Functional analysis

Lecture 3: $\mathscr{L}(X,Y)$ AND X^* AS NORMED/BANACH SPACES; THE DUAL SPACES OF c_0, ℓ_p AND $L_p([0,1])$ $(1\leq p<\infty)$; THE RIESZ REPRESENTATION THEOREM FOR $(C[0,1])^*$

We finish Section 2 with an important characterization of continuity for linear functionals on normed spaces. First, we need the following lemma.

Lemma 2.8. *Let $M\subsetneq X$ be a closed subspace of a normed space X and let $u\in X\setminus M$. Then, we have:*

(i) $\|y+\lambda u\|\geq |\lambda|\cdot \mathrm{dist}(u,M)$ *for all $y\in M$, $\lambda\in\mathbb{K}$;*
(ii) *for any $(y_n)_{n=1}^\infty\subset M$ and $(\lambda_n)_{n=1}^\infty\subset\mathbb{K}$, the sequence $(y_n+\lambda_n u)_{n=1}^\infty$ is convergent in X if and only if the both sequences $(y_n)_{n=1}^\infty$ and $(\lambda_n)_{n=1}^\infty$ are convergent in M and $\mathbb{K}$, respectively;*
(iii) *the subspace $M+\mathbb{K}u=\mathrm{lin}(M\cup\{u\})$ is closed.*

Proof. (i) For $\lambda=0$ the assertion is trivial. For $\lambda\neq 0$, we have

$$\|y+\lambda u\|=|\lambda|\cdot\big\|u-(-\lambda^{-1}y)\big\|\geq|\lambda|\cdot\mathrm{dist}(u,Y).$$

(ii) Obviously, if both $(y_n)_{n=1}^\infty$ and $(\lambda_n)_{n=1}^\infty$ are convergent, then $(y_n+\lambda_n u)_{n=1}^\infty$ converges in X, as the algebraic operations are norm continuous. For the converse, suppose we have $y_n+\lambda_n u\to z\in X$. Since the sequence $(y_n+\lambda_n u)_{n=1}^\infty$ is Cauchy, assertion (i) implies that $(\lambda_n)_{n=1}^\infty\subset\mathbb{K}$ is also Cauchy, and hence convergent to some $\lambda_0\in\mathbb{K}$ (the scalar field is complete). Now, $\lambda_n u-\lambda_0 u\to 0$ and $y_n+\lambda_n u-\lambda_0 u\to z-\lambda_0 u$, whence $y_n\to z-\lambda_0 u$. Hence, $(y_n)_{n=1}^\infty$ is also convergent and its limit belongs to M, as M is closed.

(iii) It follows readily from assertion (ii). ☐

Corollary 2.9. *Let X be a normed space. If M and F are linear subspaces of X such that M is closed and F is finite-dimensional, then the subspace $M+F$ is closed.*

Proof. Follows from Lemma 2.8(iii) by a simple induction. ☐

Proposition 2.10. *Let X be a normed space over $\mathbb{K}$ and $\varphi\colon X\to\mathbb{K}$ a linear functional. Then, φ is continuous if and only if its kernel $\ker\varphi$ is closed.*

Proof. The necessity is clear. For sufficiency, assume $\ker\varphi$ is a closed subspace of X. If $\ker\varphi=X$, there is nothing to prove. Otherwise, pick any $u\in X\setminus\ker\varphi$, so that we have $\ker\varphi+\mathbb{K}u=X$ (see **Problem 1.12(a)**). Obviously, $\varphi(y+\lambda u)=\lambda\varphi(u)$ for all $y\in\ker\varphi$ and $\lambda\in\mathbb{K}$.

Fix any sequence $(x_n)_{n=1}^\infty=(y_n+\lambda_n u)_{n=1}^\infty\subset X$ which converges to some $x_0\in X$. In view of Lemma 2.8(ii) and our assumption, there exist $y_0\in\ker\varphi$ and $\lambda_0\in\mathbb{K}$ such that $y_n\to y_0$ and $\lambda_n\to\lambda_0$. Hence, $x_0=y_0+\lambda_0 u$ and we have

$$\varphi(x_n)=\lambda_n\varphi(u)\xrightarrow[n\to\infty]{}\lambda_0\varphi(u)=\varphi(x_0)$$

which shows that φ is continuous. ☐

Remark. The above result appeared in some form in **Problem 1.13**. Combining the assertion of that problem with Proposition 2.10, we infer that if $\varphi\neq 0$ is a linear functional, then the continuity of φ is equivalent to the closedness of $\ker\varphi$ which is in turn equivalent to $\ker\varphi$ not being dense.

3 Dual spaces and Riesz representation theorems

The main goal of this section is to give several characterizations of dual spaces of the most classical Banach spaces. All these duality results are basically due to F. Riesz.

We start with an easy observation that the set of bounded linear operators/functionals forms a normed space in its own right, and that it is complete if and only if the codomain space is complete. Of course, the linear operations considered in $\mathscr{L}(X,Y)$ are the pointwise ones: $(S+T)x = Sx + Tx$ and $(\lambda T)x = \lambda Tx$, and the norm is considered to be the operator norm as in Definition 2.1.

Proposition 3.1. *Let X and Y be normed spaces. Then $\mathscr{L}(X,Y)$ is a normed space as well. It is complete if and only if Y is complete. In particular, for any normed space X, the dual space X^* is a Banach space.*

Proof. **(classes)** □

Definition 3.2. Let X, Y be normed spaces over the same scalar field $\mathbb{R}$ or $\mathbb{C}$ and let $T\colon X \to Y$ be a linear operator. We call T:

- an *isomorphism* (or *normed space isomorphism*) provided that T is a bijection such that both T and T^{-1} are bounded, i.e. $T \in \mathscr{L}(X,Y)$ and $T^{-1} \in \mathscr{L}(Y,X)$,
- an *isometry* provided that $\|Tx\| = \|x\|$ for each $x \in X$,
- an *isometric isomorphism* provided that it is a bijective isometry (so, an isometry and an isomorphism at the same time),
- an *isomorphic embedding* if it is an isomorphism onto its range, i.e. $T\colon X \to T(X)$ is an isomorphism,
- an *isometric embedding* provided it is an isometry and an isomorphic embedding,
- *bounded below* provided that there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for every $x \in X$.

Remarks. (a) Plainly, for any linear isometry T we have $\|T\| = 1$. Thus, for any isometric embedding T we have $\|T\| = \|T^{-1}\| = 1$ (the inverse being defined on the range of T).

(b) It is easy to see that $T \in \mathscr{L}(X,Y)$ is an isomorphic embedding if and only if it is bounded below, which means that for some $\delta > 0$ we have

$$\delta\|x\| \leq \|Tx\| \leq \|T\|\|x\| \quad \text{for every } x \in X. \tag{3.1}$$

Indeed, inequality (3.1) easily implies that T is one-to-one, so T is a bijection onto the range of T. Then, T^{-1} is linear and also bounded, because (3.1) yields that $\|T^{-1}\| \leq \delta^{-1}$. Conversely, if T is an isomorphic embedding, then inequality (3.1) follows from the definition of operator norm.

(c) It follows from Theorem 1.6 that all normed spaces of a given finite dimension $n \in \mathbb{N}$ are mutually isomorphic (but, of course, not isometric). Indeed, consider any normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ with $\dim X = \dim Y = n < \infty$. Picking any Hamel bases, we get the associated linear isomorphisms $\varphi\colon X \to \mathbb{K}^n$ and $\psi\colon Y \to \mathbb{K}^n$. Now, consider the linear isomorphism $\psi^{-1} \circ \varphi\colon X \to Y$. It is easily seen that the formula $\|x\|' \coloneqq \|(\psi^{-1} \circ \varphi)x\|_Y$ defines a new norm on X, therefore equivalent to $\|\cdot\|_X$. This means that for some constants $0 < c \leq C < \infty$ we have $c\|x\|_X \leq \|(\psi^{-1} \circ \varphi)x\|_Y \leq C\|x\|_X$, that is, $\psi^{-1} \circ \varphi$ is an isomorphism between the normed spaces X and Y.

(d) Any normed space isomorphism yields a linear and topological isomorphism (homeomorphism) at the same time. Thus, isomorphic normed spaces share the same structures from the point of view of algebra and topology. However, unless they are isometrically isomorphic, their geometries may be different—in particular, the shapes of their unit balls. Compare, for example, the unit balls of ℓ_1^2, ℓ_2^2 and ℓ_∞^2. We have $\ell_1^2 \cong \ell_\infty^2$ but neither of these spaces is isometrically isomorphic to ℓ_2^2 (why?).

For any normed spaces X and Y, we will use the following notation:

- $X \sim Y$ if X and Y are isomorphic, i.e. there exists an isomorphism $T \in \mathscr{L}(X,Y)$,
- $X \cong Y$ if X and Y are isometrically isomorphic, i.e. there exists an isometric isomorphism $T \in \mathscr{L}(X,Y)$,
- $X \hookrightarrow Y$ if X isomorphically embeds in Y, i.e. there exists an isomorphic embedding $T \in \mathscr{L}(X,Y)$,
- $X \overset{1}{\hookrightarrow} Y$ if X isometrically embeds into Y, i.e. there exists an isometric embedding $T \in \mathscr{L}(X,Y)$.

Proposition 3.3. *We have the following isometric isomorphisms:*

(a) $c_0^* \cong \ell_1$,

(b) $\ell_1^* \cong \ell_\infty$,

(c) $\ell_p^* \cong \ell_q$, *where* $p,q \in (1,\infty)$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

More precisely, for every functional $\varphi \in c_0^*$ *(resp.* ℓ_1^*, ℓ_p^**) there exists a unique sequence* $(a_i)_{i=1}^\infty \in \ell_1$ *(resp.* ℓ_∞, ℓ_q*) such that*

$$\varphi(x) = \sum_{i=1}^\infty a_i x_i \quad \text{for every } x=(x_i)_{i=1}^\infty \in c_0 \ (\text{resp. } \ell_1,\ \ell_p) \tag{3.2}$$

and, moreover, $\|\varphi\| = \|(a_i)_{i=1}^\infty\|_1$ *(resp.* $\|(a_i)_{i=1}^\infty\|_\infty$, $\|(a_i)_{i=1}^\infty\|_q$*). On the other hand, any such* $(a_i)_{i=1}^\infty$ *gives rise to a continuous linear functional via formula* (3.2)*, and hence the map* $\varphi \mapsto (a_i)_{i=1}^\infty$ *yields the desired isometric isomorphism.*

Proof. Clauses (a) and (b) are easier and are left as an exercise **(classes)**. We prove assertion (c).

Let $p,q \in (1,\infty)$ be conjugate exponents and fix any $\varphi \in \ell_p^*$. Define

$$a_i = \varphi(\mathrm{e}_i), \qquad \text{where } \mathrm{e}_i = (0,\dots,0,\underset{i}{1},0,0,\dots)$$

is the i^{th} vector of the canonical basis[1]. For any $x=(x_i)_{i=1}^\infty \in \ell_p$ and $n \in \mathbb{N}$ let

$$x^{(n)} = (x_1,\dots,x_n,0,0,\dots)$$

[1]The term *basis* must be understood correctly. The system $(\mathrm{e}_i\colon i\in\mathbb{N})$ is **not** a Hamel (algebraic) basis for any of the spaces c_0 or ℓ_p $(1 \le p \le \infty)$, because it spans only the space c_{00} of finitely supported sequences, i.e.

- $c_{00} = \{(x_n)_{n=1}^\infty \in \mathbb{K}^\mathbb{N}\colon x_n \neq 0 \text{ for finitely many } n\text{'s}\} = \mathrm{lin}\{\mathrm{e}_i\colon i\in\mathbb{N}\}$.

However, it forms a so-called *Schauder basis* (but not for ℓ_∞ as this one is nonseparable). We have not introduced this notion yet, but we use the word 'basis' just for terminological purposes.

and notice that $\|x^{(n)}-x\|_p\to 0$ as $n\to\infty$[2]. Since φ is linear and continuous, we have

$$\sum_{i=1}^n a_i x_i=\varphi\Big(\sum_{i=1}^n x_i \mathrm{e}_i\Big)=\varphi(x^{(n)})\xrightarrow[n\to\infty]{}\varphi(x).$$

This shows that formula (3.2) holds true, in particular, the series at the right-hand side is convergent for every $x\in\ell_p$.

Now, we want to show that $(a_i)_{i=1}^\infty\in\ell_q$ and estimate its ℓ_q-norm. For $n\in\mathbb{N}$, define

$$z^{(n)}=\Big(\sum_{i=1}^n |a_i|^q\Big)^{-1/p}\cdot\sum_{i=1}^n \overline{a_i}\,|a_i|^{q-2}\mathrm{e}_i$$

(if for some i we have $a_i=0$, then we omit the corresponding summand, so we use the convention that $0^{q-2}=0$). Note that

$$\|z^{(n)}\|_p=\Big(\sum_{i=1}^n |a_i|^q\Big)^{-1/p}\cdot\Big(\sum_{i=1}^n (|a_i|^{q-1})^p\Big)^{1/p}=1,$$

as we have $p=\frac{q}{q-1}$. We also have

$$\varphi(z^{(n)})=\Big(\sum_{i=1}^n |a_i|^q\Big)^{-1/p}\cdot\sum_{i=1}^n a_i\overline{a_i}\,|a_i|^{q-2}=\Big(\sum_{i=1}^n |a_i|^q\Big)^{1-1/p}=\|(a_i)_{i=1}^\infty\|_q.$$

Therefore, the ℓ_q-norm of $(a_i)_{i=1}^\infty$ is finite and, moreover,

$$\|(a_i)_{i=1}^\infty\|_q=\varphi(z^{(n)})\leq\|\varphi\|\|z^{(n)}\|_p=\|\varphi\|. \tag{3.3}$$

To show that the above inequality is in fact equality, consider an arbitrary $(a_i)_{i=1}^\infty\in\ell_q$ and define φ by means of formula (3.2). That series is convergent and defines a continuous linear functional on ℓ_p, because by Hölder's inequality we have

$$\sum_{i=1}^\infty |a_i x_i|\leq\|(a_i)_{i=1}^\infty\|_q\cdot\|(x_i)_{i=1}^\infty\|_p<\infty.$$

This also shows that $\|\varphi\|\leq\|(a_i)_{i=1}^\infty\|_q$ which jointly with (3.3) yields that every $\varphi\in\ell_p^*$ corresponds to an element of ℓ_q with the same norm as φ, and *vice versa*. Clearly, such an element of ℓ_q is unique, so the map $\varphi\mapsto(a_i)_{i=1}^\infty$ is well-defined, obviously linear, and yields an isometric isomorphism. □

Remark 3.4. By a similar argument, one can show that analogous statements hold true for finite-dimensional versions of the sequence spaces c_0 and ℓ_p. Namely, for each $n\in\mathbb{N}$ and any $1\leq p,q\leq\infty$ satisfying $\frac{1}{p}+\frac{1}{q}=1$ we have

$$(\ell_p^n)^*\cong\ell_q^n,$$

where for $p=1$ we take $q=\infty$ and *vice versa* (recall that c_0^n coincides with ℓ_∞^n). The duality is given in the same way as in Proposition 3.3, that is, any functional $\varphi\in(\ell_p^n)^*$ corresponds to a unique sequence $(a_i)_{i=1}^n\in\ell_q^n$ via the formula

$$\varphi(x_1,\ldots,x_n)=\sum_{i=1}^n a_i x_i.$$

[2]This is not true in ℓ_∞ which is easily seen by considering e.g. the sequence $x=(1,1,\ldots)$.

Remark 3.5. In Proposition 3.3(c) we did not include the case $p = \infty$. This is, of course, not accidental as it turns out that

$$\ell_\infty^* \ncong \ell_1.$$

The space ℓ_∞^* is considerably larger than ℓ_1; see **Problems 2.13** and also **2.20**. Another argument for this is that ℓ_∞^* must be nonseparable (see **Problem 2.22**). This dual space can be described in terms of finitely addivite bounded measures on the σ-algebra of all subsets of $\mathbb{N}$, which will be the subject of Problem **3.15**.

Observe, however, that we have

$$\ell_1 \overset{1}{\hookrightarrow} \ell_\infty^*,$$

because $\ell_\infty^* \cong \ell_1^{**}$ and every normed space X embeds isometrically into its bidual X^{**} (every $x \in X$ corresponds to an element $\iota(x) \in X^{**}$ acting as $\langle x^*, \iota(x)\rangle = \langle x, x^*\rangle$).

Now, we proceed to very important duality results for spaces of integrable and continuous functions. Just for simplicity, we formulate and prove them for the unit interval $[0, 1]$. The first result is in fact valid for any interval (even unbounded) on $\mathbb{R}$ with basically the same proof, so we have e.g. $L_p(\mathbb{R})^* \cong L_q(\mathbb{R})$ for any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. The second result, Theorem 3.9, holds true for any compact interval. Later, we will prove a much more general Riesz representation theorem for the dual of $C_0(K)$, where K is any locally compact Hausdorff space.

Recall that a real- or complex-valued function f defined on a (possibly unbounded) interval $I \subseteq \mathbb{R}$ is called *absolutely continuous* if for each $\varepsilon > 0$ there is $\delta > 0$ satisfying the following condition: for any finite collection $\{[\tau_i, t_i] \colon 1 \le i \le n\}$ of subintervals of I with mutually disjoint interiors we have

$$\sum_{i=1}^{n} (t_i - \tau_i) < \delta \implies \sum_{i=1}^{n} |f(t_i) - f(\tau_i)| < \varepsilon.$$

Absolutely continuous functions are a.e. differentiable and satisfy the Newton–Leibniz formula, that is,

$$f(x) - f(a) = \int_a^x f'(t)\,\mathrm{d}t.$$

This result is sometimes referred to as *Lebesgue's Fundamental Theorem of Calculus* (see, e.g., [W. Rudin, *Real and complex analysis*; Thm. 8.18]).

Proposition 3.6. *Let* $1 < p, q < \infty$ *be conjugate exponents, that is,* $\frac{1}{p} + \frac{1}{q} = 1$. *For every functional* $\Lambda \in (L_p[0, 1])^*$ *there exists a unique* $f \in L_q[0, 1]$ *such that*

$$\Lambda g = \int_0^1 f(x) g(x)\,\mathrm{d}x \quad \textit{for every } g \in L_p[0, 1] \tag{3.4}$$

and, moreover, $\|\Lambda\| = \|f\|_q$. *On the other hand, any* $f \in L_q[0, 1]$ *gives rise to a continuous linear functional on* $L_p[0, 1]$ *via formula* (3.4). *Consequently, the map* $\Lambda \mapsto f$ *yields an isometric isomorphism*

$$(L_p[0, 1])^* \cong L_q[0, 1].$$

Proof. We prove this theorem in the case $\mathbb{K} = \mathbb{R}$. However, it is not too difficult to modify it to work in the complex case as well, and the proof of Proposition 3.3 can provide a hint **(Problem 3.16)**.

Fix any $\Lambda \in (L_p[0,1])^*$ and define a function $\alpha\colon [0,1] \to \mathbb{R}$ by

$$\alpha(t) = \Lambda u_t, \qquad \text{where } u_t = \mathbb{1}_{[0,t)}.$$

Claim 1. α is absolutely continuous on $[0,1]$.

Fix any finite collection $\{[\tau_i, t_i]\colon 1 \leq i \leq n\}$ of subintervals of $[0,1]$ having mutually disjoint interiors. Set $\varepsilon_i = \operatorname{sgn}(\alpha(t_i) - \alpha(\tau_i))$ for $1 \leq i \leq n$ and estimate:

$$\begin{aligned}
\sum_{i=1}^{n} |\alpha(t_i) - \alpha(\tau_i)| &= \sum_{i=1}^{n} \varepsilon_i(\alpha(t_i) - \alpha(\tau_i)) = \Lambda\Big(\sum_{i=1}^{n} \varepsilon_i(u_{t_i} - u_{\tau_i})\Big) \\
&\leq \|\Lambda\| \cdot \Big\| \sum_{i=1}^{n} \varepsilon_i(u_{t_i} - u_{\tau_i}) \Big\|_{L_p} \\
&= \|\Lambda\| \cdot \Big\{ \int_0^1 \Big| \sum_{i=1}^{n} \varepsilon_i(u_{t_i}(x) - u_{\tau_i}(x)) \Big| \,\mathrm{d}x \Big\}^{1/p} \\
&= \|\Lambda\| \cdot \Big\{ \sum_{i=1}^{n} \int_{\tau_i}^{t_i} \mathrm{d}x \Big\}^{1/p} = \|\Lambda\| \cdot \Big(\sum_{i=1}^{n} (t_i - \tau_i) \Big)^{1/p}.
\end{aligned}$$

This shows that the initial expression can be arbitrarily small, provided that the sum of lengths of the intervals from our collection is sufficiently small. So, the requirement from the definition of absolute continuity is satisfied and Claim 1 has been proved.

By the Lebesgue Fundamental Theorem of Calculus, the derivative α' exists a.e. on $[0,1]$ and for every $t \in [0,1]$ we have

$$\alpha(t) = \alpha(t) - \alpha(0) = \int_0^t \alpha'(x)\,\mathrm{d}x \tag{3.5}$$

(note that $\alpha(0) = \Lambda u_0 = 0$). Consider $f = \alpha'$ as a function defined a.e.

Claim 2. $\Lambda g = \displaystyle\int_0^1 f(x)g(x)\,\mathrm{d}x$ for every bounded measurable function $g\colon [0,1] \to \mathbb{R}$.

First, note that in view of (3.5), we have

$$\Lambda u_t = \alpha(t) = \int_0^t f(x)\,\mathrm{d}x = \int_0^1 f(x)u_t(x)\,\mathrm{d}x,$$

so the required formula holds for every u_t ($t \in [0,1]$). Secondly, as Λ is linear, our formula also holds for every function g of the form

$$g_n = \sum_{k=1}^{n} c_k\big(u_{\frac{k}{n}} - u_{\frac{k-1}{n}}\big) \quad (n \in \mathbb{N},\ c_k \in \mathbb{R}).$$

Now, we use the fact that every bounded measurable function g on $[0,1]$ is an a.e. pointwise limit of a sequence of *step functions*, that is, finite linear combinations of indicator functions of intervals[3]. It is therefore possible to find a uniformly bounded sequence $(g_n)_{n=1}^{\infty}$ of measurable functions as in the formula above such that $g_n(x) \to g(x)$ a.e. Notice that by Lebesgue's Dominated Convergence Theorem, we have $\|g - g_n\|_{L_p} \to 0$, and hence $\Lambda g_n \to \Lambda g$. But using Lebesgue's theorem once again, we infer that

$$\Lambda g_n = \int_0^1 f(x) g_n(x)\,\mathrm{d}x \xrightarrow[n\to\infty]{} \int_0^1 f(x) g(x)\,\mathrm{d}x$$

which establishes Claim 2.

Claim 3. $f \in L_q[0,1]$ and $\|f\|_q \leq \|\Lambda\|$

Define a sequence $(h_n)_{n=1}^{\infty}$ of bounded measurable functions by the formula

$$h_n(x) = \begin{cases} \operatorname{sgn}(f(x)) \cdot |f(x)|^{q-1} & \text{if } |f(x)| \leq n \\ 0 & \text{if } |f(x)| > n. \end{cases}$$

Note that in view of Claim 2, f is integrable and hence $|h_n(x)| \to |f(x)|^{q-1}$ a.e. So, we can use h_n's in order to show that L_q-norm of f is finite, but first compute

$$\begin{aligned} \|h_n\|_p^p &= \int_0^1 |h_n(x)|^p\,\mathrm{d}x = \int_0^1 |h_n(x)| \cdot |h_n(x)|^{1/(q-1)}\,\mathrm{d}x \\ &\leq \int_0^1 |h_n(x)| \cdot |f(x)|\,\mathrm{d}x = \int_0^1 h_n(x) f(x)\,\mathrm{d}x = \Lambda h_n \leq \|\Lambda\| \cdot \|h_n\|_p, \end{aligned}$$

whence $\|h_n\|_p^{p/q} \leq \|\Lambda\|$. Using Fatou's lemma, we obtain

$$\|f\|_q = \left\{ \int_0^1 |f(x)|^{(q-1)p}\,\mathrm{d}x \right\}^{1/q} \leq \liminf_{n\to\infty} \left\{ \int_0^1 |h_n(x)|^p\,\mathrm{d}x \right\}^{1/q} \leq \|\Lambda\|$$

which proves Claim 3.

Claim 4. $\Lambda g = \displaystyle\int_0^1 f(x) g(x)\,\mathrm{d}x$ for every $g \in L_p[0,1]$.

Fix any $g \in L_p[0,1]$ and pick a sequence $(g_n)_{n=1}^{\infty}$ of bounded measurable functions such that $g_n(x) \to g(x)$ a.e. As we observed before, we then have $\|g_n - g\|_{L_p} \to 0$ and hence $\Lambda g_n \to \Lambda g$. Also, by Hölder's inequality and Claim 2, we have

$$\Lambda g_n = \int_0^1 f(x) g_n(x)\,\mathrm{d}x \xrightarrow[n\to\infty]{} \int_0^1 f(x) g(x)\,\mathrm{d}x,$$

so the resulting limit must be equal to Λg.

Now, every $f \in L_q[0,1]$ defines a functional $\Lambda \in (L_p[0,1])^*$ via formula (3.4). The existence of this integral, as well as continuity of Λ, follows from Hölder's inequality:

$$\int_0^1 |f(x) g(x)|\,\mathrm{d}x \leq \|f\|_{L_q} \cdot \|g\|_{L_p}.$$

[3] This is quite a standard fact from measure theory. It follows e.g. from a corollary of Luzin's theorem which implies that every bounded measurable function on $[0,1]$ is an a.e. pointwise limit of a sequence of continuous functions (see [W. Rudin, *Real and complex analysis*; Thm. 2.23 and the subsequent corollary]), and continuous functions can certainly be approximated by step functions.

Finally, the uniqueness of f is easy: If there were two functions $f, f' \in L_q[0,1]$ representing the same functional $\Lambda \in (L_p[0,1])^*$, then for every $g \in L_p[0,1]$ the integral of $(f - f')g$ would vanish, which easily implies that f and f' coincide a.e. Thus, the map $\Lambda \mapsto f$ is well-defined and yields an isometric isomorphism according to what we have shown above. □

Proposition 3.7. *For every functional $\Lambda \in (L_1[0,1])^*$ there exists a unique $f \in L_\infty[0,1]$ such that*

$$\Lambda g = \int_0^1 f(x)g(x)\,\mathrm{d}x \quad \textit{for every } g \in L_1[0,1] \tag{3.6}$$

and, moreover, $\|\Lambda\| = \|f\|_\infty$. On the other hand, any $f \in L_\infty[0,1]$ gives rise to a continuous linear functional on $L_1[0,1]$ via formula (3.6). Consequently, the map $\Lambda \mapsto f$ yields an isometric isomorphism

$$(L_1[0,1])^* \cong L_\infty[0,1].$$

Proof. The proof is similar to the one above **(classes)**. □

Remark 3.8. Similarly to Remark 3.5, we did not include $p = \infty$ in Proposition 3.6. In fact,

$$(L_\infty[0,1])^* \ncong L_1[0,1]$$

and $(L_\infty[0,1])^* = (L_1[0,1])^{**}$ is a huge space compared to $L_1[0,1]$ (which is separable).

In the future, we will prove that the duality between L_p- and L_q-spaces, for p and q being conjugate, holds true over any measure space with a σ-finite measure. But to this end, we need the Radon–Nikodym theorem which will be proved after discussing Hilbert space theory.

Now, we prove the Riesz representation theorem for the space of real-valued continuous functions on a compact interval. Before doing it, let us recall some necessary terminology and tools from real analysis.

A real- or complex-valued function f defined on an interval $[a,b]$ is said to have *bounded variation*, in which case we write $f \in \mathrm{BV}([a,b])$ provided that

$$V_a^b(f) := \sup\left\{\sum_{j=1}^n |f(x_j) - f(x_{j-1})| \colon a \le x_0 < \ldots < x_n \le b\right\} < \infty.$$

The left-hand side defines the *(total) variation* of f on the interval $[a,b]$. Two fundamental facts on functions of bounded variation are the following (see [W. Rudin, *Real and complex analysis*; Ch. 8]):

- $f \in \mathrm{BV}([a,b])$ if and only if there exist increasing functions g and h on $[a,b]$ such that $f = g - h$ (*Jordan decomposition theorem*). More precisely, we can define these functions by $g(x) = \frac{1}{2}(V_a^x(f) + f(x))$ and $h(x) = \frac{1}{2}(V_a^x(f) - f(x))$.
- If $f \in \mathrm{BV}([a,b])$, then $f'(x)$ exists a.e. on $[a,b]$ and $f' \in L_1[a,b]$, i.e. f' is integrable. However, in this case the Newton–Leibniz formula for f may fail. In fact, there exist monotone (even strictly monotone) *singular* functions f (i.e. nonconstant, continuous and such that $f'(x) = 0$ a.e.). A classical example is the *Cantor staircase function* on $[0,1]$ whose derivative vanishes outside the Cantor set.

Theorem 3.9 (Riesz). *For every $\mathbb{R}$-linear functional $\Lambda \in (C[0,1])^*$ there exists a real-valued function $f \in \mathrm{BV}([0,1])$ such that*

$$\Lambda g = \int_0^1 g \,\mathrm{d}f \quad \textit{for every } g \in C[0,1] \tag{3.7}$$

(the Riemann–Stieltjes integral) and, moreover, $\|\Lambda\| = V_0^1(f)$. On the other hand, any real-valued function $f \in \mathrm{BV}([0,1])$ gives rise to a continuous linear functional Λ on $C[0,1]$ by means of formula (3.7).

Proof. First, assume $\Lambda \in (C[0,1])^*$. Consider the space $\ell_\infty[0,1]$ of all bounded real-valued functions on $[0,1]$, equipped with the supremum norm. By standard arguments, $\ell_\infty[0,1]$ is a Banach space and $C[0,1]$ is its closed subspace. In view of the Hahn–Banach theorem, we can extend Λ to a functional $\widetilde{\Lambda}\colon \ell_\infty[0,1] \to \mathbb{R}$ preserving its norm, so that we have

$$|\widetilde{\Lambda} g| \leq \|\Lambda\| \cdot \|g\|_\infty \quad \text{for each } g \in \ell_\infty[0,1].$$

For any $t \in [0,1]$ consider $u_t = \mathbb{1}_{[0,t)} \in \ell_\infty[0,1]$ and define $f\colon [0,1] \to \mathbb{R}$ by $f(t) = \widetilde{\Lambda} u_t$.
Claim. $f \in \mathrm{BV}([0,1])$ and $V_0^1(f) \leq \|\Lambda\|$.
Indeed, fix any partition $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ and set

$$\varepsilon_i = \mathrm{sgn}(f(t_i) - f(t_{i-1})) \quad \text{for } 1 \leq i \leq n.$$

Then,

$$\begin{aligned}\sum_{i=1}^n |f(t_i) - f(t_{i-1})| &= \sum_{i=1}^n \varepsilon_i (f(t_i) - f(t_{i-1})) = \widetilde{\Lambda}\Big(\sum_{i=1}^n \varepsilon_i (u_{t_i} - u_{t_{i-1}})\Big) \\ &\leq \|\widetilde{\Lambda}\| \cdot \Big\| \sum_{i=1}^n \varepsilon_i (u_{t_i} - u_{t_{i-1}}) \Big\|_\infty = \|\widetilde{\Lambda}\| = \|\Lambda\|,\end{aligned}$$

which proves our Claim.

Fix any $g \in C[0,1]$ and consider a sequence $(g_n)_{n=1}^\infty \subset \ell_\infty[0,1]$ defined as

$$g_n = \sum_{k=1}^n g\Big(\frac{k}{n}\Big)(u_{\frac{k}{n}} - u_{\frac{k-1}{n}}) \quad (n \in \mathbb{N}).$$

As g is continuous, it is Riemann–Stieltjes integrable with respect to f. On the other hand,

$$\widetilde{\Lambda} g_n = \sum_{k=1}^n g\Big(\frac{k}{n}\Big)\Big(f\Big(\frac{k}{n}\Big) - f\Big(\frac{k-1}{n}\Big)\Big)$$

is an integral sum for g corresponding to the partion $\{\frac{k}{n}\colon 0 \leq k \leq n\}$. Hence,

$$\lim_{n\to\infty} \widetilde{\Lambda} g_n = \int_0^1 g \,\mathrm{d}f.$$

Since $\widetilde{\Lambda} \in (\ell_\infty[0,1])^*$ and $\|g_n - g\|_\infty \to 0$, we have

$$\widetilde{\Lambda} g = \widetilde{\Lambda}(\lim_{n\to\infty} g_n) = \lim_{n\to\infty} \widetilde{\Lambda} g_n = \int_0^1 g \, \mathrm{d}f,$$

and since $\widetilde{\Lambda}$ is an extension of Λ, we have proved formula (3.7). Moreover, in view of our Claim, the function f has all the desired properties.

On the other hand, from the general theory of Riemann–Stieltjes integral[4] we know that for each $f \in \mathrm{BV}([0,1])$, the functional $C[0,1] \ni g \mapsto \int_0^1 g \, \mathrm{d}f$ is linear and satisfies the estimate

$$\left| \int_0^1 g \, \mathrm{d}f \right| \le V_0^1(f) \cdot \|g\|_\infty.$$

Hence, this functional is bounded with norm not greater than (and hence equal to) $V_0^1(f)$. This completes the proof. □

[4]In the first step, the Riemann–Stieltjes integral is defined with respect to any monotone increasing function. Then, we can integrate over any function of bounded variation, as these are differences of increasing functions, in view of the Jordan decomposition. Every continuous function on a compact interval is Riemann–Stieltjes integrable; see [W. Rudin, *Principles of mathematical analysis*; Ch. 6].