Functional analysis

Lecture 5: Variation of a measure; Hahn's decomposition theorem; the Riesz–Markov–Kakutani representation theorem for $C_0(X)^*$

Proof of the Riesz–Markov–Kakutani theorem (cont.)

Part 7. $\mathfrak{M}$ is a σ-algebra containing all Borel subsets of X.

Fix any compact set $K \subset X$. If $A \in \mathfrak{M}$, then $(X \setminus A) \cap K = K \setminus (A \cap K) \in \mathfrak{M}_F$ which shows that $\mathfrak{M}$ is closed under complements. Now, assume $A = \bigcup_{i=1}^{\infty} A_i$, where $A_i \in \mathfrak{M}$. Define

$$B_1 = A_1 \cap K, \;\; (A_n \cap K) \setminus (B_1 \cup \ldots \cup B_{n-1}) \text{ for } n \geq 2.$$

According to Part 6, these are pairwise disjoint elements of $\mathfrak{M}_F$. Since $A \cap K = \bigcup_{i=1}^{\infty} B_i$, Part 4 implies that $A \cap K \in \mathfrak{M}_F$, hence $A \in \mathfrak{M}$. Therefore, $\mathfrak{M}$ is a σ-algebra.

Observe that for any closed set $F \subseteq X$, we have $F \cap K \in \mathfrak{M}_F$, thus $F \in \mathfrak{M}$. This shows that $\mathfrak{M}$ contains all closed sets, hence all Borel sets.

Part 8. $\mathfrak{M}_F = \{E \subseteq X \colon \mu(E) < \infty\}$.

If $E \in \mathfrak{M}_F$, then by Parts 2,6, we have $E \cap K \in \mathfrak{M}_F$ for every compact K and hence $E \in \mathfrak{M}$. Conversely, assume $E \in \mathfrak{M}$ satisfies $\mu(E) < \infty$. Then, there exists an open set $V \subset X$ such that $E \subset V$ and $\mu(V) < \infty$. By Parts 3,4, for any $\varepsilon > 0$ we may pick a compact set $K \subset V$ with $\mu(V \setminus K) < \varepsilon$. Since $E \cap K \in \mathfrak{M}_F$, there exists a compact set $H \subseteq E \cap K$ such that $\mu(E \cap K) < \mu(H) + \varepsilon$. We have $E \subseteq (E \cap K) \cup (V \setminus K)$, hence $\mu(E) \leq \mu(E \cap K) + \mu(V \setminus K) < \mu(H) + 2\varepsilon$. This shows that $E \in \mathfrak{M}_F$.

Note that in this way we have proved assertion (d).

Part 9. μ is σ-additive on $\mathfrak{M}$ (and hence it is a positive Borel measure).

It follows from Parts 4 and 8.

Part 10. For every $f \in C_c(X)$ we have

$$\Lambda f = \int_X f \,\mathrm{d}\mu.$$

By splitting any complex-valued function into its real and imaginary parts, we can only consider real-valued functions $f \in C_c(X)$. Moreover, it is enough to prove the inequality

$$\Lambda f \leq \int_X f \,\mathrm{d}\mu, \tag{3.1}$$

as the reverse one follows by linearity: $-\Lambda f = \Lambda(-f) \leq -\int_X f \,\mathrm{d}\mu$.

Let $K = \operatorname{supp}(f)$ and choose real numbers a, b so that $f(K) \subset [a, b]$. Fix any $\varepsilon > 0$ and pick $y_0 < a < y_1 < \ldots < y_n = b$ such that $y_i - y_{i-1} < \varepsilon$ for each $1 \leq i \leq n$. Define

$$E_i = \{x \in X \colon y_{i-1} < f(x) \leq y_i\} \cap K \quad (1 \leq i \leq n);$$

these are pairwise disjoint Borel sets whose union is K. For each $1 \leq i \leq n$, pick an open set V_i such that:

$$E_i \subset V_i, \;\; \mu(V_i) < \mu(E_i) + \frac{\varepsilon}{n} \quad \text{and} \quad f(x) < y_i + \varepsilon \text{ for } x \in V_i.$$

Applying the partition of unity to the open cover $V_1, \dots, V_n$ of K, we get continuous functions $h_i \prec V_i$ $(1 \le i \le n)$ such that $\sum_{i=1}^n h_i(x) = 1$ for $x \in K$. Hence, $f = \sum_{i=1}^n h_i f$. By Part 2, we have

$$\mu(K) \le \Lambda\Big(\sum_{i=1}^n h_i\Big) = \sum_{i=1}^n \Lambda h_i.$$

Also, observe that for each $1 \le i \le n$, we have

$$h_i(x)f(x) \le (y_i + \varepsilon)h_i(x) \quad \text{and} \quad y_i - \varepsilon < f(x) \quad \text{for } x \in E_i.$$

Therefore,

$$\begin{aligned}
\Lambda f &= \sum_{i=1}^n \Lambda(h_i g) \le \sum_{i=1}^n (y_i + \varepsilon)\Lambda h_i \\
&= \sum_{i=1}^n (|a| + y_i + \varepsilon)\Lambda h_i - |a| \sum_{i=1}^n \Lambda h_i \\
&\le \sum_{i=1}^n (|a| + y_i + \varepsilon)\big(\mu(E_i) + \frac{\varepsilon}{n}\big) - |a|\mu(K) \\
&= \sum_{i=1}^n (y_i - \varepsilon)\mu(E_i) + 2\varepsilon\mu(K) + \frac{\varepsilon}{n}\sum_{i=1}^n (|a| + y_i + \varepsilon) \\
&\le \int_X f \,\mathrm{d}\mu + \varepsilon\big(2\mu(K) + |a| + b + \varepsilon\big).
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ we obtain (3.1) and thus the proof is completed. $\square$

Every metric space M can be isometrically embedded as a dense subspace in a complete metric space which is defined in terms of classes of abstraction consisting of Cauchy sequences in M. By applying the same procedure to any normed space X, we infer that X has a *completion* which is a complete normed space, i.e. a Banach space. Of course, such a completion is unique up to an isometric isomorphism. There is also a 'canonical' way of defining the completion of X whose elements can be realized as absolutely convergent series in X (see **Problem 3.14**).

Lemma 3.17. *If X is a locally compact Hausdorff space, then the space $C_0(X)$ of functions vanishing at infinity is the completion of the space $C_c(X)$ under the supremum norm.*

Proof. Our assertion is equivalent to the conjuction that $C_0(X)$ is complete and $C_c(X)$ is dense in $C_0(X)$. The former claim follows from Proposition 1 11. For the latter one, fix any $f \in C_0(X)$ and $\varepsilon > 0$. Pick a compact set K such that $|f(x)| < \varepsilon$ for $x \notin K$. By Urysohn's lemma, there is a function $g \in C_c(X)$ such that $0 \le g \le 1$ and $g|_K = 1$. Let $h = fg$; then $h \in C_c(X)$ and $\|f - h\|_\infty < \varepsilon$. $\square$

Now, our goal is to prove a theorem characterizing the dual space $C_0(X)^*$ for any locally compact Hausdorff space X. We reduce the problem to considering positive linear functionals and then apply Theorem 3.16. Such a reduction will be possible due to decomposition (3.1) which we obtain using some general Banach lattice methods (see the discussion after Remark 3.11).

Lemma 3.18. *The dual space and the completion of a normed Riesz space are Banach lattices.*

Proof. Let E be a normed Riesz space. In the dual space E^* we introduce an order by:

$$f \geq g \quad \iff \quad f(x) \geq g(x) \quad \text{for every } x \in E^+.$$

First, it is easily seen that E^* then becomes an ordered vector space. Moreover, E^* is a Riesz space (vector lattice) with lattice operations given by

$$(f \vee g)(x) = \sup\big\{f(y) + g(z)\colon y, z \in E^+ \text{ and } y + z = x\big\} \quad (x \in E^+)$$

and

$$(f \wedge g)(x) = \inf\big\{f(y) + g(z)\colon y, z \in E^+ \text{ and } y + z = x\big\} \quad (x \in E^+).$$

Notice that so far $f \vee g$ and $f \wedge g$ are only defined on E^+, but it is not difficult to extend them to the whole of E. We prove it only for $h := f \vee g$; the proof for $f \wedge g$ is similar.

Obviously, h is positively homogeneous, i.e. $h(\lambda x) = \lambda h(x)$ for $\lambda \geq 0$, $x \in E^+$. We can rewrite the definition of h as

$$h(x) = \sup\big\{f(y) + g(x - y)\colon 0 \leq y \leq x\big\} \quad (x \in E^+).$$

To see that h is additive, fix $u, v \in E^+$. For arbitrary $0 \leq u_1 \leq u$ and $0 \leq v_1 \leq v$, we have

$$\begin{aligned}&[f(u_1) + g(u - u_1)] + [f(v_1) + g(v - v_1)]\\ &\qquad\qquad = f(u_1 + v_1) + g(u + v - (u_1 + v_1)) \leq h(u + v),\end{aligned}$$

whence $h(u) + h(v) \leq h(u + v)$. For the converse inequality, we need the following result.

Claim (the Riesz Decomposition Property). If $x_1, \ldots, x_n, y \in E$ satisfy $|y| \leq |\sum_{i=1}^n x_i|$, then there exist vectors $y_1, \ldots, y_n \in E$ such that $y = \sum_{i=1}^n y_i$ and $|y_i| \leq |x_i|$ for each $1 \leq i \leq n$. If y is positive, then all y_i's can be chosen to be positive, too.

Proof of the Claim. We prove it for $n = 2$; the rest is an easy induction. So, assuming $|y| \leq |x_1 + x_2|$, define

$$y_1 = \big[(-|x_1|) \vee y\big] \wedge |x_1| \quad \text{and} \quad y_2 = y - y_1.$$

Clearly, $-|x_1| \leq y_1 \leq |x_1|$, i.e. $|y_1| \leq |x_1|$ (note also that if $y \geq 0$, then $0 \leq y_1 \leq y$). We are to prove that also $|y_2| \leq |x_2|$. To this end, notice that in view of

$$|y| \leq |x_1 + x_2| \leq |x_1| + |x_2|,$$

we have

$$-|x_1| - |x_2| \leq y \leq |x_1| + |x_2|, \text{ i.e. } -|x_2| \leq |x_1| + y \text{ and } y - |x_1| \leq |x_2|.$$

Hence,

$$-|x_2| \leq (|x_1| + y) \wedge 0 \quad \text{and} \quad (y - |x_1|) \vee 0 \leq |x_2|. \tag{3.2}$$

Now, calculate

$$\begin{aligned} y_2 &= y - \big[(-|x_1|) \vee y\big] \wedge |x_1| \\ &= y + \big[|x_1| \wedge (-y)\big] \vee (-|x_1|) \\ &= \big[(|x_1| + y) \wedge 0\big] \vee (y - |x_1|) \end{aligned}$$

and observe that, by (3.2), we have $-|x_2| \leq y_2 \leq |x_2|$, as desired.

Now, let $0 \leq y \leq u + v$. By our Claim, there exist $y_1, y_2 \in E^+$ such that $y = y_1 + y_2$, $0 \leq y_1 \leq u$ and $0 \leq y_2 \leq v$. Hence,

$$f(y) + g((u+v) - y) = [f(y_1) + g(u - y_1)] + [f(y_2) + g(v - y_2)] \leq h(u) + h(v),$$

which proves that h is additive on E^+.

We extend h by defining $\widetilde{h}\colon E \to \mathbb{R}$ as $h(x) = h(x^+) - h(x^-)$ $(x \in E)$. Then, h is a linear functional. Indeed, for any $x, y \in E$ we have

$$(x+y)^+ - (x-u)^- = x + y = x^+ - x^- + y^+ - y^-,$$

hence $(x+y)^+ + x^- + y^- = x^+ + y^+ + (x+y)^-$. By the additivity of h, we obtain

$$\begin{aligned} \widetilde{h}(x+y) &= h((x+y)^+) - h((x+y)^-) \\ &= [h(x^+) - h(x^-)] + [h(y^+) - h(y^-)] = \widetilde{h}(x) + \widetilde{h}(y). \end{aligned}$$

Also, observe that

$$\widetilde{h}(-x) = h((-x)^+) - h((-x)^-) = h(x^-) - h(x^+) = -\widetilde{h}(x)$$

and since h is positively homogeneous, it is homogeneous, hence linear. To see that $\widetilde{h}$ is continuous, fix any $x \in E$ with $\|x\| \leq 1$. Then $0 \leq x^+, x^- \leq |x| = 1$, thus $\|x^+\|, \|x^-\| \leq 1$ (recall that E is a normed Riesz space). Given any $y, z \in E^+$ with $y + z = x^+$, we have $|f(y) + g(z)| \leq (\|f\| + \|g\|)\|x^+\|$ and similarly for x^-. Therefore,

$$|\widetilde{h}(x)| \leq |h(x^+)| + |h(x^-)| \leq (\|f\| + \|g\|)(\|x^+\| + \|x^-\|) \leq 2(\|f\| + \|g\|)$$

which proves that $h \in E^*$.

Clearly, $f \leq \widetilde{h}$ and $g \leq \widetilde{h}$. Moreover, given any $\varphi \in E^*$ satisfying $f \leq \varphi$ and $g \leq \varphi$, we have for $0 \leq y \leq x$ that

$$f(y) + g(x - y) \leq \varphi(y) + \varphi(x - y) = \varphi(x).$$

This shows that $\widetilde{h} \leq \varphi$, that is, $\widetilde{h}$ is the smallest upper bound for $\{f, g\}$.

We have shown that E^* is a normed Riesz space. That it is complete follows from Proposition 3.1. Hence, E^* is a Banach lattice.

For the completion $\widetilde{E}$ of E, observe that $\widetilde{E}$ is the closure of E in the Banach lattice E^{**} (see **Problem 3.14**), so our assertion follows from the previous part. □

Remark. In view of the above defined lattice structure on E^*, we have the following formula for the positive part of a continuous linear functional $f \in E^*$:

$$f^+ = f \vee 0 = \sup\{f(y)\colon 0 \leq y \leq x\}.$$

Lemma 3.19 (**Kantorovich**)**.** *Let E and F be Riesz spaces and assume F is Archimedean. Then, every additive map $\varphi\colon E^+ \to F^+$ extends uniquely to a positive linear functional $\Phi\colon E \to F$. Moreover, the unique positive linear extension is given by the formula*

$$\Phi(x) = \varphi(x^+) - \varphi(x^-) \qquad (x \in E).$$

Proof. Since $x = x^+ - x^-$, the uniqueness part is obvious.

For proving the announced properties of Φ, first observe that, by simple induction, $\varphi(kx) = k\varphi(x)$ for all $k \in \mathbb{N}$, $x \in E^+$. Thus, for any positive rational $r = \frac{m}{n}$ $(m, n \in \mathbb{N})$ we have

$$r\varphi(x) = \frac{m}{n}\varphi(x) = \frac{m}{n}\varphi\Big(\frac{nx}{n}\Big) = m\varphi\Big(\frac{x}{n}\Big) = \varphi(rx).$$

Now, given any $\lambda > 0$, pick sequences $(r_n)_{n=1}^\infty$ and $(t_n)_{n=1}^\infty$ of rational numbers such that $0 < r_n \nearrow \lambda$ and $t_n \searrow \lambda$. Then, for any $x \in E^+$, $0 \le r_n x \le \lambda x \le t_n x$, whence

$$r_n\varphi(x) = \varphi(r_n x) \le \varphi(\lambda x) \le \varphi(t_n x) = t_n\varphi(x). \tag{3.3}$$

Recall that F is Archimedean if $0 \le ny \le z$ for all $n \in \mathbb{N}$ implies $y = 0$. This is equivalent to saying that for each $y \in F^+$ we have $\frac{1}{n}y \downarrow 0$, i.e. the sequence $(\frac{1}{n}y)_{n=1}^\infty$ is decreasing and $\inf_n \frac{1}{n}y = 0$ in F. Hence, letting $n \to \infty$ in (3.3) we get $\varphi(\lambda x) = \lambda\varphi(x)$ for all $\lambda > 0$ and $x \in E^+$.

In the proof of Lemma 3.18, we have already shown (for $\widetilde{h}$ playing the role of Φ) that the extension Φ is an additive odd function. Hence, the positive homogeneity of φ yields the homogeneity of Φ and the proof is completed. □

We are almost ready to prove a representation theorem for the dual of $C_0(X)$ over $\mathbb{R}$, for any locally compact Hausdorff space X. As indicated earlier, the representing object are going to be regular Borel signed (i.e. with values in $\mathbb{R}$) measures on X. To get a full picture, we need to introduce the notion of variation (we do it in the general complex case) and prove a very important Hahn's decomposition theorem which nicely describes the variation of a measure in the real case.

Definition 3.20. Let μ be a complex-valued measure defined on a σ-algebra $\mathfrak{M}$ of subsets of a set X. For any $E \in \mathfrak{M}$ we denote by $\Pi(E)$ the collection of all measurable partitions of E, that is

$$\Pi(E) = \Big\{(E_1, \ldots, E_n)\colon n \in \mathbb{N},\, E_i \in \mathfrak{M},\, E_i \cap E_j = \varnothing \text{ for } 1 \le i \ne j \le n,\, \textstyle\bigcup_{i=1}^n E_i = E\Big\}.$$

We define the *variation* of μ by the formula

$$|\mu|(E) = \sup\Big\{\sum_{i=1}^n |\mu(E_i)|\colon (E_1, \ldots, E_n) \in \Pi(E)\Big\} \quad (E \in \mathfrak{M});$$

the value $|\mu|(X)$ is called the *total variation* of μ.

Proposition 3.21. *For any complex σ-additive measure μ, the variation $|\mu|$ is a positive σ-additive measure.*

Proof. (classes) □

It follows readily from Proposition 3.21 that $|\mu|$ yields the smallest positive measure majorizing μ in the sense that $|\mu(E)| \leq |\mu|(E)$ for every $E \in \mathfrak{M}$. It may seem quite surprising that $|\mu|$ happens to be always bounded, but recall that we deal with complex-valued measures, so they do not attain the values $\pm\infty$. Boundedness of μ in the general complex case will be proved later. In the real case (i.e. for signed measures) it follows from Hahn's decomposition theorem below, which also gives a simple description of $|\mu|$.

Theorem 3.22 (Hahn decomposition theorem). *Let μ be a signed measure defined on a σ-algebra $\mathfrak{M}$ of subsets of X. There exists sets $A, B \in \mathfrak{M}$ such that $A \cup B = X$, $A \cap B = \varnothing$, $\mu(A \cap E) \geq 0$ and $\mu(B \cap E) \leq 0$ for each $E \in \mathfrak{M}$. Moreover, if we define*

$$\mu^+(E) = \mu(A \cap E), \quad \mu^-(E) = -\mu(B \cap E) \qquad (E \in \mathfrak{M}),$$

then $|\mu|(E) = \mu^+(A \cap E) + \mu^-(B \cap E)$.

We refer to the measures μ^+ and μ^- as the *positive* and *negative part* of μ, respectively. The pair (A, B) is called *Hahn's decomposition* of X determined by μ. The following, surprisingly short proof is due to [R. Doss, *The Hahn decomposition theorem*, Proc. Amer. Math. Soc. 80 (1980), p. 377].

Proof. We call a set $E \in \mathfrak{M}$ *positive* (resp. *negative*) if $\mu(A) \geq 0$ (resp. $\mu(A) \leq 0$) for every $A \subseteq E$, $A \in \mathfrak{M}$.

Claim. Every set $E \in \mathfrak{M}$ contains a positive set P such that $\mu(P) \geq \mu(E)$.

Indeed, observe that for each $\varepsilon > 0$ there is $A_\varepsilon \subseteq E$ in $\mathfrak{M}$ such that

$$\mu(A_\varepsilon) \geq \mu(E) \quad \text{ and } \quad \mathfrak{M} \ni D \subseteq A_\varepsilon \implies \mu(D) > -\varepsilon.$$

Assume not. Then, there is a sequence $(D_k)_{k=1}^\infty$ of measurable sets such that $D_1 \subseteq E$ and $D_k \subseteq E \setminus (D_1 \cup \ldots \cup D_{k-1})$ for $k \geq 2$ and $\mu(D_k) \leq -\varepsilon$ for each $k \in \mathbb{N}$. As D_k's are pairwise disjoint, the set $D = \bigcup_{k=1}^\infty D_k$ would be of measure $-\infty$ which is impossible.

Choose any sequence $\varepsilon_k \searrow 0$ and let (A_{ε_k}) be a sequence of measurable sets produced by the above statement, where at the k^{th} step we apply it to $E \cap A_{\varepsilon_1} \cap \ldots \cap A_{\varepsilon_{k-1}}$ in the place of E. Then, (A_{ε_k}) is descending and the intersection $P := \bigcap_{k=1}^\infty A_{\varepsilon_k}$ is a positive set with $\mu(P) = \lim_k \mu(A_{\varepsilon_k}) \geq \mu(E)$. Thus, our Claim has been established.

Define $s = \sup\{\mu(E) \colon E \in \mathfrak{M}\}$ and pick a sequence $(P_k)_{k=1}^\infty \subset \mathfrak{M}$ with $\mu(P_k) \nearrow s$. By the Claim, we can assume that each P_k is a positive set. Therefore, the set $P := \bigcup_{k=1}^\infty P_k$ is also positive and, plainly, $\mu(P) = s$. Hence, $N := X \setminus P$ is negative, for if $E \subseteq N$ and $\mu(E) > 0$, then $\mu(P \cup E) = s + \mu(E) > s$ which is impossible. $\square$

Recall that $\mathcal{M}(X)$ is the space of regular Borel measures on a locally compact Hausdorff space X. At this point, we deal with the real case only, so $\mathcal{M}(X)$ consists here of signed measures. It is easy to verify that $\mathcal{M}(X)$ becomes a real normed space with natural operations and the total variation norm $\|\mu\| = |\mu|(X)$. (Notice that Theorem 3.22 guarantees that every signed measure is of bounded variation.)

We also introduce a lattice structure on $\mathcal{M}(X)$ by considering the pointwise ordering and lattice operations:

$$(\mu \vee \nu)(E) = \sup\big\{\mu(A) + \nu(E \setminus A) \colon A \subseteq E, \ A \text{ measurable}\big\}$$

and

$$(\mu \wedge \nu)(E) = \inf \big\{ \mu(A) + \nu(E \setminus A) \colon A \subseteq E,\ A \text{ measurable} \big\}.$$

It is an instructive exercise to verify that these formulas indeed satisfy the axioms of supremum and infimum. Of course, a glance at Hahn's theorem shows that $|\mu| \leq |\nu|$ (the moduli defined by means of the supremum operation) implies $\|\mu\| \leq \|\nu\|$, which means that $\mathcal{M}(X)$ is a normed Riesz space. Also, the positive and negative parts defined in terms of the lattice operations: $\mu \vee 0$ and $(-\mu) \vee 0$ coincide with μ^+ and μ^-, respectively, given by the Hahn decomposition. Completeness of $\mathcal{M}(X)$ (i.e. the fact that it is a Banach lattice) will follow automatically from the theorem below, where we identify this space as a dual space (see Proposition 3.1).

Theorem 3.23 **(Riesz–Markov–Kakutani for $\boldsymbol{C_0(X)^*}$).** *Let X be a locally compact Hausdorff space and $\Lambda \in C_0(X)^*$ be a continuous linear functional on the Banach space (over $\mathbb{R}$) of real-valued continuous functions on X vanishing at infinity. Then, there exists a unique regular Borel σ-additive signed measure μ on X such that*

$$\Lambda f = \int_X f \, \mathrm{d}\mu \quad \textit{for every } f \in C_0(X). \tag{3.4}$$

Moreover, we have $\|\Lambda\| = |\mu|(X)$. On the other hand, every $\mu \in \mathcal{M}(X)$ gives rise to an element Λ of $C_0(X)^*$ via formula (3.4)*. Consequently, the map $\Lambda \mapsto \mu$ is an isometric isomorphism*

$$C_0(X)^* \cong \mathcal{M}(X).$$

Proof. In view of Lemma 3.17, every (positive) continuous linear functional on $C_c(X)$ has a unique (positive) extension to an element of $C_0(X)^*$. Hence, the positive part of $C_0(X)^*$ can be identified with the collection of all positive linear functionals on $C_c(X)$. By virtue of the Riesz–Markov–Kakutani theorem for positive functionals (Theorem 3.16), there is a linear surjective isometry

$$\mathcal{M}(X)^+ \xrightarrow{\ \varphi\ } [C_0(X)^*]^+. \tag{3.5}$$

Surjectivity follows from the fact that every positive functional Λ is represented by a positive measure μ. That it is an isometry follows from the observation that if μ is positive, we have

$$\|\mu\| = \mu(X) = \sup \big\{ \Lambda f \colon f \prec X \big\} = \sup \big\{ |\Lambda f| \colon f \in C_0(X),\ \|f\|_\infty \leq 1 \big\} = \|\Lambda\|.$$

As explained above, $\mathcal{M}(X)$ is a (normed) Riesz space and, by Lemma 3.18, so is $C_0(X)^*$. It is easy to verify that $C_0(X)^*$ is also Archimedean. Therefore, according to the Kantorovich Lemma 3.19, the isometry φ indicated in (3.5) has a unique extension to a positive linear map

$$\mathcal{M}(X) \xrightarrow{\ \Phi\ } C_0(X)^*. \tag{3.6}$$

For every $\mu \in \mathcal{M}(X)$ we have $\Phi(\mu) = \varphi(\mu^+) - \varphi(\mu^-)$ which means that the functional $\Lambda = \Phi(\mu)$ is decomposed as

$$\Lambda f = \int_X f \, \mathrm{d}\mu^+ - \int_X f \, \mathrm{d}\mu^- \quad (f \in C_0(X)). \tag{3.7}$$

Claim. $\Phi(\mu) \geq 0$ if and only if $\mu \geq 0$.

If $\mu \geq 0$, then $\Phi(\mu) \geq$ as Φ is positive. For the converse, assume $\Lambda = \Phi(\mu) \geq 0$, that is, $\int_X f \, \mathrm{d}\mu \geq 0$ for every $f \geq 0$, $f \in C_0(X)$. Fix any open set $V \subseteq X$. Since μ is regular, we just need to show that $\mu(V) \geq 0$. Given any $\varepsilon > 0$, pick a closed set $F \subset V$ with $\mu^-(V) - \mu^-(F) < \varepsilon$. By Urysohn's lemma, there is a function f with $F \prec f \prec V$. We have

$$0 \leq \int_X f \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu^+ - \int_X f \,\mathrm{d}\mu^- \leq \mu^+(V) - \mu^-(F) \leq \mu^+(V) - \mu^-(V) + \varepsilon = \mu(V) + \varepsilon,$$

which proves our Claim.

It follows that Φ is one-to-one: the pre-image of the zero functional consists of measure which are simultaneously positive and negative and there is just one such measure, the zero measure. It also follows from our Claim that Φ is a lattice isomorphism, in particular, it preserves the modulus: $\Phi(|\mu|) = |\Phi(\mu)|$. Hence, for $\Lambda = \Phi(\mu)$, we have

$$\|\Lambda\| = \||\Lambda|\| = \|\varphi(|\mu|)\| = \|\mu\|,$$

thus Φ is an isometry. Finally, Φ is surjective because so is φ and every $\Lambda \in (C_0(X))^*$ is the difference of two positive operators. $\square$

Remark. Given any $\Lambda \in C_0(X)^*$, the Hahn decomposition $\mu = \mu^+ - \mu^-$ of the representing measure μ of Λ yields the decomposition given by (3.7). Moreover,

$$\Lambda^+ f = \int_X f \,\mathrm{d}\mu^+, \quad \Lambda^- f = \int_X f \,\mathrm{d}\mu^- \quad \text{and} \quad \|\Lambda\| = \|\Lambda^+\| + \|\Lambda^-\|.$$

The proof of Theorem 3.23 in the complex case is postponed till we prove the Radon–Nikodym theorem.