Functional analysis

Lecture 7: Proof of the Riesz–Schauder theorem – stabilization of kernels and ranges; Fredholm operators and the Fredholm alternative

Proof of Proposition 4.15 (cont.) Recall that, with no loss of generality, we assumed that $\lambda = 1$ and denoted $S = T - I$. We have already proved that $N = \ker S$ is finite-dimensional and, if X_1 is a closed subspace of X which is complementary to N, then $S_1 = S|_{X_1}$ is bounded below and thus has a closed range.

Define the iterates S^k for $k = 0, 1, 2, \dots$ by $S^0 = I$ and $S^k = S \circ S^{k-1}$ for $k \in \mathbb{N}$. Let $N_k = \ker S^k$ and observe that since $\mathscr{K}(X)$ is a two-sided ideal in $\mathscr{L}(X)$, we have $S^k = (T - I)^k = T_k \pm I$, where $T_k \in \mathscr{K}(X)$. Therefore, by the first part of the proof, we infer that $\dim N_k < \infty$. Define $M_k = S^k(X) = S^k(X_1)$ and notice that:

- $\{0\} = N_0 \subseteq N_1 \subseteq N_2 \subseteq \dots,$
- $X = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$

Claim 2. There exists $n \in \mathbb{N}$ such that $M_n = M_{n+1}$ and, similarly, there exists $m \in \mathbb{N}$ such that $N_m = N_{m+1}$.

Suppose that for every $n \in \mathbb{N}$ we have $M_{n+1} \subsetneq M_n$. Then, we can apply Lemma 4.12 to the operator $S|_{M_n} \colon M_n \to M_n$ and we infer that there is $y_n \in B_{M_n}$ such that

$$\operatorname{dist}(Ty_n, T(M_{n+1})) \geq \frac{1}{2}.$$

Thus, we get a sequence $(y_n)_{n=1}^{\infty} \subset B_X$ satisfying $\|Ty_n - Ty_m\| \geq \frac{1}{2}$ for all $m \neq n$. This is plainly impossible, since it would mean that $T(B_X)$ is not totally bounded, that is, T would not be a compact operator.

Similarly, assuming that for every $m \in \mathbb{N}$ we have $N_m \subsetneq N_{m+1}$, we can apply Lemma 4.12 to the operator $S|_{N_{m+1}} \colon N_{m+1} \to N_{m+1}$. There exists $z_m \in B_{N_{m+1}}$ such that

$$\operatorname{dist}(Tz_m, T(N_m)) \geq \frac{1}{2}.$$

Again, we have a bounded sequence $(z_m)_{m=1}^{\infty}$ satisfying $\|Tz_m - Tz_n\| \geq \frac{1}{2}$ for all $m \neq n$, which contradicts the assumption that T is compact.

Consequently, there are $m, n \in \mathbb{N}$ such that $M_n = M_{n'}$ for all $n' \geq n$ and $N_m = N_{m'}$ for all $m' \geq m$. Let $p = \max\{m, n\}$.

Claim 3. $X = M_p \oplus N_p$.

Fix $x \in X$. If $x \in M_p \cap N_p$, then $S_p(x) = 0$ and $x = S^p(y)$ for some $y \in X$. Hence, $0 = S^{2p}(y)$, i.e. $y \in N_{2p} = N_p$ which gives $x = 0$. Now, observe that $S^p(x) \in M_p$ but $S^p(M_p) = S^p(S^p(X)) = S^{2p}(X) = S^p(X) = M_p$ which imples that there is $y \in M_p$ satisfying $S^p(x) = S^p(y)$. Therefore, $x - y \in N_p$ and we have the decomposition $x = y + (x - y)$ which shows that $x \in M_p + N_p$ and proves our Claim.

Now, the result follows because by Claim 3, we have $\operatorname{codim} M_p = \dim N_p < \infty$ and $M_1 \supseteq M_p$, so $M_1 = S(X)$ is also of finite codimension. □

The above proposition says that compact perturbations of nonzero multiples of the identity operator are, in a sense, 'almost inveritble'. This property distinguishes another important class of operators.

Definition 4.16. Let X and Y be Banach spaces. An operator $T \in \mathscr{L}(X, Y)$ is called a *Fredholm operator* if it has a closed range[1] and satisfies:

$$\dim \ker T < \infty \quad \text{and} \quad \operatorname{codim} T(X) < \infty. \tag{4.1}$$

The integer number

$$i(T) \coloneqq \dim \ker T - \operatorname{codim} T(X)$$

is called the *index* of T.

Corollary 4.17. *If X is a Banach space and $T \in \mathscr{K}(X)$, then for every $\lambda \neq 0$ the operator $T - \lambda I$ is Fredholm.*

The next assertion is needed in the proof of the Riesz–Schauder theorem, but it is also very important in its own right.

Theorem 4.18 (**Fredholm alternative**). *Let X be a Banach space, $T \in \mathscr{K}(X)$ and let $\lambda \neq 0$. Then, the equation $Tx - \lambda x = y$ has a solution for every $y \in X$ if and only if the equation $Tx = \lambda x$ has only the trivial solution $x = 0$. In other words, $T - \lambda I$ is one-to-one if and only if it is surjective*[2].

Proof. With no loss of generality, we assume that $\lambda = 1$ and let $S = T - I$. First, suppose that $\ker S = \{0\}$. Then, as we saw in the proof of Proposition 4.15, S is an isomorphism onto $S(X)$. We shall prove that $S(X) = X$. As before, for $k = 0, 1, 2, \ldots$ we denote $M_k = S^k(X)$, the range of the k^{th} iterate of S. Referring again to the proof of Proposition 4.15, we infer that there is $n \in \mathbb{N}$ such that $M_n = M_{n'}$ for each $n' \geq n$. In fact, we have $M_1 = M_0 = X$. If not, pick the smallest $k \in \mathbb{N}$ for which $M_{k-1} \neq M_k = M_{k+1}$ and take any $u \in M_{k-1} \setminus M_k$. Since $S(u) \in M_k = M_{k+1}$, there exists $v \in M_k$ with $S(u) = S(v)$, but then we have $0 \neq u - v \in \ker S$; a contradiction.

In the second part, we assume that $S(X) = X$. Define $N_k = \ker S^k$ and suppose, towards a contradiction, that there exists $0 \neq x_0 \in N_1$. By induction, we can construct a sequence $(x_k)_{k=1}^{\infty} \subset X$ such that for every $k \in \mathbb{N}$ we have:

- $S(x_{k+1}) = x_k$,
- $x_k \in N_k \setminus N_{k-1}$.

Indeed, if $x_1, \ldots, x_k$ have been already constructed, then we use the assumption that S is onto and hence we get $x_{k+1} \in X$ with $S(x_{k+1}) = x_k$. Then, we have

$$S^k(x_{k+1}) = S^{k-1}(x_k) = S^{k-2}(x_{k-1}) = \ldots = x_1 \neq 0$$

and hence $S^{k+1}(x_{k+1}) = S(x_1) = 0$, i.e. $x_{k+1} \in N_{k+1} \setminus N_k$.

Having constructed the sequence $(x_k)_{k=1}^{\infty} \subset X$, we clearly obtain a contradiction in view of $N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \ldots$ which, as we have shown in the proof of Proposition 4.15, is impossible. □

[1] That the range $T(X)$ is closed follows automatically from conditions (4.1) by the Open Mapping Theorem. Since we have not proved this theorem yet, we include the condition of the closedness of $T(X)$ in our definition. Also, the Open Mapping Theorem implies that for any Fredholm operator $T \in \mathscr{L}(X, Y)$ there is a decomposition $X = \ker T \oplus X_1$ such that $T|_{X_1}$ is an isomorphism onto its range. Regarding T roughly as an infinite-dimensional matrix, we can thus say that T is invertible up to some finite-dimensional blocks, which justifies calling Fredholm operators 'almost isomorphisms'.

[2] In fact, we have a stronger statement: for every $T \in \mathscr{K}(X)$ and $\lambda \neq 0$, the index $i(T - \lambda I) = 0$. The proof, however, requires the machinery of adjoint operators.

Lemma 4.19. *Let X be a Banach space, $T \in \mathscr{L}(X)$ and assume that $\lambda_1, \ldots, \lambda_n \in \sigma_{\mathrm{p}}(T)$ are pairwise distinct eigenvalues of T. If $e_i \in X$ is a nonzero eigenvector corresponding to λ_i, for $1 \le i \le n$, then the set $\{e_1, \ldots, e_n\}$ is linearly independent.*

Proof. Induction on n. Assume $e_1, \ldots, e_{n-1}$ are linearly independent and $e_n = \sum_{j=1}^{n-1} \alpha_j e_j$ for some scalars α_j. Then

$$\lambda_n e_n = T(e_n) = \sum_{j=1}^{n-1} \alpha_j \lambda_j e_j \quad \text{as well as} \quad \lambda_n e_n = \sum_{j=1}^{n-1} \lambda_n \alpha_j e_j.$$

Hence,

$$\sum_{j=1}^{n-1} \alpha_j (\lambda_n - \lambda_j) e_j = 0;$$

a contradiction. □

Proof of Theorem 4.11. (a) That $0 \in \sigma(T)$ is clear, as otherwise T would be a compact and invertible operator which cannot happen on an infinite-dimensional space. (T would be bounded below and hence $T(B_X)$ would contain some ball, but no ball is relatively compact in view of Corollary 1.8.)

(b) By virtue of the Fredholm alternative, if $\lambda \neq 0$ is not an eigenvalue of T, then $T - \lambda I$ is both one-to-one and surjective. Hence, it is invertible; its inverse is bounded because $T - \lambda I$ is bounded below, as we have shown in the proof of Proposition 4.15 (it also follows from the Open Mapping Theorem; see the footnote at the beginning of Section 4). This means that $\lambda \notin \sigma(T)$ which proves that $\sigma(T) = \{0\} \cup \sigma_{\mathrm{p}}(T)$.

(c) Suppose that $\varepsilon > 0$ and there are infintely many eigenvalues $\{\lambda_i \colon i \in \mathbb{N}\}$ of T with $|\lambda_i| \geq \varepsilon$. For every $i \in \mathbb{N}$ pick an eigenvector $x_i \neq 0$ corresponding to λ_i and define $X_n = \mathrm{lin}\{x_1, \ldots, x_n\}$ for $n \in \mathbb{N}$. We have $T(X_n) = X_n$ and $X_{n-1} \subsetneq X_n$ by Lemma 4.19. So, the Riesz lemma produces $y_n \in X_n$ such that $\|y_n\| = 1$ and $\mathrm{dist}(y_n, X_{n-1}) \geq \frac{1}{2}$. Let $z_n = \lambda_n^{-1} y_n$; then $\|z_n\| \leq \varepsilon^{-1}$ and $T(z_n) \in X_n$. Notice also that $y_n - T(z_n) \in X_{n-1}$. For, write $y_n = \sum_{j=1}^{n} c_j x_j$ and note that

$$y_n - T(z_n) = \sum_{j=1}^{n} \Big(1 - \frac{\lambda_j}{\lambda_n}\Big) c_j x_j = \sum_{j=1}^{n-1} \Big(1 - \frac{\lambda_j}{\lambda_n}\Big) c_j x_j \in X_{n-1}.$$

Now, if $n > m$, then $T(z_m) \in X_m \subseteq X_{n-1}$ and $y_n - T(z_n) \in X_{n-1}$. Therefore,

$$\begin{aligned} \|T(z_n) - T(z_m)\| &\geq \mathrm{dist}(T(z_n), X_{n-1}) \\ &= \mathrm{dist}(T(z_n) + y_n - T(z_n), X_{n-1}) = \mathrm{dist}(y_n, X_{n-1}) \geq \frac{1}{2}, \end{aligned}$$

which plainly contradicts the fact that $T(\varepsilon^{-1} B_X)$ is totally bounded (and as T is compact, the range of every ball is totally bounded).

(d) That $\dim \ker(T - \lambda I) < \infty$ for every $\lambda \in \sigma(T)$, $\lambda \neq 0$ follows from Proposition 4.15. □

We finish this section with an instructive example which illustrates how the Riesz–Schauder theorem works. But, before that, we provide a useful criterion of compactness of integral operators on spaces of continuous functions.

Proposition 4.20. *Let $K\colon [0,1]\times[0,1]\to\mathbb{R}$ and T_K be an endomorphism of $C[0,1]$ defined by*

$$T_K f(t) = \int\limits_0^1 K(t,s)f(s)\,\mathrm{d}s.$$

Assume that:

(i) *$K(t,\cdot)$ is integrable on $[0,1]$, for each $t\in[0,1]$;*

(ii) *the map $[0,1]\ni t\mapsto K(t,\cdot)\in L_1[0,1]$ is continuous.*

Then $T_K\in\mathscr{K}(C[0,1])$.

Proof. First, note that T_K is bounded because for every $t\in[0,1]$ we have $|T_K f(t)|\leq \|K(t,\cdot)\|_{L_1}$ and hence

$$\|T_K\|\leq \max\nolimits_{0\leq t\leq 1}\|K(t,\cdot)\|_{L_1}$$

which is finite in view of conditions (i) and (ii).

Fix any bounded sequence $(f_n)_{n=1}^\infty\subset C[0,1]$, that is, $M\coloneqq \sup_n\|f_n\|_\infty<\infty$. We want to show that the sequence $(T_K f_n)_{n=1}^\infty$ contains a norm convergent (i.e. uniformly convergent) subsequence. Once we do this, the assertion follows because $(f_n)_{n=1}^\infty$ was an arbitrary bounded sequence and that would imply that $T_K(B_{C[0,1]})$ is relatively compact. First, we have

$$\sup_n\|T_K f_n\|_\infty\leq M\|T_K\|<\infty$$

which means that the set $\{T_K f_n\colon n\in\mathbb{N}\}$ is uniformly bounded. Secondly, for any $s,t\in[0,1]$, $s\neq t$ and each $n\in\mathbb{N}$ we have

$$\begin{aligned}|T_K f_n(s)-T_K f_n(t)| &= \Big|\int_0^1 \big(K(s,u)-K(t,u)\big)f_n(u)\,\mathrm{d}u\Big|\\ &\leq \|K(s,\cdot)-K(t,\cdot)\|_{L_1}\cdot\|f_n\|_\infty\end{aligned}$$

which, in view of the uniform continuity of the map in condition (ii), converges to zero when $s\to t$ (uniformly with respect to $n\in\mathbb{N}$). This means that the set $\{T_K f_n\colon n\in\mathbb{N}\}$ is equicontinuous. Consequently, the Arzela–Ascoli theorem implies that $(T_K f_n)_{n=1}^\infty$ contains a uniformly convergent subsequences, as desired. $\square$

Example 4.21. Define an operator $T\in\mathscr{L}(C[0,1])$ on the real Banach space $C[0,1]$ by the formula

$$Tf(x) = \int\limits_0^1 G(x,y)f(y)\,\mathrm{d}y,$$

where

$$G(x,y) = \begin{cases} x(1-y) & \text{for } 0\leq x\leq y\leq 1\\ y(1-x) & \text{for } 0\leq y\leq x\leq 1.\end{cases}$$

Claim 1. $T\in\mathscr{K}(C[0,1])$.

It follows directly from Proposition 4.20 (note that G is continuous on $[0,1]\times[0,1]$, so both assumptions (i) and (ii) are easily verified).

Claim 2. For every $f\in C[0,1]$ we have $Tf(0)=Tf(1)=0$ and

$$(Tf)''(x) = -f(x)\quad\text{for each } x\in[0,1].$$

We use the following classical result on differentiation under the integral sign: Let $H(x,y)$ be defined on a rectangle $[a,b]\times[c,d]$ and assume that:

- for every $x\in[a,b]$ there is a measure zero set $Z_x\subset[c,d]$ such that the partial derivative of H with respect to x exists at all point (x,y) with $y\notin Z_x$;
- there exists an integrable function $\psi\colon[c,d]\to\mathbb{R}$ such that for every $x\in[a,b]$ we have
$$\Big|\frac{\partial H}{\partial x}(x,y)\Big|\le\psi(y)\quad\text{a.e. on } [c,d].$$

Then, the map
$$I(x)=\int_c^d H(x,y)\,\mathrm{d}y$$
is differentiable and we have
$$I'(x)=\int_c^d \frac{\partial H}{\partial x}(x,y)\,\mathrm{d}y\quad\text{for every } x\in[a,b].$$

Since
$$\frac{\partial G}{\partial x}(x,y)=\begin{cases}1-y & \text{for } 0<x<y<1\\ -y & \text{for } 0<y<x<1\end{cases}$$
exists a.e. and is a bounded function, the above mentioned result implies that Tf is differentiable and for every $x\in[0,1]$ we have
$$\begin{aligned}(Tf)'(x)&=\int_0^1\frac{\partial G}{\partial x}(x,y)f(y)\,\mathrm{d}y=-\int_0^x yf(y)\,\mathrm{d}y+\int_x^1(1-y)f(y)\,\mathrm{d}y\\&=-\int_0^1 yf(y)\,\mathrm{d}y+\int_x^1 f(y)\,\mathrm{d}y.\end{aligned}$$

The first term is constant and to the second one we apply the Fundamental Theorem of Calculus which yields $(Tf)''(x)=-f(x)$.

Claim 3. $\sigma_{\mathrm{p}}(T)=\Big\{\dfrac{1}{\pi^2k^2}\colon k=1,2,\dots\Big\}$.
Let $\lambda\neq0$ and $\lambda\in\sigma_{\mathrm{p}}(T)$. Then $Tf=\lambda f$ for some nonzero function $f\in C[0,1]$. Taking the second derivative we get $(Tf)''=\lambda f''$, but Claim 1 yields that $(Tf)''=-f$. Therefore, every eigenvector f of λ satisfies the differential equation
$$\lambda f''=f. \tag{4.2}$$

The characteristic polynomial is λX^2+1; let ξ_1 and ξ_2 be the two complex roots of this polynomial.

Case 1. $\lambda<0$. Then $\xi_1=\sqrt{-\frac{1}{\lambda}}$ and $\xi_2=-\sqrt{-\frac{1}{\lambda}}$. The general solution of (4.2) is
$$f(x)=A\exp\Big(\sqrt{-\frac{1}{\lambda}}x\Big)+B\exp\Big(-\sqrt{-\frac{1}{\lambda}}x\Big).$$

But $f(0) = f(1) = 0$ (which follows from $Tf(0) = Tf(1) = 0$ and the fact that f is an eigenvector of $\lambda \neq 0$). From this we have $A + B = 0$ and, if $A \neq 0$, $\sqrt{-\frac{1}{\lambda}} = -\sqrt{-\frac{1}{\lambda}}$ which is impossible. Hence, there are no negative eigenvalues.

Case 2. $\lambda > 0$. Then $\xi_1 = \mathrm{i}\sqrt{\frac{1}{\lambda}}$ and $\xi_2 = -\mathrm{i}\sqrt{\frac{1}{\lambda}}$. Again, from the general form of solutions and the initial conditions $f(0) = f(1) = 0$ we get $B = -A$ and

$$\exp\Big(\mathrm{i}\sqrt{\frac{1}{\lambda}}\Big) = \exp\Big(-\mathrm{i}\sqrt{\frac{1}{\lambda}}\Big) \iff \sin\sqrt{\frac{1}{\lambda}} = -\sin\sqrt{\frac{1}{\lambda}} \iff \sqrt{\frac{1}{\lambda}} = k\pi \ (k \in \mathbb{Z}).$$

This yields $\lambda = \pi^{-2}k^{-2}$ and for each such λ, equation (4.2) has a nonzero solution which means that λ is an eigenvalue of T and proves Claim 3.

Claim 4. Each eigenvalue $\lambda_k = \pi^{-2}k^{-2}$ $(k \in \mathbb{N})$ has multiplicity one and the eigenspace

$$\ker(T - \lambda_k I) = \mathrm{lin}\{\sin(k\pi x)\}.$$

Putting $\lambda = \lambda_k$ in the formula for the general solution

$$f(x) = A\exp\Big(\mathrm{i}\sqrt{\frac{1}{\lambda}}x\Big) + B\exp\Big(\mathrm{i} - \sqrt{\frac{1}{\lambda}}x\Big),$$

and remembering that $B = -A$, we infer that one solution f is given by

$$f(x) = \exp(\mathrm{i}k\pi x) - \exp(-\mathrm{i}k\pi x) = 2\mathrm{i}\sin(k\pi x)$$

and any other solution is proportional to this one. Hence, the space of (in our case, real-valued) solutions is $\mathrm{lin}\{\sin(k\pi x)\}$ which proves our Claim.