Functional analysis — Exercises*

Part 1: General properties of norms; distances between subspaces; hyperplanes; linear functionals

● **Problem 1.1.** Show that the normed spaces: c_0, c and ℓ_p for $1 \leqslant p < \infty$ are all separable, while ℓ_∞ is not separable.

● **Problem 1.2.** Let Y be a finite-dimensional subspace of a normed space X. Show that for every $x \in X$ there exists $y \in Y$ such that

$$\|x - y\| = \operatorname{dist}(x, Y).$$

● **Problem 1.3.** Define

$$M = \Big\{ f \in L_1([0,1]) \colon \int_0^1 f(t)\,\mathrm{d}t = 1 \Big\}.$$

Show that M is a closed and convex subset of $L_1([0,1])$ which contains infinitely many elements of minimal norm, i.e. there are infinitely many $f \in M$ satisfying $\|f\|_1 = \min_{g\in M} \|g\|_1$.

Note that $L_1([0,1])$ stands for the normed space $L_1(\mu)$ as defined in the lecture, where μ is the Lebesgue measure on $[0,1]$. In other words, the elements of $L_1([0,1])$ are the (classes of abstraction of) integrable scalar-valued functions on $[0,1]$ with the norm $\|f\|_1 = \int_0^1 |f(t)|\,\mathrm{d}t$.

● **Problem 1.4.** Define

$$M = \Big\{ f \in C[0,1] \colon \int_0^{1/2} f(t)\,\mathrm{d}t - \int_{1/2}^1 f(t)\,\mathrm{d}t = 1 \Big\}.$$

Prove that M is a closed and convex subset of $C[0,1]$ which does not contain any element of minimal norm, i.e. for each $f \in M$ we have $\|f\|_\infty > \inf_{g\in M} \|g\|_\infty$.

● **Problem 1.5.** Consider the following two subspaces of c_0:

$$Y = \Big\{ (\alpha_n)_{n=1}^\infty \in c_0 \colon \alpha_{2n-1} = 0 \text{ for each } n \in \mathbb{N} \Big\},$$

$$Z = \Big\{ (\alpha_n)_{n=1}^\infty \in c_0 \colon \alpha_{2n} = n\alpha_{2n-1} \text{ for each } n \in \mathbb{N} \Big\}.$$

Show that Y and Z are closed subspaces of c_0, whereas $Y + Z \subsetneq c_0$ is a proper dense subspace of c_0 (and therefore fails to be closed).

● **Problem 1.6.** Let X be a normed space and $M \subset X$ be any nonempty set. Show that the function

$$f(x) = \operatorname{dist}(x, M) = \inf\{\|x - y\| \colon y \in M\}$$

is 1-Lipschitz, that is, $|f(x) - f(y)| \leqslant \|x - y\|$ for all $x, y \in X$.

● **Problem 1.7.** Prove that any two vectors x, y of a normed space satisfying $\|x + y\| = \|x\| + \|y\|$ also satisfy

$$\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\| \quad \text{for any } \alpha, \beta \geqslant 0.$$

*Evaluation: ●=2pt, ●=3pt, ●=4pt

● **Problem 1.8.** Prove that any nonzero vectors x, y in a normed space satisfy the inequality

$$\|x+y\| \leqslant \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \cdot \min\{\|x\|, \|y\|\}.$$

● **Problem 1.9.** Let X be a normed space over $\mathbb{C}$ and let $X_{\mathbb{R}}$ be the same space treated as a normed space over $\mathbb{R}$. Show that for every $\mathbb{R}$-linear functional $\varphi\colon X_{\mathbb{R}} \to \mathbb{R}$ there exists a unique $\mathbb{C}$-linear functional $\widetilde{\varphi}\colon X \to \mathbb{C}$ satisfying $\varphi(x) = \operatorname{Re}\widetilde{\varphi}(x)$ for each $x \in X$ and it is given by the formula

$$\widetilde{\varphi}(x) = \varphi(x) - \mathrm{i}\varphi(\mathrm{i}x) \quad (x \in X).$$

● **Problem 1.10.** Let E be a linear space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$; we denote by E' its (algebraic) conjugate space, i.e. the space of all linear functionals acting on E. Suppose $e_1, \ldots, e_n \in E$ are linearly independent and $e_1^*, \ldots, e_n^* \in E'$ satisfy

$$e_i^*(e_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases}$$

Show that

$$E = \operatorname{lin}\{e_1, \ldots, e_n\} \oplus \bigcap_{i=1}^{n} \ker(e_i^*).$$

● **Problem 1.11.** Let X be a normed space and let $x, y \in X$ be nonzero vectors such that $\|x\|, \|y\| \leqslant 1$ and $\|x - y\| \geqslant 1$. Prove that

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \geqslant \|x - y\|.$$

● **Problem 1.12.** Let E be a linear space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. By a *hyperplane* in E we mean any linear subspace $M \subset E$ of codimension 1, i.e. a subspace such that $E = M \oplus \operatorname{lin}\{u\}$ for some $0 \neq u \in E$.

(a) Prove that for every nonzero linear functional x^* on E, the kernel $\ker(x^*)$ forms a hyperplane, and $E = \ker(x^*) + \operatorname{lin}\{u\}$ for any $u \in E$ with $x^*(u) \neq 0$.

(b) Show that every hyperplane in E is the kernel of some linear functional on E.

(c) Prove that for each hyperplane $M \subset E$ the linear functional x^* satisfying $\ker(x^*) = M$ is determined uniquely up to a scalar constant. More precisely, if linear functionals x^*, y^* satisfy $\ker(x^*) = \ker(y^*) = M$, then there exists $\alpha \in \mathbb{K}$ such that $x^* = \alpha y^*$.

● **Problem 1.13.** (a) Prove that on every infinite-dimensional normed space one can define a discontinuous linear functional.

(b) Show that for any discontinuous linear functional x^* defined on a normed space X we have

$$\overline{\ker(x^*)} = X,$$

that is, the kernel of x^* is a dense subspace of X.

● **Problem 1.14.** The *Hilbert cube* $\mathcal{Q}$ is a subset of ℓ_2 defined as

$$\mathcal{Q} = \Big\{(x_n)_{n=1}^{\infty} \in \ell_2 \colon |x_n| < 2^{-n} \text{ for each } n \in \mathbb{N}\Big\}.$$

Show that $\mathcal{Q}$ is a compact subset of ℓ_2.

● **Problem 1.15.** Consider the space $C[0,1]$ over $\mathbb{R}$ and its closed subspace X consisting of those functions x for which $x(0)=0$. Next, define $Y\subset X$ by $Y=\{y\in X\colon \int_0^1 y(t)\,dt=0\}$. Show that Y is a closed subspace of X, yet there is no $x\in S_X$ satisfying $\operatorname{dist}(x,Y)=1$.

This assertion means that in general we cannot avoid the $\varepsilon>0$ in Riesz' lemma (although we can do it if the subspace in question is finite-dimensional — why?).

● **Problem 1.16.** Let $a,b\in\mathbb{R}$, $a<b$, $k\in\mathbb{N}$ and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We denote by $C^{(k)}([a,b])$ the space of all k-times continuously differentiable $\mathbb{K}$-valued functions on $[a,b]$ (at the endpoints we consider one-sided derivatives). Prove that $C^{(k)}([a,b])$ is a Banach space when equipped with each of the following two norms:

$$\|f\|_{(k)}=\sum_{i=0}^{k}\|f^{(i)}\|_\infty,$$

$$\|f\|'_{(k)}=|f(a)|+\max_{1\leqslant i\leqslant k}\|f^{(i)}\|_\infty,$$

where $\|\cdot\|_\infty$, as usual, stands for the supremum norm. Are the two norms $\|\cdot\|_{(k)}$ and $\|\cdot\|'_{(k)}$ equivalent?

● **Problem 1.17.** Prove the following Riesz' theorem which characterizes Banach spaces among normed spaces: A normed space X is complete if and only if every absolutely convergent series in X is convergent, that is, for every sequence $(x_n)_{n=1}^\infty\subset X$ the condition

$$\sum_{n=1}^{\infty}\|x_n\|<\infty$$

implies that the series $\sum_{n=1}^\infty x_n$ is convergent (with respect to the norm topology) in X.

● **Problem 1.18.** Let (M,ρ) be a metric space and $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. For a Lipschitz function $f\colon M\to\mathbb{K}$ we denote by $L(f)$ its Lipschitz constant, i.e.

$$L(f)=\sup\left\{\frac{|f(x)-f(y)|}{\rho(x,y)}\colon x,y\in M,\, x\neq y\right\}.$$

Define $\operatorname{Lip}(M)$ to be the set of all bounded $\mathbb{K}$-valued Lipschitz functions on M. Also, for any distinguished point $0\in M$, we define $\operatorname{Lip}_0(M)$ to be the set of all Lipschitz functions $f\colon M\to\mathbb{K}$ with $f(0)=0$. With the standard algebraic operations on functions these sets form vector spaces over $\mathbb{K}$. Show that the formula

$$\|f\|_L=\max\{\|f\|_\infty,L(f)\}$$

defines a norm on $\operatorname{Lip}(M)$, and that $L(\cdot)$ is a norm on $\operatorname{Lip}_0(M)$. Next, prove that both $(\operatorname{Lip}(M),\|\cdot\|_L)$ and $(\operatorname{Lip}_0(M),L(\cdot))$ are Banach spaces.

Hint. You are allowed to use the assertion of Problem 1.17.

● **Problem 1.19.** Let X be a normed space and $M\subset X$. For $\varepsilon>0$, we say that a set $A\subseteq M$ is an ε-*net* in M provided that for every $x\in M$ there is $y\in A$ such that $\|x-y\|<\varepsilon$. We say that A is ε-*separated* if $\|x-y\|\geqslant\varepsilon$ for all $x,y\in A$, $x\neq y$. Suppose that $\dim X=n<\infty$. Show that every ε-net in the unit ball B_X must contain at least ε^{-n} elements. On the other hand, show that there exists an ε-net in B_X with at most $(1+\frac{2}{\varepsilon})^n$ elements.

● **Problem 1.20.** Prove that for every infinite dimensional normed space X there exists an infinite sequence $(x_n)_{n=1}^\infty$ on the unit sphere S_X such that $\|x_m - x_n\| > 1$ for all $m, n \in \mathbb{N}$, $m \neq n$.

● **Problem 1.21.** Prove that for every compact subset K of a normed space X there exists a sequence $(x_n)_{n=1}^\infty \subset X$ such that $\lim_n \|x_n\| = 0$ and $K \subseteq \overline{\text{co}}\{x_n \colon n \in \mathbb{N}\}$.

Here, $\overline{\text{co}}(A)$ denotes the closed convex hull of A, that is, the closure of the smallest convex set containing A, that is, the closure of $\{\lambda_1 x_1 + \ldots + \lambda_n x_n \colon n \in \mathbb{N},\, x_i \in A,\, \lambda_i \in [0,1],\, \lambda_1 + \ldots + \lambda_n = 1\}$.