Complex analysis

I. Topology and geometry of $\mathbb{C}$

Problem 1. Let $\zeta, z_1, z_2, \dots, z_n$ be complex numbers satisfying

$$\sum_{k=1}^{n} \frac{1}{\zeta - z_k} = 0.$$

Show that ζ belongs to the convex hull of $\{z_1, z_2, \dots, z_n\}$.

Problem 2. Denote by $B(z, r)$ the open disk on the complex plane of centre z and radius r. Decide whether there exists a convergent sequence $(z_n)_{n=1}^{\infty}$ of points in $\mathbb{C}$ such that the disks $B(z_n, 1/n)$ are pairwise disjoint.

Problem 3. Let $n \in \mathbb{N}$. Find the number of these $z \in \mathbb{C} \setminus \mathbb{R}$ for which both z^n and $(z+1)^n$ are real numbers.

Problem 4. Suppose that $z_1, z_2, \dots, z_n \in \mathbb{C}$ (considered as points in the complex plane) lie on the same side of some straight line passing through the origin. Prove that

$$z_1 + z_2 + \ldots + z_n \neq 0 \quad \text{and} \quad \frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n} \neq 0.$$

II. Inequalities on the complex plane

Problem 5. Let A be a subset of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ with k elements, where $n > 6^k$. For each $r \in \mathbb{Z}_n$ define

$$f(r) = \sum_{a \in A} \exp\left(\frac{2\pi i}{n} ra\right)$$

(which may be called the rth *Fourier coefficient* of A). Show that for at least one $r \neq 0$ we have $|f(r)| \geq k/2$.

Problem 6. Let D be the closed unit disc on the plane and let $p_1, p_2, \dots, p_n$ be fixed points in D. Show that there exists a point $p \in D$ such that the sum of the distances of p to each of $p_1, p_2, \dots, p_n$ is greater than or equal to n.

Problem 7. Let A be a finite set of complex numbers. Show that there exists a set $B \subset A$ such that

$$\left| \sum_{z \in B} z \right| \geq \frac{1}{\pi} \sum_{z \in A} |z|.$$

Problem 8. Consider n distinct complex numbers $z_1, \dots, z_n$ satisfying

$$\min_{i \neq j} |z_i - z_j| \geq \max_{1 \leq i \leq n} |z_i|.$$

What is the greatest possible value of n?

Problem 9. Let $z \in \mathbb{C}$. Show that $|z| - \Re z \leq 1/2$ if and only if $z = ac$ for some $a, c \in \mathbb{C}$ satisfying $|\bar{c} - a| \leq 1$.

Problem 10. Let $(a_n)_{n=1}^{\infty}$ be a sequence of non-zero complex numbers with the property that $|a_k - a_\ell| > 1$ for all $k, \ell \in \mathbb{N}$, $k \neq \ell$. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n^3}$$

converges.

Problem 11. Let $a, b, c, d \in \mathbb{C}$, $ac \neq 0$. Show that

$$\frac{\max(|ac|, |ad + bc|, |bd|)}{\max(|a|, |b|) \cdot \max(|c|, |d|)} \geq \frac{-1 + \sqrt{5}}{2}.$$

III. Complex polynomials

Problem 12 (*Gauss–Lucas theorem*)**.** Prove that all complex roots of the derivative of a complex polynomial P belong to the convex hull of the set of all complex roots of P.

Problem 13. Let all roots of an nth degree polynomial P with complex coefficients lie on the unit circle of the complex plane. Prove that all roots of the polynomial

$$2zP'(z) - nP(z)$$

also lie on the unit circle.

Problem 14. Let P be a non-constant complex polynomial of degree n. Prove that there exist at least $n + 1$ complex numbers z such that $P(z) \in \{0, 1\}$.

Problem 15. Let k, n be positive integers and P a polynomial of degree n each of whose coefficients equals -1, 1, or 0, and which is divisible by $(x-1)^k$. Let q be a prime number such that

$$\frac{q}{\log q} < \frac{k}{\log(n+1)}.$$

Prove that the complex qth roots of unity are roots of the polynomial P.

Problem 16. Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial with real coefficients. Prove that if all roots of f lie in the left half-plane $\{z \in \mathbb{C} : \Re z < 0\}$, then

$$a_k a_{k+3} < a_{k+1} a_{k+2}$$

holds true for every $k \in \{0, 1, \ldots, n-3\}$.

Problem 17. How many non-zero coefficients can a polynomial P have, if all its coefficients are integers and $|P(z)| \leq 2$ for all $z \in \mathbb{C}$ such that $|z| = 1$?

Problem 18. Let $p(z) = a_0 + a_1 z + a_2 z^2 \ldots + a_n z^n$ be a complex polynomial. Suppose that $1 = c_0 \geq c_1 \geq \ldots \geq c_n \geq 0$ is a sequence of real numbers which is convex (i.e. $2c_k \leq c_{k-1} + c_{k+1}$ for each $k = 1, 2, \ldots, n-1$). Consider the polynomial

$$q(z) = c_0 a_0 + c_1 a_1 z + c_2 a_2 z^2 + \ldots + c_n a_n z^n.$$

Prove that

$$\max_{|z| \leq 1} |q(z)| \leq \max_{|z| \leq 1} |p(z)|.$$

Problem 19. Show that all complex roots of the polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$, where $0 < a_0 < a_1 < \ldots < a_n$, satisfy $|z| > 1$.

Problem 20. Let S_n denote the set of polynomials $p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ with $a_i \in \mathbb{C}$. Evaluate

$$\min_{p \in S_n} \max_{|z|=1} |p(z)|.$$

Problem 21. Let $P(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$, where $a_i \in \mathbb{C}$. Prove that if $|P(z)| = 1$ for all $z \in \mathbb{C}$ satisfying $|z| = 1$, then $a_1 = \ldots = a_n = 0$.

Problem 22. Let $f(z) = p(z)/q(z)$, where p and q are complex polynomials such that $\deg p > \deg q$ and all roots of p lie inside the unit circle, whereas all roots of q lie outside the unit circle. Prove that

$$\max_{|z|=1} |f'(z)| > \frac{\deg p - \deg q}{2} \max_{|z|=1} |f(z)|.$$

Problem 23. Let $m, n \in \mathbb{N}$, $m < n$ and

$$P(z) = z^m \prod_{k=1}^{n-m} (z - z_k),$$

where $|z_k| \geq 1$ for $1 \leq k \leq n - m$. Show that $P'(z) \neq 0$ for $z \in \mathbb{C}$ with $0 < |z| < m/n$.

Problem 24. Consider a polynomial

$$P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n$$

with complex coefficients.

(a) Show that if $a_i \leq 0$ for each $1 \leq i \leq n$ and $a_1 + a_2 + \ldots + a_n < 0$, then P has exactly one positive root.

(b) Show that for an arbitrary complex root ξ of the polynomial P the absolute value $|\xi|$ is not larger than the only positive zero of the polynomial

$$z^n - |a_1| z^{n-1} - |a_2| z^{n-2} - \ldots - |a_n|.$$

(c) Let $c_1, c_2, \ldots, c_n$ be arbitrary positive numbers satisfying

$$c_1 + c_2 + \ldots + c_n \leq 1.$$

Prove that the absolute value of every complex root of the polynomial P does not exceed the number

$$M = \max\left(\frac{|a_1|}{c_1}, \sqrt{\frac{|a_2|}{c_2}}, \ldots, \sqrt[n]{\frac{|a_n|}{c_n}} \right).$$

(d) Prove that the absolute values of the roots of P

(i) are not larger than the largest among the numbers

$$\sqrt[k]{n|a_k|}, \quad k = 1, 2, \ldots, n;$$

(ii) are not larger than the largest of the numbers

$$\sqrt[k]{\frac{2^n - 1}{\binom{n}{k}} |a_k|}, \quad k = 1, 2, \ldots, n;$$

(iii) are smaller than the largest of the numbers

$$2\sqrt[k]{|a_k|}, \quad k = 1, 2, \ldots, n.$$

IV. Holomorphic functions

Problem 25. For every complex number $z \notin \{0,1\}$ define

$$f(z) = \sum (\log z)^{-4},$$

where the sum is over all branches of the complex logarithm.

(a) Show that there are two polynomials P and Q such that $f(z) = P(z)/Q(z)$ for all $z \in \mathbb{C} \setminus \{0,1\}$.
(b) Show that for all $z \in \mathbb{C} \setminus \{0,1\}$ we have

$$f(z) = z \cdot \frac{z^2 + 4z + 1}{6(z-1)^4}.$$

Problem 26. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function with the property that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$. Prove that there are $\vartheta \in \mathbb{R}$ and $k \in \{0,1,2,\ldots\}$ such that

$$f(z) = \mathrm{e}^{i\vartheta} z^k \quad \text{for } z \in \mathbb{C}.$$

Problem 27. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function such that for some $k \in \mathbb{N}$ and some positive numbers $a, b \in \mathbb{R}$ we have $|f(z)| \le a + b|z|^k$. Show that f is a complex polynomial.

Problem 28. Suppose γ is a closed path on the complex plane and $a, b \in \mathbb{C} \setminus \gamma^*$ satisfy $\mathrm{Ind}_\gamma(a) = \mathrm{Ind}_\gamma(b)$. Prove that for all $m, n \in \mathbb{N}$ we have

$$\int_\gamma (z-a)^{-m}(z-b)^{-n} dz = 0.$$

Problem 29. Let $n \in \mathbb{N}$, $r > 0$ and let $C(r)$ stand for the circle centered in the origin and with radius r. Show that if P is a complex polynomial with $\deg P \le n$ and $a \in \mathbb{C}$ lies inside $C(r)$, then

$$\int_{C(r)} \frac{P(z)dz}{z^{n+1}(z-a)} = 0.$$

Problem 30. Determine all holomorphic functions $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $|f(z)|$ is constant on every circle given by the equation $x^2 + y^2 - ax = 0$, where $a \in \mathbb{R}$.