Functional analysis

Lecture 4: Normalized functions of bounded variation; regular Borel measures; the Riesz–Markov–Kakutani theorem;

Remark 3.10. Assume that the functional Λ above is *positive* (i.e. $\Lambda f \geq 0$ for $f \geq 0$). We observe that the extension $\widetilde{\Lambda}$ is also positive and hence the constructed function f is then monotone increasing. Indeed, suppose that there is $g \in \ell_\infty[0,1]$ with $g(t) \geq 0$ for each $t \in [0,1]$ but $\widetilde{\Lambda} g < 0$. By normalizing, we can assume that $\|g\|_\infty = 1$, thus $0 \leq g(t) \leq 1$ for each $t \in [0,1]$. Let $\mathbf{1}$ be the constant 1 function. As Λ is positive and linear, it is monotone (i.e. $\Lambda g \leq \Lambda h$ if $g \leq h$), and hence $\Lambda \mathbf{1} = \|\Lambda\|$. We have $\|\mathbf{1} - g\|_\infty \leq 1$ and

$$\widetilde{\Lambda}(\mathbf{1} - g) = \Lambda \mathbf{1} - \widetilde{\Lambda} g = \|\Lambda\| - \widetilde{\Lambda} g > \|\Lambda\| = \|\widetilde{\Lambda}\|,$$

which gives a contradicition.

Therefore, positive continuous functionals on $C[a,b]$ correspond to increasing functions on $[a,b]$, while general continuous functionals correspond to functions of bounded variation.

Remark 3.11. Since every $f \in \mathrm{BV}([a,b])$ is the difference of two increasing functions, we see that every $\Lambda \in (C[a,b])^*$ can be written as

$$\Lambda = \Lambda^+ - \Lambda^-, \quad \text{where } \Lambda^+, \Lambda^- \text{ are positive.} \tag{3.1}$$

The conclusion of Remark 3.11 can be justified without appealing to Riesz' Theorem 3.9. To this end, let us use a more general language which will be also useful in the sequel.

By an *ordered vector space* we mean any vector space E over $\mathbb{R}$ equipped with a partial order relation $\geq$ compatible with the algebraic operations in the sense that $x \geq y$ implies $x + z \geq y + z$ for all $x, y, z \in E$ and $x \geq y$ implies $\lambda x \geq \lambda y$ for all $x, y \in E$, $\lambda \geq 0$. We denote by E^+ the *positive cone* of E defined by $\{x \in E \colon x \geq 0\}$. An ordered vector space which is also a lattice with respect to the given order is called a *Riesz space* or a *vector lattice*.

If E is a Riesz space and $\vee$, $\wedge$ denote the lattice operations, then for any vector $x \in E$ we define its *positive part* x^+, *negative part* x^- and its modulus $|x|$ by

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad |x| = x \vee (-x).$$

We then have $x = x^+ - x^-$ and $|x| = x^+ + x^-$, so this gives a decomposition of x as the difference between two positive elements. A Riesz space is called *Archimedean* if whenever $0 \leq nx \leq y$ for all $n \in \mathbb{N}$ and some $y \in E^+$, then $x = 0$.

If E is a Riesz space and a normed space with a norm $\|\cdot\|$ satisfying the condition

$$|x| \leq |y| \implies \|x\| \leq \|y\| \quad \text{for all } x, y \in E,$$

then E is called a *normed Riesz space*, and in the case where the norm is complete, we call E a *Banach lattice*. There are plenty of examples of Banach lattices: $\mathbb{R}^n$ with the pointwise ordering and any of the norms $\|\cdot\|_p$ $(1 \leq p \leq \infty)$, the space $C(X)$ of continuous functions, and $C_b(X)$ of bounded continuous functions on any topological space X, also considered with the pointwise ordering and the supremum norm, as well as $L_p(\mu)$-spaces under the a.e. pointwise ordering.

If E is a normed Riesz space, then its dual E^* can be ordered in a natural way by defining $f \geq g$ if and only if $f(x) \geq g(x)$ for every $x \in E^+$ ($f, g \in E^*$). We will see (Lemma 3.18 below) that in this way we equip E^* with a Banach lattice structure[1]. In particular, the decomposition (3.1) is possible in the Banach lattice $(C[a,b])^*$.

Actually, decomposition (3.1) holds true with a further requirement that

$$\|\Lambda\| = \|\Lambda^+\| + \|\Lambda^-\|$$

and in a much more general setting of the so-called *order unit normed Riesz spaces* (see **Problem 3.22**). For $C[a,b]$ we will obtain that equality with the aid of Hahn's decomposition theorem.

As we know that every continuous linear functional on the real space $C[a,b]$ is represented by a function $f \in \mathrm{BV}[a,b]$, let us consider the problem of uniqueness of this representation.

Obviously, f is not unique as stated in Theorem 3.9, because adding any constant function to f does not change the Riemann–Stieltjes integral. So, one normalization would be to require that e.g. $f(a) = 0$. By Jordan's decomposition, we have $f = g - h$ for some increasing functions g and h, and therefore f has at most countably many points of discontinuity in (a,b). Observe that by modifying the values of f at each of these points, but keeping the variation bounded, we do not change the value of the integral in formula (3.7). Indeed, any $g \in C[a,b]$ is Riemann–Stieltjes integrable, hence its integral can be approximated by any sequence of integral sums over partitions with diameters converging to zero[2]. In particular, we can avoid each point of discontinuity of f (except the endpoints) in these partitions. Consequently, another normalization condition which does not affect the validity of (3.7) is that $f(t+) = f(t)$ for each $t \in (a,b)$, i.e. f is right-continuous inside the interval (a,b) (the choice of the left or right limit is arbitrary). Every $f \in \mathrm{BV}([a,b])$ satisfying the above two normalization conditions is called a *normalized* function of bounded variation, and we denote the class of all such functions by

- $\mathrm{NBV}([a,b]) = \{f \in \mathrm{BV}([a,b])\colon\, f(a) = 0 \text{ and } f \text{ is right-continuous in } (a,b)\}$.

By modifying the representing function f in Theorem 3.9 as described above, we can guarantee that it belongs to the class $\mathrm{NBV}([a,b])$. Moreover, it turns out that in this class the function f is *unique*. We will obtain this as a particular case of the 'uniqueness part' of Theorem 3.16 below. Before proceeding to that result, let us build a 'bridge' between the special Riesz Theorem 3.9 and the much more general Riesz–Markov–Kakutani Theorem 3.16.

Given any function $f \in \mathrm{BV}([a,b])$, we define a measure μ_f on the field $\mathcal{F}$ consisting of finite unions of the intervals $(y,x]$ ($a \leq y < x \leq b$) which are all right-closed and left-open unless $y = a$ in which case we take the closed interval $[a,x]$. Namely:

- $\mu_f([a,x]) = f(x) - f(a)$,
- $\mu_f((y,x]) = f(x) - f(y)$ for $a < y < x \leq b$.

[1] An interested reader may consult e.g. [C.D. Aliprantis, K.C. Border, *Infinite dimensional analysis. A hitchhiker's guide*, 3rd edition, Springer 2006; Ch. 8 and 9].

[2] See [W Rudin, *Principles of mathematical analysis*; Thms. 6.6 and 6.8].

Notice that the measure μ_f is finitely additive and satisfies

$$\mu_f(E) = \sum_{j=1}^{n}(f(b_j) - f(a_j)) \quad \text{for } E = |a_1, b_1] \cup \bigcup_{j=2}^{n}(a_j, b_j] \in \mathcal{F},$$

where $a \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n \le b$. Moreover, μ_f is bounded because f is of bounded variation. But, as we will see, μ_f has also another nice property of being *regular*, provided that f belongs to the class $\mathrm{NBV}([a, b])$.

Definition 3.12. Let $\mathcal{F}$ be a field of subsets of a topological space X and $\mu\colon \mathcal{F} \to \mathbb{K}$ be a finitely additive set function, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We say μ is *regular* if for every $E \in \mathcal{F}$ and $\varepsilon > 0$ there exist a set $F \in \mathcal{F}$ with $\overline{F} \subseteq E$ and a set $G \in \mathcal{F}$ with $\operatorname{int} G \supseteq E$ satisfying

$$|\mu(C)| < \varepsilon \quad \text{for every } C \in \mathcal{F},\ C \subseteq G \setminus F.$$

If $f \in \mathrm{NBV}([a, b])$, it is possible to extend μ_f to a regular σ-additive measure defined on all Borel subsets of $[a, b]$. It follows from the following general result on extensions of measures. (Actually, we only need a simplified version of this result, as the measure μ_f is trivially σ-additive in the sense that there are no countably infinite, but not finite, unions of the intervals $|c, d]$ in $\mathcal{F}$.)

Extension theorem. *Let $\mathcal{F}$ be a field of subsets of some topological space X and let $\mu\colon \mathcal{F} \to \mathbb{K}$ be a bounded, regular, finitely additive set function, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then, μ has a unique regular, σ-additive extension $\widetilde{\mu}\colon \sigma(\Sigma) \to \mathbb{K}$ to the σ-algebra $\sigma(\mathcal{F})$ generated by $\mathcal{F}$.*

For the proof see e.g. [N. Dunford, J.T. Schwartz, *Linear operators* (vol. 1: *General theory*), Wiley–Interscience 1988; Ch. III.5]. This theorem is a combination of two classical measure theory results:

- the Alexandroff theorem which says that every bounded regular finitely additive (real- or complex-valued) set function on a field of subsets of a compact topological space is σ-additive;
- the Carathéodory extension theorem which guarantees that positive σ-additive measures defined on a ring $\mathcal{R}$ can be extended to positive σ-additive measures on the σ-algebra generated by $\mathcal{R}$.

(It should also be noted that a measure is regular if and only the positive and negative parts of its real and imaginary parts are regular.)

Lemma 3.13. *If $f \in \mathrm{BV}([a, b])$, then for every $t \in [a, b)$ we have*

$$\lim_{\varepsilon \to 0+} V(f, [t, t + \varepsilon]) = 0.$$

Proof. **(classes)** □

Now, assume that $f \in \mathrm{NBV}([a, b])$. Since f is right-continuous on (a, b), we have $\lim_{\varepsilon \to 0} f(t + |\varepsilon|) = f(t)$ for every $t \in (a, b)$ and it follows that for every interval $|c, d] \in \mathcal{F}$,

$$V(f, [c, d]) = \sup\Big\{ \sum_{j=1}^{n} |\mu_f(I_j)| \colon I_j \in \mathcal{F},\ I_j \cap I_k = \varnothing \text{ for } j \ne k,\ \textstyle\bigcup_{j=1}^{n} I_j \subseteq |c, d] \Big\}.$$

(The right-hand side defines the *variation* of μ_f on the set $|c, d]$; we will introduce this notion later on.) Plainly, we can extend this formula to any set $E \in \mathcal{F}$ in the sense that if $E = I_1 \cup \ldots \cup I_m$, with $I_j = |c_j, d_j] \in \mathcal{F}$ $(1 \le j \le m)$, then the left-hand side is $\sum_{j=1}^{m} V(f, [a_j, b_j])$, whereas the right-hand side is the variation of μ_f on E, denoted by $|\mu_f|(E)$. For any such E we can assume that $a \le c_1 < d_1 < c_2 < d_2 < \ldots < c_m < d_m \le b$. Pick

$$0 < \varepsilon < \min_{1 \le j \le m} (d_j - c_j), \min_{1 \le j < m} (c_{j+1} - d_j)$$

and define

$$E_1(\varepsilon) = \bigcup_{j=1}^{m} (c_j + \varepsilon, d_j], \quad E_2(\varepsilon) = \bigcup_{j=1}^{m} (c_j, d_j + \varepsilon].$$

(The interval $(c_1, d_1 + \varepsilon]$ should be replaced by $[a, d_1 + \varepsilon]$ in the case where $c_1 = a$.) Then, in view of the formula above and Lemma 3.13, we have

$$|\mu_f|(E \setminus E_1(\varepsilon)) = \sum_{j=1}^{m} V(f, [c_j, c_j + \varepsilon]) \xrightarrow[\varepsilon \to 0+]{} 0$$

and

$$|\mu_f|(E_2(\varepsilon) \setminus E) = \sum_{j=1}^{m} V(f, [d_j, d_j + \varepsilon]) \xrightarrow[\varepsilon \to 0+]{} 0.$$

This shows that μ_f is regular and thus the extension theorem applies.

For σ-additive measures we shall use the following common definition of regularity. It does not completely coincide with Definition 3.12 in general, but it does if X is a locally compact, σ-compact[3] Hausdorff space. In this case, a positive Borel measure on X is regular if and only if for every Borel set $E \subseteq X$ and each $\varepsilon > 0$ there exist a closed set F and an open set V such that $F \subseteq E \subseteq V$ and $\mu(V \setminus F) < \varepsilon$ **(classes)**.

Definition 3.14. Let μ be a positive Borel measure on a locally compact Hausdorff space X. A Borel set $E \subseteq X$ is called *outer regular* (resp. *inner regular*) if

$$\mu(E) = \inf \{\mu(V) \colon E \subseteq V,\ V \text{ is open}\}$$

$$(\text{resp.} \quad \mu(E) = \sup \{\mu(K) \colon K \subseteq E,\ K \text{ is compact}\}).$$

The measure μ is called *regular* if every Borel subset of X is both outer and inner regular.

We have proved that any $f \in \mathrm{NBV}([a, b])$ gives rise to a bounded, regular, finitely additive measure on $\mathcal{F}$ to which we can apply the extension theorem. Also, it is not difficult to see that, in the converse direction, the regularity of μ_f implies that f is right-continuous on (a, b). Summarizing, we have the following result.

Proposition 3.15. *Let $f \in \mathrm{BV}([a, b])$ (real- or complex-valued), $f(a) = 0$, let $\mathcal{F}$ be the field of finite unions of the intervals of the form $[a, d]$ and $(c, d]$ $(a < c < d \le b)$, and let $\mu_f \colon \mathcal{F} \to \mathbb{R}$ be the finitely additive (real- or complex-valued) measure associated with f as above. Then, μ_f is regular if and only if $f \in \mathrm{NBV}([a, b])$ in which case μ_f can be uniquely extended to a Borel, regular, σ-additive measure.*

[3] X is *σ-compact* if it is a countable union of compact sets.

In the Riesz–Markov–Kakutani theorem we deal with positive functionals on the space of compactly supported continuous functions. We use the following notation:

- $\mathrm{supp}(f) = \overline{\{x \in X \colon f(x) \neq 0\}}$
- $C_c(X) = \{f \colon X \to \mathbb{K} \mid \mathrm{supp}(f) \text{ is compact}\}$

Also, when given a topological space X, we write:

- $K \prec f$ if $K \subseteq X$ is compact, $f \in C_c(X)$, $0 \leq f \leq 1$ and $f|_K = 1$,
- $f \prec V$ if $V \subseteq X$ is open, $f \in C_c(X)$, $0 \leq f \leq 1$ and $\mathrm{supp}(f) \subset V$,
- $K \prec f \prec V$ if $K \prec f$ and $f \prec V$.

Two basic topological tools which are essential in the proof are the following results:

Urysohn's lemma. *Let X be a locally compact Hausdorff space, $V \subseteq X$ be an open set and $K \subset V$ a compact set. Then, there exists a function $f \in C_c(X)$ such that*

$$K \prec f \prec V.$$

Partition of unity. *Let X be a locally compact Hausdorff space. Suppose $V_1, \ldots, V_n \subseteq X$ are open set and K is a compact set, $K \subset V_1 \cup \ldots \cup V_n$. Then, there exist functions $h_i \prec V_i$ (for $1 \leq i \leq n$) satisfying*

$$h_1(x) + \ldots + h_n(x) = 1 \quad \text{for every } x \in K.$$

In what follows, the space $C_c(X)$ can consist of real- or complex-valued continuous functions with compact support. Correspondingly, Λ can be real-valued and $\mathbb{R}$-linear, as well as complex-valued and $\mathbb{C}$-linear. However, the crucial assumption is that Λ is positive. Notice that the measure μ produced by the theorem below can be infinite. For example, one can think of $X = \mathbb{R}$ (or, more generally $\mathbb{R}^k$) and $\Lambda f = \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$ in which case the measure μ is the Lebesgue measure on $\mathbb{R}$.

Theorem 3.16 (Riesz–Markov–Kakutani). *Let X be a locally compact Hausdorff space and Λ be a positive linear functional on $C_c(X)$. Then, there exist: a σ-algebra $\mathfrak{M}$ in X which contains all Borel subsets of X, and a positive measure μ on $\mathfrak{M}$ satisfying the following conditions:*

(a) $\Lambda f = \int_X f\,\mathrm{d}\mu$ *for every* $f \in C_c(X)$*;*

(b) *for every compact set $K \subset X$ we have $\mu(K) < \infty$;*

(c) *for every $E \in \mathfrak{M}$,*

$$\mu(E) = \inf\big\{\mu(V) \colon E \subseteq V,\ V \text{ is open}\big\};$$

(d) *for every open set E and every $E \in \mathfrak{M}$ with $\mu(E) < \infty$,*

$$\mu(E) = \sup\big\{\mu(K) \colon K \subseteq E,\ E \text{ is compact}\big\};$$

(e) *$(X, \mathfrak{M}, \mu)$ is complete, i.e. if $E \in \mathfrak{M}$, $\mu(E) = 0$ and $A \subset E$, then $A \in \mathfrak{M}$.*

Moreover, the measure μ is unique in the class of positive measures on $\mathfrak{M}$ satisfying conditions (a)–(d).

Proof. The proof consists of three main steps. First, we show that there is at most one measure μ satisfying the announced properties. Next, we provide definitions of μ and $\mathfrak{M}$, together with an auxiliary class $\mathfrak{M}_F$. We do it in such a way that assertion (c) follows from the very definition, while (e) is trivial. Finally, and this will be the toughest part, we shall show that $\mathfrak{M}$ is a σ-algebra containing all Borel set, μ is σ-additive on $\mathfrak{M}$ and satisfies (a), (b) and (d). For clarity, the last, most complicated step will be split into ten parts.

Uniqueness. If $\mathfrak{M}$ and μ satisfy (c) and (d), then μ is completely determined by its values on compact sets. Hence, if ν is another positive measure on $\mathfrak{M}$ satisfying (a)–(d), it is enough to show that $\mu(K) = \nu(K)$ for every compact set $K \subset X$. For any $\varepsilon > 0$, conditions (b) and (c) imply that there exists an open set V such that $K \subset V$ and $\nu(V) < \nu(K) + \varepsilon$. By Urysohn's lemma, there is a function f with $K \prec f \prec V$, and hence

$$\mu(K) = \int_X \mathbb{1}_K \, \mathrm{d}\mu \leq \int_X f \, \mathrm{d}\mu = \Lambda f = \int_X f \, \mathrm{d}\nu \leq \int_X \mathbb{1}_V \, \mathrm{d}\nu = \nu(V) < \nu(K) + \varepsilon.$$

Therefore, $\mu(K) \leq \nu(K)$ and, by symmetry, $\mu(K) = \nu(K)$.

Construction of $\mathfrak{M}$ and μ. We define:

$$\mu(V) = \sup \{\Lambda f \colon f \prec V\} \qquad \text{for any open set } V \subseteq X, \tag{3.2}$$

$$\mu(E) = \inf \{\mu(V) \colon E \subseteq V,\ V \text{ is open}\} \quad \text{for an arbitrary } E \subseteq X. \tag{3.3}$$

Notice these two definitions are compatible. For, if E is open, then for any open set $U \supseteq E$, the values of μ defined by (3.2) satisfy $\mu(E) \leq \mu(U)$. Hence, $\mu(E)$ defined by formula (3.3) is the same as that defined by (3.2). Next, define $\mathfrak{M}_F$ to be the collection of all sets $E \subset X$ such that

$$\mu(E) < \infty \quad \text{and} \quad \mu(E) = \sup \{\mu(K) \colon K \subseteq E,\ K \text{ is compact}\}.$$

Finally, let

$$\mathfrak{M} = \{E \subset X \colon E \cap K \in \mathfrak{M}_F \text{ for every compact set } K \subset X\}.$$

Observe that by formula (3.2) we have guaranteed condition (c). Of course, the function μ is monotone, i.e. $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ and it implies that condition (e) is also satisfied.

Proof of properties (a)*,* (b) *and* (d).

Part 1. μ is an outer measure, that is, for any sequence $(E_i)_{i=1}^{\infty}$ of subsets of X we have

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

First, we show this inequality for two open sets. So, assume $V_1, V_2 \subseteq X$ are open and pick a function g with $g \prec V_1 \cup V_2$. Using the partition of unity, we produce functions h_1, h_2 such that $h_1 \prec V_1$, $h_2 \prec V_2$ and $h_1(x) + h_2(x) = 1$ for each $x \in \operatorname{supp}(g)$. Hence, $h_1 g \prec V_1$, $h_2 g \prec V_2$ and $g = h_1 g + h_2 g$ which yields

$$\Lambda g = \Lambda(h_1 g) + \Lambda(h_2 g) \leq \mu(V_1) + \mu(V_2).$$

Passing to supremum over all the functions g with $g \prec V_1 \cup V_2$, we obtain $\mu(V_1 \cup V_2) \le \mu(V_1) + \mu(V_2)$. The rest of the proof is left for the **classes**.

<u>*Part 2*</u>. For any compact set $K \subset X$ we have $K \in \mathfrak{M}_F$ and

$$\mu(K) = \inf\big\{\Lambda f \colon K \prec f\big\}.$$

Consider any function f with $K \prec f$ and $\alpha \in (0,1)$. Define $V_\alpha = \{x \in X \colon f(x) > \alpha\}$ and note that $K \subset V_\alpha$ and $\alpha g \le f$ provided that $g \prec V_\alpha$. Hence,

$$\mu(K) \le \mu(V_\alpha) = \sup\big\{\Lambda g \colon g \prec V_\alpha\big\} \le \alpha^{-1}\Lambda f.$$

Passing to the limit as $\alpha \to 1$, we get $\mu(K) \le \Lambda f < \infty$. Therefore, K satisfies the requirements of the definition of $\mathfrak{M}_F$, thus $K \in \mathfrak{M}_F$.

We have established assertion (b).

<u>*Part 3*</u>. Every open set $V \subseteq X$ satisfies

$$\mu(V) = \sup\big\{\mu(K) \colon K \subseteq V,\ K \text{ is compact}\big\},$$

and hence $\mathfrak{M}_F$ contains all open sets V for which $\mu(V) < \infty$.

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<u>*Part 4*</u>. Let $(E_i)_{i=1}^\infty$ be a sequence of mutually disjoint sets from the class $\mathfrak{M}_F$. Then,

$$\mu\Big(\bigcup_{i=1}^\infty E_i\Big) = \sum_{i=1}^\infty \mu(E_i).$$

Moreover, if $\mu(\bigcup_{i=1}^\infty E_i) < \infty$, then $\bigcup_{i=1}^\infty E_i \in \mathfrak{M}_F$.

We will prove that

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2) \quad \text{for compact } K_1, K_2 \subset X,\ K_1 \cap K_2 = \varnothing. \tag{3.4}$$

Fix any $\varepsilon > 0$. By Urysohn's lemma, there is a function $f \in C_c(X)$ such that $f|_{K_1} = 1$ and $f|_{K_2} = 0$. In view of Part 2, we can pick a function g with

$$K_1 \cup K_2 \prec g \quad \text{and} \quad \Lambda g < \mu(K_1 \cup K_2) + \varepsilon.$$

Notice that $K_1 \prec fg$ and $K_2 \prec (1-f)g$. Since in Part 2 we have proved that $\mu(K) \le \Lambda h$ for any compact K and any h with $K \prec h$, we have

$$\mu(K_1) + \mu(K_2) \le \Lambda(fg) + \Lambda(g - fg) = \Lambda g < \mu(K_1 \cup K_2) + \varepsilon.$$

Therefore, we obtain formula (3.4) by the fact that $\varepsilon > 0$ was arbitrary. The rest of the proof is left for the **classes**.

<u>*Part 5*</u>. For every $E \in \mathfrak{M}_F$ and any $\varepsilon > 0$ there exist a compact set K and an open set V such that $K \subseteq E \subseteq V$ and $\mu(V \setminus K) < \varepsilon$.

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<u>*Part 6*</u>. If $A, B \in \mathfrak{M}_F$, then $A \cup B$, $A \cap B$, $A \setminus B \in \mathfrak{M}_F$.

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