Functional analysis

Lecture 2: BANACH SPACE; COMPLETENESS OF $C_b(X)$ AND $L_p(\mu)$; BOUNDED LINEAR FUNCTIONALS AND OPERATORS; THE HAHN–BANACH THEOREM

Definition 1.9. A sequence $(x_n)_{n=1}^{\infty}$ in a normed space X is called a *Cauchy sequence* provided it satisfies the Cauchy condition:

$$\forall_{\varepsilon>0}\ \exists_{N_\varepsilon\in\mathbb{N}}\ \forall_{m,n\geq N_\varepsilon}\quad \|x_m-x_n\|<\varepsilon.$$

Plainly, this notion coincides with the usual notion of Cauchy sequence in a metric space, where the metric considered is given by the norm: $\rho(x,y)=\|x-y\|$. As for metric spaces, we call a normed space $(X,\|\cdot\|)$ *complete* if every Cauchy sequence in X is convergent in X (with respect to norm), which means that (X,ρ) is complete as a metric space.

Definition 1.10. A complete normed space is called a *Banach space.*

Now, we are going to show that the two important classes of normed spaces: the $C_b(X)$-spaces and the $L_p(\mu)$-spaces (see Example 1.2, (3) and (4)) are complete.

Proposition 1.11. *Let X be any Hausdorff topological space. Then, the space $C_b(X)$ of continuous bounded scalar-valued functions on X is complete.*

Proof. Fix any Cauchy sequence $(f_n)_{n=1}^{\infty}\subset C_b(X)$, that is, for each $\varepsilon>0$ there is $N_\varepsilon\in\mathbb{N}$ such that $\|f_m-f_n\|_\infty<\varepsilon$ for all $m,n\geq N_\varepsilon$. Obviously, for every fixed $x\in X$ the sequence $(f_n(x))_{n=1}^{\infty}$ is Cauchy and since the scalar field ($\mathbb{R}$ or $\mathbb{C}$) is complete, there exists the limit $f(x)\coloneqq\lim_n f_n(x)$. Now, observe that for m,n as above, we have

$$|f(x)-f_n(x)|\leq|f(x)-f_m(x)|+\|f_m-f_n\|_\infty<|f(x)-f_m(x)|+\varepsilon\xrightarrow[m\to\infty]{}\varepsilon.$$

Hence $|f(x)-f_n(x)|\leq\varepsilon$ provided $n\geq N_\varepsilon$. Recall that the choice of $N_\varepsilon\in\mathbb{N}$ was independent of x, thus $(f_n)_{n=1}^{\infty}$ converges uniformly to f, i.e. $\lim_n\|f_n-f\|_\infty=0$. As a uniform limit of continuous functions, f must be also continuous. Since, for n large enough we have $\|f\|_\infty\leq\|f_n\|_\infty+1$, it is also a bounded function. Therefore, $f\in C_b(X)$ is the limit of our Cauchy sequence. □

Theorem 1.12. *Let $(X,\mathfrak{M},\mu)$ be a measure space with a positive measure μ, and let $1\leq p\leq\infty$. Then, the normed space $L_p(\mu)$ is complete.*

Proof. First, we deal with the easier case $p=\infty$. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L_\infty(\mu)$. Recalling that the norm in $L_\infty(\mu)$ is given by the essential supremum, we infer that the sets $A_n\coloneqq\{x\in X\colon|f_n(x)|>\|f_n\|_\infty\}$ and $B_{m,n}\coloneqq\{x\in X\colon|f_m(x)-f_n(x)|>\|f_m-f_n\|_\infty\}$ (for $m,n\in\mathbb{N}$) are of measure zero. Hence, $C\coloneqq\bigcup_n A_n\cup\bigcup_{m,n}B_{m,n}$ is of measure zero and on the set $X\setminus C$, the sequence $(f_n)_{n=1}^{\infty}$ is Cauchy with respect to the usual supremum norm and thus uniformly convergent to a measurable function $f\colon X\setminus C\to\mathbb{K}$. Extending f to X arbitrarily, we get that $f\in L_\infty(\mu)$ (it is essentially bounded because all f_n's were bounded outside C) and $\lim_n\|f_n-f\|_\infty=0$.

Now, we assume $1\leq p<\infty$. The first part of the proof consists of showing quite an important assertion:

Claim. Any Cauchy sequence $(f_n)_{n=1}^\infty$ in $L_p(\mu)$ contains an almost everywhere convergent subsequence. That is, there is a subsequence $(f_{n_k})_{k=1}^\infty$ such that the limit $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ exists a.e. and it defines a function $f \in L_p(\mu)$.

Proof of the claim. By a simple induction we construct a subsequence $(f_{n_i})_{i=1}^\infty$ such that

$$\left\|f_{n_{i+1}} - f_{n_i}\right\|_p < 2^{-i} \quad \text{for each } i \in \mathbb{N}. \tag{1.1}$$

Of course, $(f_{n_i})_{i=1}^\infty$ is a.e. convergent if and only the series

$$f_{n_1}(x) + \sum_{i=1}^{\infty} (f_{n_{i+1}}(x) - f_{n_i}(x)) \tag{1.2}$$

is a.e. convergent (note that its k^{th} partial sum is exactly $f_{n_k}(x)$). To show that it is, in fact, absolutely convergent a.e., consider

$$s_k = \sum_{i=1}^{k} \left|f_{n_{i+1}} - f_{n_i}\right| \quad \text{and} \quad s = \sum_{i=1}^{\infty} \left|f_{n_{i+1}} - f_{n_i}\right|.$$

In view of (1.1) and the triangle (Minkowski) inequality in $L_p(\mu)$, we have $\|s_k\|_p \leq 2^{-1} + \ldots + 2^{-k} < 1$ for each $k \in \mathbb{N}$. Therefore, by the Fatou lemma applied to the sequence $(s_k^p)_{k=1}^\infty$, we get

$$\|s\|_p^p = \int_X s^p \,\mathrm{d}\mu = \int_X \lim_{k\to\infty} s_k^p \,\mathrm{d}\mu \leq \liminf_{k\to\infty} \int_X s_k^p \,\mathrm{d}\mu = \liminf_{k\to\infty} \|s_k\|_p^p \leq 1.$$

So, $\|s\|_p \leq 1$ which implies that $s(x) < \infty$ a.e. and hence the series (1.2) is absolutely convergent a.e. It also implies that its a.e. defined sum $f(x) = \lim_k f_{n_k}(x)$ belongs to $L_p(\mu)$ (note that $\|f\|_p \leq \|f_{n_1}\|_p + \|s\|_p < \infty$).

Having established the claim, we need to show that f is the L_p-limit of $(f_n)_{n=1}^\infty$, i.e.

$$\lim_{n\to\infty} \|f - f_n\|_p = 0. \tag{1.3}$$

For any $\varepsilon > 0$ pick $N_\varepsilon \in \mathbb{N}$ so that $\|f_m - f_n\|_p < \varepsilon$ for all $m, n \geq N_\varepsilon$. By the Fatou lemma, for each $m \geq N_\varepsilon$, we have

$$\|f - f_m\|_p^p = \int_X |f - f_m|^p \,\mathrm{d}\mu \leq \liminf_{i\to\infty} \int_X |f_{n_i} - f_m|^p \,\mathrm{d}\mu \leq \varepsilon^p,$$

which shows that (1.3) holds true and completes the proof. □

Lemma 1.13. *Let X be a Banach space and $Y \subseteq X$ be its linear subspace. Then, Y is a Banach space (with the norm inherited from X) if and only if Y is closed in X.*

Proof. (classes) □

Now, we extend our list of normed spaces presented in Example 1.2 to show that Banach spaces appear naturally in various branches of mathematics.

Example 1.14. Let, as usual, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The following are examples of normed spaces and, in fact (as we will see below), Banach spaces:

(1) $C^{(k)}([a,b])$, the space of k-times continuously differentiable functions $f\colon [a,b]\to\mathbb{K}$, supplied with e.g. one of the norms:

$$\|f\|_{(k)}=\sum_{i=0}^{k}\big\|f^{(i)}\big\|_\infty,\qquad \|f\|'_{(k)}=|f(a)|+\max\nolimits_{1\le i\le k}\big\|f^{(i)}\big\|_\infty.$$

(2) $\mathrm{Lip}_0(M)$, the space of Lipschitz functions $f\colon M\to\mathbb{K}$ vanishing at a distinguished point $0\in M$, where (M,ρ) is a metric space. We equip it with the optimal Lipschitz constant norm $L(\cdot)$, i.e.

$$L(f)=\sup\left\{\frac{|f(x)-f(y)|}{\rho(x,y)}\colon x,y\in M,\ x\neq y\right\}.$$

(3) $\mathrm{Lip}(M)$, the space of bounded $\mathbb{K}$-valued Lipschitz functions on a metric space M, supplied with the norm

$$\|f\|_L=\max\{\|f\|_\infty, L(f)\}.$$

(4) $H^\infty(D)$, the *Hardy space* of bounded holomorphic (analytic) functions $f\colon D\to\mathbb{C}$ on the unit disc $D=\{z\in\mathbb{C}\colon |z|<1\}$, equipped with the supremum norm $\|f\|_\infty=\sup_{z\in D}|f(z)|$.

(5) $A(D)$, the *disc algebra*[1], that is, the space of all holomorphic functions $f\colon D\to\mathbb{C}$ which admit a continuous extension to the closed disc $\overline{D}$. In other words, $A(D)=H^\infty(D)\cap C(\overline{D})$ and the space is equipped again with the supremum norm which, according to the maximum principle, is the maximum over the unit circle:

$$\|f\|_\infty=\sup_{z\in D}|f(z)|=\max\nolimits_{|z|=1}|f(z)|.$$

(In the previous two examples we set $\mathbb{K}=\mathbb{C}$.)

(6) $\mathcal{M}(X)$, the space of regular Borel measures on a locally compact Hausdorff space X, supplied with the *total variation* norm $\|\mu\|=|\mu|(X)$ which can be defined with the aid of Hahn's decomposition theorem for signed measures.[2]

In the next proposition, we use the notation introduced in Examples 1.2 and 1.14, and every space considered is equipped with one of the norms defined above.

Proposition 1.15. *All the following spaces (with any $1\le p\le\infty$):*

$$\ell_p^n,\ c_0,\ c,\ \ell_p,\ L_p(\mu),\ C(K),\ C_0(L),\ C_b(X),\ C^{(k)}([a,b]),$$
$$\mathrm{Lip}_0(M),\ \mathrm{Lip}(M),\ H^\infty(D),\ A(D),\ \mathcal{M}(X)$$

are Banach spaces.

[1]The term *algebra* comes from the fact that $A(D)$, with the natural operations, becomes a *Banach algebra*. From this point of view, the conclusion of the maximum principle can be formulated by saying that the unit circle is the *Shilov boundary* of $A(D)$; we shall not go further into this topic.

[2]This is an extremely important example which we will elaborate on in Section 3, where we explain all the required terminology (regularity, variation etc.) in details, and prove the Riesz representation theorem for positive/continuous linear functionals on $C_0(X)$.

Proof. For every $1 \le p \le \infty$, $L_p(\mu)$ is a Banach space according to Theorem 1.12, and so are the spaces ℓ_p^n ($n \in \mathbb{N}$) and ℓ_p as particular cases.

Since c_0 and c are closed subspaces of ℓ_∞ **(classes)**, they are Banach spaces in view of Lemma 1.13.

By Proposition 1.11, $C_b(X)$ is complete for every Hausdorff space. If X is also locally compact, then $C_0(X)$ as a closed subspace of $C_b(X)$ **(classes)** is Banach as well. Of course, for K compact Hausdorff, we have $C(K) = C_0(K)$, so $C(K)$ is also a Banach space.

The proof that $C^{(k)}([a,b])$, $\mathrm{Lip}_0(M)$ and $\mathrm{Lip}(M)$ are Banach spaces is left as an exercise **(Problems 1.16 and 1.18)**.

Every sequence in $(f_n)_{n=1}^\infty \subset H^\infty(D)$ which satisfies the Cauchy condition with respect to the supremum norm is uniformly convergent on D. By the Weierstrass theorem, the limit function os holomorphic on D and, of course bounded as each f_n is bounded. Hence, every Cauchy sequence in $H^\infty(D)$ has a limit in $H^\infty(D)$.

The space $A(D)$ can be treated as a subspace of $C(\overline{D})$, the Banach space of continuous functions on the closed unit disc (formally we identify each member of $A(D)$ with its unique continuous extension to $\overline{D}$). Again, by the Weierstrass theorem, it is a closed subspace, thus complete in view of Lemma 1.13.

The proof that $\mathcal{M}(X)$ is a Banach space (together with a detailed analysis of this space) is **postponed** to Section 3. □

2 Bounded linear operators

Definition 2.1. Let X and Y be normed spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $T\colon X \to Y$ be a linear operator. We define the *norm* of T by the formula

$$\|T\| = \sup\big\{\|Tx\|\colon x \in B_X\big\} = \sup\bigg\{\frac{\|Tx\|}{\|x\|}\colon x \in X,\, x \neq 0\bigg\}.$$

We say that T is *bounded* provided that $\|T\| < \infty$.

Proposition 2.2. *For any linear operator T acting between normed spaces X and Y, the following assertions are equivalent:*

(i) *T is bounded;*
(ii) *T is continuous;*
(iii) *T is continuous at a single point.*

Proof. **(classes)** □

Remark. It is quite common to omit the brackets and write Tx instead of $T(x)$ when dealing with a linear operator T. From now on, considering any linear operator between linear (normed) space we assume without mentioning that they are over the same scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. In particular, if the codomain is just the scalar field, i.e. $T\colon X \to \mathbb{K}$, then we call such an operator T a *functional*. Usually, we will be denoting operators by S, T, Λ, Φ etc. and functionals by f, φ, x^*, y^* etc. Another common notation, used when evaluating a functional on a vector, is

- $\langle x, x^*\rangle$, or $\langle x^*, x\rangle$ instead of $x^*(x)$.

Note that the term *bounded* in reference to operators means literally *bounded on the unit ball* (and hence, on any ball). In view of Proposition 2.2, we can use the terms *bounded* and *continuous* interchangeably, although we prefer using *bounded* in reference to operators, and *continuous* to functionals. For any normed spaces X and Y, we introduce the following important notation:

- $\mathscr{L}(X,Y)=\{T\colon X\to Y \mid\ T \text{ is a bounded linear operator}\}$,
- $X^*=\{x^*\colon X\to\mathbb{K} \mid\ x^* \text{ is a continuous linear functional}\}$.

Proposition 2.3. *If X and Y are normed spaces and $\dim X<\infty$, then every linear operator $T\colon X\to Y$ is bounded.*

Proof. It is easily seen that T is bounded with respect to one norm on X if and only if it is bounded with respect to any other equivalent norm. By Theorem 1.6, we may thus assume that the norm on X is given by $\|\sum_{j=1}^n\alpha_j\mathrm{e}_j\|=\max_{1\leq j\leq n}|\alpha_j|$, where $n=\dim X$ and $(\mathrm{e}_1,\dots,\mathrm{e}_n)$ is any Hamel basis of X. Then, for any $x=\sum_{j=1}^n\alpha_j\mathrm{e}_j\in X$, we have

$$\|Tx\|=\Big\|\sum_{j=1}^n\alpha_jT\mathrm{e}_j\Big\|\leq\sum_{j=1}^n|\alpha_j|\|T\mathrm{e}_j\|\leq C\|x\|,$$

where $C\coloneqq\sum_{j=1}^n\|T\mathrm{e}_j\|$. Hence, T is bounded and $\|T\|\leq C$. □

Lemma 2.4. *Let X be a vector space over $\mathbb{C}$. Then:*

(a) *if $\varphi\colon X\to\mathbb{C}$ is a $\mathbb{C}$-linear functional and $u=\operatorname{Re}f$, then*

$$\varphi(x)=u(x)-\mathrm{i}u(\mathrm{i}x)\quad \textit{for every } x\in X; \tag{2.1}$$

(b) *if $u\colon X\to\mathbb{R}$ is any $\mathbb{R}$-linear functional, then formula* (2.1) *defines a $\mathbb{C}$-linear functional φ on X;*

(c) *if X is also a normed space and linear functionals $\varphi\colon X\to\mathbb{C}$ and $u\colon X\to\mathbb{R}$ satisfy equation* (2.1)*, then $\|\varphi\|=\|u\|$.*

Proof. Clauses (a) and (b) are left as an exercise **(Problem 1.9)**. For proving (c), observe that obviously $\|\varphi\|\geq\|u\|$. For the converse inequality, fix any $x\in X$ and pick $\alpha\in\mathbb{C}$ with $|\alpha|=1$ and such that $|\varphi(x)|=\alpha\varphi(x)$. Then,

$$|\varphi(x)|=\alpha\varphi(x)=\varphi(\alpha x)=\operatorname{Re}\varphi(\alpha x)=u(\alpha x)\leq\|u\|\|\alpha x\|=\|u\|\|x\|,$$

whence $\|\varphi\|\leq\|u\|$. □

Theorem 2.5 (Hahn–Banach theorem). *Let X be a normed space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ and let $M\subset X$ be a subspace of X. Every continuous linear functional $f\colon M\to\mathbb{K}$ admits a norm preserving extension, that is, there exists a continuous linear functional $F\colon X\to\mathbb{K}$ such that $F|_M=f$ and $\|f\|=\|F\|$.*

Proof. In the first part of the proof we assume that $\mathbb{K} = \mathbb{R}$. First, we show how to extend any functional to a subspace larger by one dimension while preserving its norm.

Suppose $M' \subsetneq X$ is a proper linear subspace of X and $f' \colon M' \to \mathbb{R}$ is a continuous linear functional. Let also $x_0 \in X \setminus M'$. We seek for a continuous linear functional $\widetilde{f} \colon \mathrm{lin}(M' \cup \{x_0\}) \to \mathbb{R}$ such that

$$\widetilde{f}|_{M'} = f' \quad \text{ and } \quad \|\widetilde{f}\| = \|f'\|. \tag{2.2}$$

Noticing that every vector from $\mathrm{lin}(M' \cup \{x_0\})$ is of the form $x + \lambda x_0$ for some $x \in M'$ and $\lambda \in \mathbb{R}$, we set

$$\widetilde{f}(x + \lambda x_0) = f'(x) + \lambda \xi \quad (x \in M',\ \lambda \in \mathbb{R}),$$

where ξ needs to be picked in such a way that the second condition in (2.2) holds true (the first one is already satisfied by the definition). Obviously, $\|\widetilde{f}\| \geq \|f'\|$ as $\widetilde{f}$ extends f'; we have to guarantee that the reversed inequality holds true.

By homogeneity, we can assume that $\|f'\| = 1$. Note that

$$\begin{aligned}
\|\widetilde{f}\| \leq 1 \quad &\iff \quad \forall_{x \in M',\, \lambda \in \mathbb{R}} \quad |\widetilde{f}(x + \lambda x_0)| \leq \|x + \lambda x_0\| \\
&\iff \quad \forall_{x \in M',\, \lambda \in \mathbb{R}} \quad |f'(x) + \lambda \xi| \leq \|x + \lambda x_0\| \\
&\iff \quad \forall_{x \in M'} \quad |f'(x) - \xi| \leq \|x - x_0\|,
\end{aligned}$$

where in the last step we divided the formula by $-\lambda$ (for $\lambda = 0$ it is trivially satisfied) and used the fact that $-\lambda^{-1} f'(x) = f'(-\lambda^{-1} x)$ and that $-\lambda^{-1} x \in M'$ if and only if $x \in M'$. Therefore, in order to have $\|\widetilde{f}\| \leq 1$ we must guarantee that the number ξ satisfies

$$f'(x) - \|x - x_0\| \leq \xi \leq f'(y) + \|y - x_0\| \quad \text{for all } x, y \in M'.$$

Such a number exists if and only if

$$\sup\big\{f'(x) - \|x - x_0\| \colon x \in M'\big\} \leq \inf\big\{f'(y) + \|y - x_0\| \colon y \in M'\big\}. \tag{2.3}$$

Observe that for any $x, y \in M'$ we have

$$f'(x) - f'(y) = f'(x - y) \leq \|x - y\| \leq \|x - x_0\| + \|y - x_0\|$$

which shows that (2.3) is indeed true. We have thus shown that it is possible to extend f' to the larger subspace $\mathrm{lin}(M' \cup \{x_0\})$ without increasing its norm.

Let $\mathcal{P}$ be the family of all pairs (M', f'), where $M' \subseteq X$ is a linear subspace of X with $M \subseteq M'$ and $f' \colon M' \to \mathbb{R}$ is a continuous linear functional extending f (i.e. $f'|_M = f$) and such that $\|f'\| = \|f\|$. Then $\mathcal{P}$ is partially ordered by the relation

$$(M', f') \preceq (M'', f'') \quad \iff \quad M' \subseteq M'' \text{ and } f''|_{M'} = f'.$$

It is easy to see that every chain $\mathcal{C}$ in $\mathcal{P}$ has an upper bound. Indeed, the upper bound is $(\mathcal{M}, F)$, where

$$\mathcal{M} = \bigcup_{(M', f') \in \mathcal{C}} M' \quad \text{and} \quad F = \bigcup_{(M', f') \in \mathcal{C}} f'.$$

(Notice that defining F as the union of functions from $\mathcal{C}$ makes sense, because all the functions in $\mathcal{C}$ are each other's extensions/restrictions, so that for $x \in \mathcal{M}$ we have $F(x) = f'(x)$ for any f' whose domain contains x.) By virtue of the Kuratowski–Zorn lemma, there exists a maximal element $(M_{\max}, f_{\max})$ in $\mathcal{P}$. But in view of the first part of the proof, $M_{\max}$ must be the whole of X because otherwise we could extend $f_{\max}$ further to obtain a new pair from $\mathcal{P}$ which is $\preceq$-larger than $(M_{\max}, f_{\max})$. Thus, $f_{\max}$ is the desired extension of f.

Now, we proceed to the complex case, $\mathbb{K} = \mathbb{C}$. If f is a $\mathbb{C}$-linear continuous functional on M, let $u = \operatorname{Re} f$ and apply the just-proved real version of the theorem to u. We obtain an $\mathbb{R}$-linear functional $U\colon X \to \mathbb{R}$ such that $U|_M = u$ and $\|U\| = \|u\|$. Define $F(x) = U(x) - \mathrm{i}U(\mathrm{i}x)$ $(x \in X)$ and note that, in view of Lemma 2.4(a) and (b), F is a $\mathbb{C}$-linear functional which extends f. Moreover, by assertion (c) of that lemma, we also have $\|F\| = \|U\| = \|u\| = \|f\|$. □

The following corollary may be very useful in situations when we want to show that a certain element lies in the closure of a given subspace, and we know the general form of continuous linear functionals.

Corollary 2.6. *Let X be a normed space, $M \subseteq X$ a linear subspace of X and $x_0 \in X$. Then, $x_0 \in \overline{M}$ if and only if there does not exist $x^* \in X^*$ such that $x^*|_M = 0$ and $\langle x_0, x^* \rangle = 1$.*

Proof. **(classes)** □

Corollary 2.7. *Let X be a normed space. For every $x \in X$ there exists $x^* \in X^*$ with $\|x^*\| = 1$ and such that $\langle x, x^* \rangle = \|x\|$.*

Proof. **(classes)** □

Notice that Corollary 2.7 gives us the following useful formula:

$$\|x\| = \sup\{\langle x, x^* \rangle \colon x^* \in S_{X^*}\},$$

which may be regarded as a *dual* version of $\|x^*\| = \sup\{\langle x, x^* \rangle \colon x \in S_X\}$, the latter being just the definiton of the norm of a functional.