself. In particular, they are both continuous. By the fact that $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is a dense subspace of $L_2(\mathbb{R})$, it is thus enough to prove formula (9.1) for every $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.

If f is such a function and additionally satisfies $\widehat{f} \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, then (9.1) follows directly from the Fourier inversion formula (Theorem 9.5). For any $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ we consider the convolutions $f * h_\lambda$ ($\lambda > 0$) which all satisfy the condition

$$\widehat{f * h_\lambda} \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}).$$

Indeed, by Lemma 9.2(c), $\widehat{f * h_\lambda} = \widehat{f} \cdot \widehat{h_\lambda}$, whereas formula (9.8) (which is nothing but the inverse formula for $H(\lambda x)$) gives $\widehat{h_\lambda}(x) = H(\lambda x) = e^{-\lambda|x|}$. Thus, the product $\widehat{f} \cdot \widehat{h_\lambda}$ is obviously integrable.

Therefore, $\Psi\Phi(f * h_\lambda) = f * h_\lambda$ for every $\lambda > 0$. By Proposition 9.4(d), we have $\|f * h_\lambda - f\|_2 \to 0$ as $\lambda \to 0$ and since $\Psi\Phi$ is an isometry, we obtain formula (9.1) for every $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, and hence for every $f \in L_2(\mathbb{R})$. We have thus proved assertion (d) completely.

Part 4. $f \mapsto \widehat{f}$ is a unitary operator on $L_2(\mathbb{R})$ (assertion (c)).

Swapping Φ and Ψ in Part 3, whose roles are symmetrical up to the minus sign, we obtain $\Phi\Psi(f) = f$ for every $f \in L_2(\mathbb{R})$. This, together with formula (9.1), implies that our extension $f \mapsto \widehat{f}$, i.e. the map Φ, is both injective and surjective.

It remains to show that it preserves the inner product in $L_2(\mathbb{R})$ (cf. **Problem 5.11**). This follows easily once we recall the polarization identity:

$$4(f,g) = \|f+g\|^2 - \|f-g\|^2 + \mathrm{i}\|f+\mathrm{i}g\|^2 - \mathrm{i}\|f-\mathrm{i}g\|^2 \quad (f,g \in L_2(\mathbb{R})).$$

Since Φ is an isometry, we thus obtain

$$(f,g) = (\Phi f, \Phi g) \quad (f,g \in L_2(\mathbb{R})),$$

which proves assertion (c) and completes the whole proof. □

Remark 9.8. Let us stress the difference between the Fourier transform $\widehat{f}$ for $f \in L_1(\mathbb{R})$ and for $f \in L_2(\mathbb{R}) \setminus L_1(\mathbb{R})$. In the former case, $\widehat{f}(x)$ is defined *everywhere* on $\mathbb{R}$ by means of formula (9.3) and is a continuous functions. In the latter case, $\widehat{f}(x)$ is just defined *almost everywhere* $\mathbb{R}$ with the aid of the approximation procedure described in assertion (d) above.

However, for every $f \in L_2(\mathbb{R})$, even if $f \notin L_1(\mathbb{R})$, the Fourier inversion formula is valid, provided that $\widehat{f} \in L_1(\mathbb{R})$. Indeed, the integral which defines $\psi_A(t)$ in assertion (d) is then convergent as $A \to \infty$ and since $(\psi_A)_{A\in\mathbb{N}}$ converges to f in $L_2(\mathbb{R})$, we have

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(x) e^{itx} \, \mathrm{d}x \quad \text{a.e. on } \mathbb{R}.$$

Of course, the Plancherel theorem has countless applications whenever we deal with Fourier transforms for L_2-functions. Let us just show one, where we calculate an improper integral in quite an elegant way.

Example 9.9. By Plancherel's theorem, we have

$$\int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x = \pi.$$

Proof. We verify easily that

$$\widehat{\mathbb{1}_{[-1,1]}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x} \quad (x \in \mathbb{R})$$

(see the comments after Definition 9.1). Comparing the squares of the L_2-norms of these functions (recall that we divide the Lebesgue measure by $\sqrt{2\pi}$), we get

$$\frac{1}{\pi^2} \int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x = \|\mathbb{1}_{[-1,1]}\|_2^2 = \frac{1}{2\pi} \int_{-1}^{1} \mathrm{d}x = \frac{1}{\pi}. \qquad \square$$