Complex analysis – solutions

I. Topology and geometry of $\mathbb{C}$

P1. From our hypothesis it follows that

$$\sum_{k=1}^{n} \frac{\overline{\zeta - z_k}}{|\zeta - z_k|} = 0.$$

Taking the conjugation and denoting $\sigma_k = 1/|\zeta - z_k|$ and $\sigma = \sigma_1 + \ldots + \sigma_k$, we obtain

$$\zeta = \sum_{k=1}^{n} \frac{\sigma_k}{\sigma} z_k \in \operatorname{conv}\{z_1, \ldots, z_n\}.$$

P2. Such a sequence **does exist**. We arrange the sequence of open squares S_n with sides of length $2/n$ in some bigger square $\mathcal{Q}$ (to be fixed later on). Namely, we put them into columns starting from the bottom side of $\mathcal{Q}$. In the nth column we put, in the upward direction, the squares S_j for $n^2 \leq j \leq (n+1)^2 - 1$. Moreover, we want all the squares to be pairwise disjoint and we want consecutive columns to touch each other, as well as consecutive squares lying in one column. Now, we put z_n in the center of S_n for each $n \in \mathbb{N}$. The total width of all columns equals $2\sum_{n=1}^{\infty} n^{-2} = \pi^2/3$ and the height of the nth column is

$$2 \sum_{j=n^2}^{n^2+2n} \frac{1}{j} \leq \frac{2(2n+1)}{n^2} \xrightarrow[n\to\infty]{} 0.$$

This shows that $\mathcal{Q}$ may be chosen to be a bounded square such that the sequence (z_n) converges to its right-bottom vertex.

P3. The problem considered is invariant under complex conjugation, hence we may restrict ourselves to complex numbers z satisfying $\Im z > 0$. Such a number fulfills the conditions $z^n \in \mathbb{R}$ and $(z+1)^n \in \mathbb{R}$ if and only if there exist real positive numbers r_1, r_2, and natural numbers $k, \ell \in \{1, \ldots, n-1\}$ such that

$$z = r_1 \mathrm{e}^{k\pi\mathrm{i}/n} \quad \text{and} \quad z + 1 = r_2 \mathrm{e}^{\ell\pi\mathrm{i}/n}.$$

Obviously, this is impossible in the case when $\ell \geq k$. On the other hand, for each pair (k, ℓ) with $\ell < k$ there is exactly one pair (r_1, r_2) of positive numbers which makes the above equalities valid (to see this observe, e.g., that the function, which to each $t > 0$ assigns the distance between the rays $\arg z = k$ and $\arg z = \ell$ at the level t, is strictly increasing from 0 to ∞). Consequently, the final answer is twice (we shall take care of the points lying in the lower half-plane as well) the number of all pairs (k, ℓ) with $k, \ell \in \{1, \ldots, n-1\}$, $\ell < k$, that is

$$2\binom{n-1}{2} = (n-1)(n-2).$$

P4. By a suitable rotation, we may assume that the line considered is $\Re z = 0$ and that all points $z_1, \ldots, z_n$ lie on the right half-plane. Then the real parts of every z_i and $1/z_i$ are positive, thus we have also

$$\Re(z_1 + z_2 + \ldots + z_n) > 0 \quad \text{and} \quad \Re\left(\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right) > 0.$$

II. Inequalities on the complex plane

P6. Since

$$\sum_{k=1}^{n}|p-p_k| \geq \left|np-\sum_{k=1}^{n}p_k\right| = n\left|p-\frac{1}{n}\sum_{k=1}^{n}p_k\right|$$

and the arithmetic mean $\overline{p}=1/n\sum_{k=1}^{n}p_k \in D$, we see that any point $p\in D$ satisfying $|p-\overline{p}|\geq 1$ will do the job.

P7. **(a)** First, we will show how a simple partition trick may be used to get the desired inequality with the constant $1/\pi$ replaced by $\sqrt{2}/8$. Namely, the two diagonal lines split the complex plane into four parts, hence for at least one of them the sum of the absolute values of all points lying there is at least $1/4$ of the whole sum of absolute values. By rotating by a suitable angle, we may assume that the part in question is the one bounded by the two rays: $\arg z = \pm\pi/4$. Let B be the set of these $z\in A$ which lie in that part. Since for each $z\in B$ we have $\Re z \geq \sqrt{2}/2|z|$, we infer that

$$\left|\sum_{z\in B} z\right| \geq \sum_{z\in B}\Re z \geq \frac{\sqrt{2}}{2}\sum_{z\in B}|z| \geq \frac{\sqrt{2}}{8}\sum_{z\in A}|z|.$$

(b) We will use the so-called *isoperimetric inequality*. Let $\mathcal{P}$ be a convex polygon lying in the plane and denote by $d(\mathcal{P})$ and $p(\mathcal{P})$ the diameter and the perimeter of $\mathcal{P}$, respectively. The inequality just mentioned asserts that

$$p(\mathcal{P}) \leq \pi d(\mathcal{P}). \tag{1}$$

Proof of inequality (1). The main idea is to consider the average width of a given polygon $\mathcal{P}$. To make it precise let us say that for any direction angle $\vartheta\in[0,\pi)$ the width $d(\vartheta,\mathcal{P})$ is the least possible distance between two parallel lines of direction ϑ which define a strip containing the polygon $\mathcal{P}$. By the average width of $\mathcal{P}$ we mean the integral mean

$$\overline{d(\mathcal{P})} = \frac{1}{\pi}\int_0^{\pi} d(\vartheta,\mathcal{P})d\vartheta.$$

What we will show is the equality

$$p(\mathcal{P}) = \pi\overline{d(\mathcal{P})}, \tag{2}$$

which obviously would imply (1), since $\overline{d(\mathcal{P})}\leq d(\mathcal{P})$.

By the convexity of $\mathcal{P}$, it is easily seen that $2d(\vartheta,\mathcal{P})$ is nothing else but the sum of the lengths of the projections in the direction ϑ of all sides of $\mathcal{P}$. For the ith side such a projection has the length

$$\delta_i(\vartheta) = \delta_i \sin\vartheta,$$

where δ_i is the length of that side. Consequently,

$$\overline{d(\mathcal{P})} = \frac{1}{2\pi}\sum_i \delta_i \int_0^{\pi}\sin\vartheta d\vartheta = \frac{1}{\pi}p(\mathcal{P}),$$

which completes the proof of (2) and (1). □

Now, we proceed to the solution of our problem. Let $A=\{z_1,\ldots,z_n\}$; obviously we may assume that $z_i\neq 0$ for each i. The right-hand side of the desired inequality suggests to construct a convex polygon with the lengths of its sides equal to $|z_i|$. To this end let us re-order the sequence $z_1, z_2, \dots, z_n, -(z_1 + \dots + z_n)$ in a sequence $w_1, \dots, w_{n+1}$ such that the sequence $\arg w_1, \dots, \arg w_{n+1}$ is increasing (in the case when $z_1 + \dots + z_n = 0$, and its argument is not well-defined, we may consider the shorter sequence $z_1, \dots, z_n$ and proceed similarly as below). Then the points

$$w_1, w_1 + w_2, \dots, w_1 + \dots + w_{n+1}$$

are the vertices of a convex polygon $\mathcal{P}$. Moreover,

$$p(\mathcal{P}) = \sum_{i=1}^{n+1} |w_i| = \sum_{i=1}^{n} |z_i| + \left|\sum_{i=1}^{n} z_i\right|$$

whereas the diameter $d(\mathcal{P})$, being the distance between some two point w_i and w_j with $i < j$, satisfies

$$d(\mathcal{P}) = \left|\sum_{z \in B} z\right|$$

for some set $B \subset A$. Therefore, it remains to apply inequality (1).

P11. Let $\varphi = (-1 + \sqrt{5})/2$; of course we have $\varphi = 1 - \varphi^2$. After dividing the numerator and the denominator by $|ac|$, and substituting $u = b/a$, $v = d/c$, our assertion reduces to

$$\frac{\max(1, |u+v|, |uv|)}{\max(1, |u|) \cdot \max(1, |v|)} \geq \varphi.$$

If either $|u| \geq 1 \leq |v|$, or $|u| < 1 > |v|$, the situation is clear, since then the fraction at the left-hand side is at least 1 which is greater than φ. It remains to consider, e.g., the case when $|u| \geq 1$ and $|v| < 1$. Dividing the nominator and denominator by $|u|$ we arrive at the expression

$$\alpha := \max\left(\frac{1}{|u|}, \left|1 + \frac{v}{u}\right|, |v|\right).$$

Since the geometric mean (or any other mean) never exceeds the maximum of the arguments, we infer that $\alpha \geq \sqrt{|v/u|}$ so in the case when $|v/u| \geq \varphi^2$ we are done. Otherwise,

$$\alpha \geq \left|1 + \frac{v}{u}\right| \geq 1 - \left|\frac{v}{u}\right| > 1 - \varphi^2 = \varphi.$$

III. Complex polynomials

P12 (*Gauss–Lucas theorem*)**.** Write $P(z) = a(z - a_1) \cdot \ldots \cdot (z - a_n)$ and

$$\frac{P'(z)}{P(z)} = (\log P(z))' = \sum_{k=1}^{n} \frac{1}{z - a_k}.$$

Hence, if $P'(z) = 0$ for some $z \notin \{z_1, \dots, z_n\}$, it is enough to recall **P1**.

P13. Let $P(z) = (z - a_1) \cdot \ldots \cdot (z - a_n)$ (we may assume that the leading coefficient is 1). Then

$$2zP'(z) - nP(z) = 2z \sum_{k=1}^{n} \frac{P(z)}{z - a_k} - nP(z) = P(z) \sum_{k=1}^{n} \left(\frac{2z}{z - a_k} - 1\right) = P(z) \sum_{k=1}^{n} \frac{z + a_k}{z - a_k}.$$

Now, an application of the formula

$$\Re\frac{x+y}{x-y}=\frac{|x|^2-|y|^2}{|x-y|^2}\quad\text{for } x,y\in\mathbb{C}, x\neq y,$$

yields

$$\Re(2zP'(z)-nP(z))=\sum_{k=1}^{n}\frac{|z|^2-1}{|z-1|^2}\quad\text{for } z\neq 1.$$

Hence, the left-hand side may vanish only for $|z|=1$.

P 14. Denote $S_i=\{z\in\mathbb{C}:\,P(z)=i\}$ for $i\in\{0,1\}$ and for an arbitrary complex polynomial p and any $z_0\in\mathbb{C}$ let $\mu(z_0,p)$ be the multiplicity of z_0 as a root of p, namely,

$$\mu(z_0,p)=\max\{k\in\mathbb{N}\cup\{0\}:\,(z-z_0)^k\mid p(z)\}.$$

We have

$$\begin{aligned}|S_0|+|S_1|&=\sum_{z\in S_0}\big(\mu(z,P)-\mu(z,P')\big)+\sum_{z\in S_1}\big(\mu(z,P-1)-\mu(z,P')\big)\\&=2n-\sum_{z\in S_0\cup S_1}\mu(z,P')\geq 2n-\deg P'=n+1.\end{aligned}$$

P15. This problem is in fact more algebraic in nature. We may write $P(x)=(x-1)^k r(x)$ for some polynomial $r(x)\in\mathbb{Q}[x]$. Denote all non-trivial qth complex roots of unity by ε_j, that is

$$\varepsilon_j=\mathrm{e}^{\frac{2\pi j}{q}\mathrm{i}}\quad\text{for } j=1,\ldots,q-1.$$

We wish to prove that $r(\varepsilon_j)=0$ for each j. However, all ε_j's are roots of the polynomial $\Phi_q(x)=1+x+\ldots+x^{q-1}$ which is irreducible over $\mathbb{Q}[x]$, by virtue of the Eisenstein criterion[1]. This means that if **at least one** of ε_j's is a root of the polynomial $r(x)$, then necessarily **all of them** must be roots of $r(x)$ (if it was not true, the greatest common divisor of the polynomials $\Phi_q(x)$ and $r(x)$ would be a non-trivial divisor of $\Phi_q(x)$ in $\mathbb{Q}[x]$).

Now, the crucial argument comes into play. If we had $r(\varepsilon_j)\neq 0$ for each $1\leq j\leq q-1$, then the product $\prod_{j=1}^{q-1}r(\varepsilon_j)$ would be a non-zero integer. Indeed, this product may be represented as a finite sum of some symmetric $(q-1)$-variables polynomials on $\varepsilon_1,\ldots,\varepsilon_{q-1}$. However, every such term vanishes, in view of the fact that every symmetric polynomial is a combination of the basic symmetric polynomial which in turn all vanish on $\varepsilon_1,\ldots,\varepsilon_{q-1}$, by virtue of Viète's formulas. Consequently,

$$\begin{aligned}(n+1)^{q-1}&\geq\left|\prod_{j=1}^{q-1}P(\varepsilon_j)\right|=\left|\prod_{j=1}^{q-1}(1-\varepsilon_j)\right|^k\cdot\left|\prod_{j=1}^{q-1}r(\varepsilon_j)\right|\\&\geq\left|\prod_{j=1}^{q-1}(1-\varepsilon_j)\right|^k=(1+1^2+\ldots+1^{q-1})^k=q^k,\end{aligned}$$

[1] **Eisenstein's criterion.** Let $\varphi(x)=a_nx^n+\ldots+a_1x+a_0$ be a polynomial with integer coefficients. If there exists a prime number p such that $p\mid a_0, p\mid a_1,\ldots,p\mid a_{n-1}, p\nmid a_n$ and $p^2\nmid a_0$, then the polynomial $\varphi(x)$ is irreducible in $\mathbb{Q}[x]$.

To show that $\Phi_q(x)$ is irreducible over $\mathbb{Q}[x]$ consider instead the polynomial $\varphi(x)=\Phi_q(x+1)$. We have

$$\varphi(x)=\frac{(x+1)^q-1}{x}=x^{q-1}+\binom{q}{q-1}x^{q-2}+\ldots+\binom{q}{2}x+q,$$

thus, by Eisenstein's criterion, $\varphi(x)$ is irreducible over $\mathbb{Q}[x]$ and, obviously, neither is $\Phi_q(x)$.

which gives $(q-1)\log(n+1) \geq k \log q$; a contradiction.

P22. Let $m = \deg p$, $n = \deg q$, $p(z) = a(z-a_1) \cdot \ldots \cdot (z-a_m)$ and $q(z) = b(z-b_1) \cdot \ldots \cdot (z-b_n)$. With no loss of generality we may suppose that the maximum of $|f(z)|$ on the unit circle is attained for $z = 1$. We will show that

$$\left| \frac{f'(1)}{f(1)} \right| > \frac{m-n}{2}\,. \tag{3}$$

The left-hand side of (3) equals

$$\left|(\log f(z))'\right|_{z=1} = \left| \sum_{k=1}^{m} \frac{1}{1-a_k} - \sum_{k=1}^{n} \frac{1}{1-b_k} \right| \geq \sum_{k=1}^{m} \Re \frac{1}{1-a_k} - \sum_{k=1}^{n} \Re \frac{1}{1-b_k}\,. \tag{4}$$

For each $1 \leq k \leq m$ we have $|a_k| < 1$, thus $1-a_k$ lies inside the circle C centered in 1 of radius 1. The inversion $z \mapsto z^{-1}$ transforms this circle into the straight line $\Re z = 1/2$, hence the open ball with boundary C is mapped onto the open half-plane $\Re z > 1/2$. Similarly, since for each $1 \leq k \leq n$ we have $|b_k| > 1$, the inverse of $1-b_k$ lies in the half-plane $\Re z < 1/2$. Hence, putting $z = 1$ in (4) gives (3).