# Functional analysis

**Lecture 6:** Invertibility; basic properties of the spectrum; compact operators and the Riesz–Schauder theorem

## 4 Compact operators and their spectral properties

In this section, we generally consider Banach spaces over either real or complex numbers, however, there are important exceptions (Proposition 4.7 and Theorem 4.8), where the assumption that the scalar field $\mathbb{K} = \mathbb{C}$ is crucial.

Recall that by $\mathscr{L}(X)$ we denote the space of all bounded linear operators from $X$ into itself. By $I$ we denote the identity operator. Usually, it will be clear from the context what is the domain of that operator. Otherwise, we will indicate it by writing $I_X$ if $X$ is the domain space. An operator $T \in \mathscr{L}(X)$ is *invertible* if it is one-to-one, surjective and the inverse $T^{-1} \in \mathscr{L}(X)$, so $T$ is an isomorphism of $X$ onto itself[1].

The space $\mathscr{L}(X)$ is not only a Banach space (if $X$ is a Banach space), but also has an algebra structure, where the multiplication is given by composition: $(TS)x = T(Sx)$. Obviously, we have $\|TS\| \le \|T\|\|S\|$ for all $T, S \in \mathscr{L}(X)$. In particular $\|T^n\| \le \|T\|^n$ for every $n \in \mathbb{N}$. Also, notice that if $T_n \to T$ and $S_n \to S$ in $\mathscr{L}(X)$, then

$$\begin{aligned}\|TS - T_nS_n\| &= \|TS - T_nS + T_nS - T_nS_n\| \\ &\le \|T - T_n\| \cdot \|S\| + \|S - S_n\| \cdot \|T_n\| \xrightarrow[n\to\infty]{} 0,\end{aligned}$$

which means that the multiplication in $\mathscr{L}(X)$ is continuous.

**Proposition 4.1.** *Let $X$ be a Banach space and $T \in \mathscr{L}(X)$. If $\|T\| < 1$, then $I + T$ is invertible and its inverse is expressed as the sum of an absolutely convergent series,*

$$(I + T)^{-1} = \sum_{n=0}^{\infty} (-1)^n T^n. \tag{4.1}$$

*Moreover,*

$$\big\|(I + T)^{-1} - I + T\big\| \le \frac{\|T\|^2}{1 - \|T\|}. \tag{4.2}$$

*Proof.* Since $\|T^n\| \le \|T\|^n$ for each $n \in \mathbb{N}$ and $\|T\| < 1$, the series (4.1) is absolutely convergent, and hence convergent in $\mathscr{L}(X)$. Indeed, let

$$S_N = I - T + T^2 - \ldots + (-1)^N T^N \quad (N \in \mathbb{N})$$

and observe that $(S_N)_{N=1}^{\infty}$ is a Cauchy sequence in $\mathscr{L}(X)$. Define $S = \lim_{N\to\infty} S_N$, i.e. $S$ is the sum of the series in (4.1). We have

$$(I + T)S_N = I + (-1)^N T^{N+1} = S_N(I + T)$$

[1] It follows from the Open Mapping Theorem (which will be proved later) that if $X$ is a Banach space and $T \in \mathscr{L}(X)$ is bijective, then $T^{-1}$ is automatically bounded, hence $T^{-1} \in \mathscr{L}(X)$. Therefore, in order to verify that $T$ is invertible it is enough to know that it is one-to-one and onto.

and, since the multiplication in $\mathscr{L}(X)$ is continuous, $(I+T)S = I = S(I+T)$, which means that $S = (I+T)^{-1}$ (note that $S$ is bounded because $\|S\| \le \sum_{n=0}^{\infty} \|T\|^n < \infty$). Inequality (4.2) follows from the estimate:

$$\Big\| \sum_{n=2}^{\infty} (-1)^n T^n \Big\| \le \sum_{n=2}^{\infty} \|T\|^n = \frac{\|T\|^2}{1-\|T\|}. \qquad \square$$

From the above assertion we infer that the set of invertible operators in $\mathscr{L}(X)$ contains an open ball centered at the identity $I$ and with radius 1. This can be generalized as follows.

**Corollary 4.2.** *Let $X$ be a Banach space and let $T \in \mathscr{L}(X)$ be invertible with $\|T^{-1}\| = \frac{1}{\alpha}$. Assume $S \in \mathscr{L}(X)$ satisfies $\|S\| = \beta < \alpha$. Then $T+S$ is invertible and*

$$\big\|(T+S)^{-1} - T^{-1} + T^{-1}ST^{-1}\big\| \le \frac{\beta^2}{\alpha^2(\alpha-\beta)}. \tag{4.3}$$

*Proof.* Note that $T+S = T(I+T^{-1}S)$, hence it is enough to show that $I+T^{-1}S$ is invertible. To this end, we observe that $\|T^{-1}S\| \le \|T^{-1}\|\|S\| = \frac{\beta}{\alpha} < 1$ and appeal to Proposition 4.1. Notice that

$$(T+S)^{-1} - T^{-1} + T^{-1}ST^{-1} = \big[(I+T^{-1}S)^{-1} - I + T^{-1}S\big]T^{-1},$$

thus inequality (4.2) yields

$$\big\|(T+S)^{-1} - T^{-1} + T^{-1}ST^{-1}\big\| \le \frac{\|T^{-1}S\|^2\|T^{-1}\|}{1-\|T^{-1}S\|} \le \frac{\beta^2}{\alpha^3(1-\beta/\alpha)} = \frac{\beta^2}{\alpha^2(\alpha-\beta)}. \qquad \square$$

For any Banach space $X$, we denote:

- $\mathscr{G}(X) = \{T \in \mathscr{L}(X) \colon T \text{ is invertible}\}$

**Corollary 4.3.** *For every Banach space $X$, $\mathscr{G}(X)$ is an open subset of $\mathscr{L}(X)$. Moreover, the map $\mathcal{J} \colon T \mapsto T^{-1}$ is a $C^1$-diffeomorphism of $\mathscr{G}(X)$ onto itself and its Fréchet derivative $\mathrm{d}\,\mathcal{J}(T) \in \mathscr{L}(\mathscr{L}(X))$ is given by*

$$\mathrm{d}\,\mathcal{J}(T)S = -T^{-1}ST^{-1} \quad (S \in \mathscr{L}(X)). \tag{4.4}$$

*Proof.* In view of Corollary 4.2, every invertible operator $T$ is contained in $\mathscr{G}(X)$ together with a certain open ball centered at $T$. Moreover, inequality (4.3) can be rewritten as

$$\mathcal{J}(T+S) - \mathcal{J}(T) + T^{-1}ST^{-1} = o(\|S\|),$$

where $S \mapsto -T^{-1}ST^{-1}$ is a bounded linear operator on $\mathscr{L}(X)$. Hence, the Fréchet derivative is indeed given by formula (4.4). In particular, $\mathcal{J}$ is continuous and hence it is a homeomorphism of $\mathscr{G}(X)$ into itself as $\mathcal{J}^{-1} = \mathcal{J}$.

In order to show that the map $\mathscr{L}(X) \ni T \mapsto \mathrm{d}\mathcal{J}(T) \in \mathscr{L}(\mathscr{L}(X))$ is continuous (i.e. that $\mathcal{J}$ is of class $C^1$), notice that it is the composition of the inverse map $T \mapsto T^{-1}$, which we already know to be continuous, and the map $\Phi(T)S = -TST$. Since

$$\begin{aligned} \|T_nST_n - TST\| &\le \|T_nST_n - TST_n\| + \|TST_n - TST\| \\ &\le (\|T_n\| + \|T\|)\|T_n - T\|\|S\|, \end{aligned}$$

we have

$$\|\Phi(T_n) - \Phi(T)\| \le (\|T_n\| + \|T\|)\|T_n - T\| \xrightarrow[n\to\infty]{} 0$$

whenever $T_n \to T$ in $\mathscr{L}(X)$. Hence, $\Phi$ is continuous and so is the map $T \mapsto \mathrm{d}\mathcal{J}(T)$. $\square$

**Definition 4.4.** Let $X$ be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $T \in \mathscr{L}(X)$. We define the *spectrum* of $T$ by

$$\sigma(T) = \big\{\lambda \in \mathbb{K} \colon T - \lambda I \notin \mathscr{G}(X)\big\}.$$

The set $\rho(T) = \mathbb{K} \setminus \sigma(T)$ is called the *resolvent set* for $T$, and the function

$$R(\lambda) = (T - \lambda I)^{-1} \qquad (\lambda \in \rho(T))$$

is called the *resolvent* of $T$.

Notice that $T - \lambda I$ may be noninvertible for two reasons; that it is not one-to-one, which means that $\ker(T - \lambda I) \neq \{0\}$, or that it is not surjective[2], i.e. $(T - \lambda I)(X)$ is a proper subspace of $X$. The latter situation admits two cases: the range of $T - \lambda I$ is dense but not closed or it is not dense in $X$. Therefore, we can distinguishe three mutually exclusive types of elements of the spectrum.

**Definition 4.5.** Let $X$ be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $T \in \mathscr{L}(X)$. We define:

- the *point spectrum* of $T$,

  $$\sigma_{\mathrm{p}}(T) = \big\{\lambda \in \mathbb{K} \colon \ker(T - \lambda I) \neq \{0\}\big\},$$

- the *continuous spectrum* of $T$,

  $$\sigma_{\mathrm{c}}(T) = \big\{\lambda \in \sigma(T) \setminus \sigma_{\mathrm{p}}(T) \colon (T - \lambda I)(X) \text{ is a dense proper subspace of } X\big\},$$

- the *residual spectrum* of $T$,

  $$\sigma_{\mathrm{r}}(T) = \big\{\lambda \in \sigma(T) \setminus \sigma_{\mathrm{p}}(T) \colon (T - \lambda I)(X) \text{ is not dense in } X\big\}.$$

Every element $\lambda$ of $\sigma_p(T)$ is called an *eigenvalue* of $T$, and every vector $x \in X$ satisfying $Tx = \lambda x$ is called an *eigenvector* corresponding to the eigenvalue $\lambda$. Also, the dimension $\dim \ker(T - \lambda I)$ is called the *multiplicity* of the eigenvalue $\lambda$.

*Remark.* In general, for any Banach space $X$ and $T \in \mathscr{L}(X)$ we have

$$\sigma(T) = \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T) \cup \sigma_{\mathrm{r}}(T). \tag{4.5}$$

This follows from the already mentioned Open Mapping Theorem which guarantees that any bijective bounded operator between Banach spaces has a bounded inverse. This theorem will be proved later, but we do not really need to refer to decomposition (4.5) at this point.

**Proposition 4.6.** *For any Banach space $X$ and any $T \in \mathscr{L}(X)$, the spectrum $\sigma(T)$ is a compact subset of $\mathbb{K}$. Moreover, for every $\lambda \in \sigma(T)$ we have $|\lambda| \leq \|T\|$.*

*Proof.* If $|\lambda| > \|T\|$ then $I - \lambda^{-1}T \in \mathscr{G}(X)$ in view of Proposition 4.1. Hence, $\lambda I - T \in \mathscr{G}(X)$ which means that $\lambda \notin \sigma(T)$ and proves the last assertion.

As we already know that $\sigma(T)$ is bounded, we are to prove that it is closed. Notice that $\lambda \in \sigma(T)$ if and only if $T - \lambda I \in \mathscr{L}(X) \setminus \mathscr{G}(X)$ which, by Corollary 4.3, is a closed subset of $\mathscr{L}(X)$. Of course, the map $\mathbb{K} \ni \lambda \mapsto T - \lambda I$ is norm continuous, whence $\sigma(T)$ is closed as the preimage of a closed set by a continuous map. $\square$

[2] Again, we silently use the Open Mapping Theorem.

In the next two results, we assume $X$ to be a complex Banach space. Recall that a function $f\colon \Omega \to \mathbb{C}$ defined on an open set $\Omega \subseteq \mathbb{C}$ is *holomorphic* provided that it has a complex derivative at every point, that is, for every $z \in \Omega$ there exists a complex limit

$$f'(z) \coloneqq \lim_{w\to z} \frac{f(w)-f(z)}{w-z}.$$

**Proposition 4.7.** *Let $X$ be a Banach space over $\mathbb{C}$, $T \in \mathscr{L}(X)$ and let $\Phi \in \mathscr{L}(X)^*$. Then, the function*

$$f(\lambda) = \Phi\big[(T-\lambda I)^{-1}\big] \quad (\lambda \in \rho(T)) \tag{4.6}$$

*is holomorphic on $\rho(T)$ and $f(\lambda) \to 0$ as $|\lambda| \to \infty$.*

*Proof.* Fix $\lambda \in \rho(T)$. We apply Corollary 4.2 to the invertible operator $T - \lambda I$ instead of $T$ and to $S$ being of the form $S = (\lambda - \mu)I$ for $\mu \in \mathbb{C}$ sufficiently close to $\lambda$. We obtain that for suitable $\mu$'s, the operator $T - \mu I \in \mathscr{G}(X)$ and inequality (4.3) yields

$$\big\|(T-\mu I)^{-1} - (T-\lambda I)^{-1} + (\lambda-\mu)(T-\lambda I)^{-2}\big\| \leq C|\mu-\lambda|^2$$

for some positive constant $C = C(T, \lambda)$ which is independent of $\mu$ if $|\mu - \lambda|$ is sufficiently small. Hence,

$$\lim_{\mu\to\lambda} \frac{(T-\mu I)^{-1} - (T-\lambda I)^{-1}}{\mu-\lambda} = (T-\lambda I)^{-2}$$

and since $\Phi$ is continuous, we also get

$$\lim_{\mu\to\lambda} \frac{f(\mu)-f(\lambda)}{\mu-\lambda} = \Phi\big[(T-\lambda I)^{-2}\big],$$

which proves that $f$ is holomorphic on $\rho(T)$.

Now, if $|\lambda| \to \infty$, then

$$\lambda f(\lambda) = \Phi\big[\lambda(T-\lambda I)^{-1}\big] = \Phi\Big[\Big(\frac{T}{\lambda} - I\Big)^{-1}\Big] \longrightarrow -\Phi(I).$$

Since $\lambda f(\lambda)$ has a finite limit in infinity, we must have $f(\lambda) \to 0$. □

**Theorem 4.8.** *If $X$ is a Banach space over $\mathbb{C}$ and $T \in \mathscr{L}(X)$, then $\sigma(T) \neq \varnothing$.*

*Proof.* By Proposition 4.6, $\sigma(T)$ is compact. Take any $\lambda_0 \in \mathbb{C} \setminus \sigma(T)$. We have $T - \lambda_0 I \in \mathscr{G}(X)$, so obviously $(T - \lambda_0 I)^{-1} \neq 0$. An appeal to the Hahn–Banach theorem produces a functional $\Phi \in \mathscr{L}(X)^*$ such that $\Phi[(T - \lambda_0 I)^{-1}] \neq 0$. Hence, the function $f\colon \rho(T) \to \mathbb{C}$ defined by (4.6) satisfies $f(\lambda_0) \neq 0$ and is holomorphic according to Proposition 4.7.

Suppose $\sigma(T) = \varnothing$. Then $f$ is an entire function (i.e. holomorphic on the whole complex plane $\mathbb{C}$) and, moreover, it is bounded in view of the fact that $f(\lambda) \to 0$ as $|\lambda| \to \infty$. But then Liouville's theorem would imply that $f$ is the constant zero function which is not the case; a contradiciton. □

**Example 4.9.** The assumption that $\mathbb{K} = \mathbb{C}$ in Theorem 4.8 is, of course, essential as we can see from the first of the following examples:

(1) Let $T \in \mathscr{L}(\mathbb{R}^2)$ be the rotation of angle $\pi/2$ on the real plane, i.e. $T$ is given by the matrix
$$T \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
Plainly, $\sigma(T) = \varnothing$. The reason is that the only potential elements of the spectrum are eigenvalues of $T$ and there are none such, because the characteristic polynomial $P(\lambda) = \det(T - \lambda I) = \lambda^2 + 1$ has no real roots.

(2) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $n \in \mathbb{N}$ and let $X = \mathbb{K}^n$ be equipped with any norm[3]. Every operator $T \in \mathscr{L}(X)$ can be described by an $n \times n$ matrix of $T$ in any fixed basis of $\mathbb{K}^n$. Hence, in this finite-dimensional case, we can write $\mathscr{L}(\mathbb{K}^n) = \mathbb{M}_n(\mathbb{K})$. By a classical fact from linear algebra, an operator $T \in \mathscr{L}(\mathbb{K}^n)$ is one-to-one if and only if it is an isomorphism of $\mathbb{K}^n$; in other words, injectivity is equivalent to surjectivity. Therefore, the only elements of the spectrum of $T$ are its eigenvalues: $\sigma(T) = \sigma_{\mathrm{p}}(T)$.

(3) In infinite-dimensional case the point spectrum can be far away from the whole spectrum which can be seen from the following example. Let $T \in \mathscr{L}(C[0,1])$ be a multiplication operator on the real Banach space $C[0,1]$ given by $Tf(t) = tf(t)$. Then $\sigma(T) = [0,1]$ although $\sigma_{\mathrm{p}}(T) = \varnothing$, i.e. $T$ has no eigenvalues at all **(classes)**.

As it is mentioned in Example 4.9 (2), an operator on a finite-dimensional space is invertible if and only if it is injective, which considerably simplifies calculating the spectrum. Now, we introduce compact operators which somehow resemble finite-rank operators, i.e. those with a finite-dimensional range. In particular, their perturbations by any nonzero multiple of identity are injective if and only if they are surjective and hence invertible. This is the so-called Fredholm alternative which is a part of a more general Riesz–Schauder theorem (Theorem 4.11 below). Compact operators constitute an extremely important class of operators which appear naturally in applications, e.g. differential and integral equations

**Definition 4.10.** Let $X$ and $Y$ be Banach spaces. An operator $T \in \mathscr{L}(X,Y)$ is called *compact* if the range of the unit ball by $T$ is a relatively compact subset of $Y$, that is, $\overline{T(B_X)}$ is compact in the norm topology of $Y$.

We will use the following notation:

- $\mathscr{K}(X,Y) = \{T \in \mathscr{L}(X,Y) \colon T \text{ is compact}\}$,
- $\mathscr{K}(X) = \mathscr{K}(X,X)$.

It is well-known that a metric space $(M,d)$ is compact if and only if it is complete and *totally bounded*, the latter means that for each $\varepsilon > 0$ there is a finite $\varepsilon$-net in $M$, i.e. a finite set $F \subset M$ such that for every $p \in M$ there exists $q \in F$ with $d(p,q) < \varepsilon$. Consequently, in order to check whether an operator $T \in \mathscr{L}(X,Y)$ between Banach spaces is compact, it is enough to verify whether $T(B_X)$ is totally bounded. Using this topological tool one can show that $\mathscr{K}(X,Y)$ is a closed subspace of $\mathscr{L}(X,Y)$ (see **Problem 3.11**). In the case $X = Y$ the subspace $\mathscr{K}(X)$ forms in fact a closed two-sided ideal of $\mathscr{L}(X)$ which means that it is a closed linear subspace and $TS \in \mathscr{K}(X)$ whenever at least one of the operators

[3]We know from Proposition 2.3 that every linear operator on $\mathbb{K}^n$ is bounded, and since the spectrum is defined by purely algebraic means, it is not affected by which norm we consider.

$T$ and $S$ is in $\mathscr{K}(X)$. Note also that for $\dim Y < \infty$ we have $\mathscr{K}(X,Y) = \mathscr{L}(X,Y)$, because every bounded subset of a finite-dimensional space is relatively compact.

Now, our main goal is to prove the following.

**Theorem 4.11 (Riesz–Schauder theorem).** *Let $X$ be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with $\dim X = \infty$ and let $T \in \mathscr{K}(X)$. Then:*

(a) *$0 \in \sigma(T)$;*

(b) *$\sigma(T) = \{0\} \cup \sigma_{\mathrm{p}}(T)$, i.e. each nonzero element of $\sigma(T)$ is an eigenvalue of $T$;*

(c) *for every $\varepsilon > 0$ there are only finitely many $\lambda \in \sigma(T)$ with $|\lambda| \geq \varepsilon$;*

(d) *every $\lambda \in \sigma(T)$, $\lambda \neq 0$ has a finite-dimensional eigenspace, i.e. $\dim \ker(T - \lambda I) < \infty$.*

We start with some auxiliary lemmas.

**Lemma 4.12.** *Let $T \in \mathscr{L}(X)$, denote $S = T - I$ and $Y = S(X)$. If $Y$ is a proper closed subspace of $X$, then for every $\varepsilon > 0$ there is $x_0 \in B_X$ such that*

$$\operatorname{dist}(Tx_0, T(Y)) > 1 - \varepsilon.$$

*Proof.* By Riesz' lemma (Lemma 1.3), there exists $x_0 \in S_X$ such that $\operatorname{dist}(x_0, Y) > 1 - \varepsilon$. Since $Sx_0 \in Y$ and $T(Y) = (S + I)(Y) \subseteq Y$, we have

$$\operatorname{dist}(Tx_0, T(Y)) \geq \operatorname{dist}(Tx_0 + Sx_0, Y) = \operatorname{dist}(x_0, Y) > 1 - \varepsilon. \qquad \square$$

**Definition 4.13.** Let $M$ be a closed subspace of a normed space $X$. We say that $M$ is *complemented* in $X$ if there exists another closed subspace $N \subseteq X$ such that $M \cap N = \{0\}$ and $M + N = X$ in which case we write $X = M \oplus N$.[4]

**Lemma 4.14.** *Let $M$ be a closed subspace of a normde space $X$.*

(a) *If $\dim M < \infty$, then $M$ is complemented.*

(b) *If $\dim X/M < \infty$, then $M$ is complemented.*

*Proof.* We show only assertion (a) which is needed for the proof of the Riesz–Schauder theorem; assertion (b) is left for **classes**.

Let $n = \dim M$ and $(\mathrm{e}_1, \dots, \mathrm{e}_n)$ be a Hamel basis of $M$. Every $x \in M$ can be uniquely written as $x = \sum_{i=1}^{n} \alpha_i(x) \mathrm{e}_i$ and $\alpha_1, \dots, \alpha_n$ defined in this way are continuous linear functionals on $M$ (see Proposition 2.3). By the Hahn–Banach theorem, we can extend each $\alpha_i$ to a functional $\Lambda_i \in X^*$ with the same norm. Define $N = \bigcap_{i=1}^{n} \ker(\Lambda_i)$; it is obviously a closed subspace of $X$ and we have $X = M \oplus N$ (see **Problem 1.10**). $\square$

**Proposition 4.15.** *Let $X$ be a Banach space, $T \in \mathscr{K}(X)$ and let $\lambda \neq 0$. Then:*

(a) *$\dim \ker(T - \lambda I) < \infty$;*

(b) *$(T - \lambda I)(X)$ is closed and finite-codimensional, i.e. $\dim X/(T - \lambda I)(X) < \infty$.*

[4] This is an important and quite subtle concept. Although, as we shall see later, in Hilbert spaces all subspaces are complemented, the situation is much more difficult for general Banach spaces. Perhaps the most classical result on noncomplementability is the Phillips–Sobczyk theorem which says that $c_0$ is not complemented in $\ell_\infty$.

*Proof.* Without loss of generality we may assume that $\lambda = 1$. Denote $N = \ker(T - I)$ and $S = T - I$. Notice that $x \in N$ if and only if $Tx = x$ which means that $T|_N$ is an isomorphism onto its range. But since $T$ is also compact, we infer that $\dim N < \infty$ (otherwise the range of the unit ball would contain an infinite-dimensional ball which, as we know from Corollary 1.8, is not relatively compact). Hence, we have proved (a).

Now, by Lemma 4.14, there exists a closed subspace $X_1 \subseteq X$ with $X = N \oplus X_1$. Define $S_1 = S|_{X_1}$; of course, $S(X) = S(X_1) = S_1(X_1)$ and $\ker S_1 = N \cap X_1 = \{0\}$, thus $S_1$ is one-to-one.

*Claim 1.* $S_1$ is bounded below (see Definition 3.2) and hence $S_1(X_1) = S(X)$ is closed.

If $S_1$ was not bounded below, there would exist a sequence $(x_n)_{n=1}^{\infty} \subset S_{X_1}$ such that $\|S_1 x_n\| \to 0$. Since $T$ is compact, by passing to a subsequence, we may assume that $Tx_n \to y$ for some $y \in X$. As we have $x_n = (T - S)(x_n) \to y$, we infer that $\|y\| = 1$ (all $x_n$'s are on the unit sphere). On the other hand, $S_1 x_n \to S_1 y$ which yields $S_1 y = 0$ and this is impossible as $S_1$ is one-to-one.

Since $S_1$ is bounded below, it has a closed range. Indeed, let $(z_n)_{n=1}^{\infty} \subset X_1$ be any sequence with $(S_1 z_n)_{n=1}^{\infty}$ convergent to some $y \in X$. Then, $(S_1 z_n)_{n=1}^{\infty}$ is a Cauchy sequence and since $S_1$ is bounded below, we have $\|S_1 z\| \geq \delta \|z\|$ for every $z \in X_1$, with some $\delta > 0$. In particular, $\|z_m - z_n\| \leq \delta^{-1} \|S_1 z_m - S_1 z_n\|$ for all $m, n \in \mathbb{N}$ which shows that $(z_n)_{n=1}^{\infty}$ is also a Cauchy sequence. As $X_1$ is complete, there is $z_0 \in X$ such that $z_n \to z_0$. Hence, $S_1 z_n \to S_1 z_0$, thus $y = S_1 z_0 \in S_1(X_1)$ which proves that the range $S_1(X_1)$ is closed.

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