Functional analysis — Exercises*

Part 2: Basic properties of bounded operators; The Hahn–Banach theorem; Minkowski functionals; separation theorems

Note that whenever we do not mention the scalar field you should work with both real and complex case (usually, it does not make any difference). Of course, considering any linear operator/functional between two vector spaces, we assume them to be over the same scalar field.

● **Problem 2.1.** For a given normed space X and a functional $\varphi \in X^*$, determine $\|\varphi\|$.

(a) $X = \ell_2, \quad \varphi(x_1, x_2, \dots) = \displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n}$

(b) $X = c_0, \quad \varphi(x_1, x_2, \dots) = \displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n^2}$

(c) $X = \ell_p, \quad \varphi(x_1, x_2, \dots) = \displaystyle\sum_{k=1}^{N} \frac{x_k}{N^{1/q}} \quad (N \in \mathbb{N},\ 1 < p, q < \infty,\ p^{-1} + q^{-1} = 1)$

(d) $X = \ell_1, \quad \varphi(x_1, x_2, \dots) = \displaystyle\sum_{n=1}^{\infty} (x_{2n-1} - 3x_{2n})$

● **Problem 2.2.** For a given subspace M of a normed space X, decide whether M is a closed hyperplane (i.e. a closed subspace of codimension 1). If so, find a functional $x^* \in X^*$ for which $\ker x^* = M$.

(a) $X = c, \qquad M = c_0$

(b) $X = \ell_1, \qquad M = \Big\{ (x_n)_{n=1}^{\infty} \in \ell_1 \colon \displaystyle\sum_{n=2}^{\infty} x_n = 2x_1 \Big\}$

(c) $X = C[0,2], \qquad M = \big\{ f \in C[0,2] \colon f(1) = f(2) \big\}$

(d) $X = C[-1,1], \quad M = \Big\{ f \in C[-1,1] \colon \displaystyle\int_0^1 f(t)\,\mathrm{d}t = \int_{-1}^1 f(t)\,\mathrm{d}t \Big\}$

● **Problem 2.3.** Consider a linear endomorphism T acting on the real vector space $\mathbb{R}^2$ and defined by

$$T(x,y) = \Big(\frac{x-y}{2}, \frac{x+y}{2} \Big).$$

Of course, its norm depends on which norms we consider on the domain and codomain. Recall that by $\|\cdot\|_p$ we denote the 'ℓ_p-norm', that is, $\|(x,y)\|_p = (|x_1|^p + |x_2|^p)^{1/p}$ (for $p = \infty$ it is the maximum norm). For each of the nine possible combinations of pairs (i,j), with $i,j \in \{1,2,\infty\}$, calculate the norm of the operator

$$T \colon (\mathbb{R}^2, \|\cdot\|_i) \longrightarrow (\mathbb{R}^2, \|\cdot\|_j).$$

● **Problem 2.4.** For any given function $g \in L^1([0,1])$ we define a linear functional on $C[0,1]$ by the formula

$$\varphi(f) = \int_{[0,1]} f(t) g(t)\,\mathrm{d}t.$$

Determine $\|\varphi\|$.

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*Evaluation: ●=2pt, ●=3pt, ●=4pt

● **Problem 2.5.** Let $K\colon [0,1]\times[0,1] \to \mathbb{R}$ be any (jointly) continuous functions of two variables. Define an operator $\Phi_K\colon C[0,1] \to C[0,1]$ by the formula

$$\Phi_K f(t) = \int\limits_0^1 K(s,t)f(s)\,\mathrm{d}s.$$

(a) Show that $\|\Phi_K\| = \sup\Big\{\int_0^1 |K(s,t)|\,\mathrm{d}s\colon t\in[0,1]\Big\}$.

(b) Determine $\|\Phi_K\|$ in the case where $K(s,t)=\cos\pi(s-t)$.

● **Problem 2.6.** In the Banach space $C[0,1]$ consider the subspace X consisting of all continuously differentiable functions on $[0,1]$, i.e. C^1-functions (at the points 0 and 1 we consider one-sided derivatives). Show that the differentiation operator

$$\mathrm{D}\colon X \longrightarrow C[0,1], \quad \mathrm{D}f = f'$$

is not bounded, despite the fact that its graph $\{(f,\mathrm{D}f)\colon f\in C[0,1]\}$ forms a closed subspace of $(C[0,1]\oplus C[0,1], \|\cdot\|_{\max})$, where on the direct sum $C[0,1]\oplus C[0,1]$ we consider the norm $\|(f,g)\| = \max\{\|f\|_\infty, \|g\|_\infty\}$.

For any subsets A, B of a vector space X over $\mathbb{K}$, and for any $\lambda\in\mathbb{K}$, we use the following standard notation for the algebraic operations on sets:

$$A+B=\{a+b\colon a\in A,\, b\in B\}, \quad \lambda A=\{\lambda a\colon a\in A\}.$$

Recall that A is called:

- *convex* provided that $\lambda A+(1-\lambda)A\subseteq A$ for every $\lambda\in[0,1]$,
- *balanced* if $\lambda A\subseteq A$ for every $\lambda\in\mathbb{K}$ with $|\lambda|\leqslant 1$,
- *absorbing* if $X=\bigcup_{\lambda>0}\lambda A$ (plainly, every neighborhood of 0 is absorbing).

Let C be an absorbing subset of a normed space X. We define the *Minkowski functional* of C by

$$\mu_C(x) = \inf\big\{\lambda>0\colon x\in\lambda C\big\}.$$

(Note that $\mu_C(x)<\infty$ because C is absorbing.) In particular, $\mu_{B_X}=\|\cdot\|$.

The next exercise is to show that for C absorbing and convex, the functional μ_C has nice properties: it is subadditive and positively homogeneous. Hence, if C is also balanced, μ_C yields a *seminorm* on X, i.e. μ_C is then a nonnegative homogeneous function satisfying the triangle inequality (although μ_C can vanish on some nonzero vectors and it is the only reason for it not being a norm). In fact, for any vector space X over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, the collection of all seminorms coincides with the class of Minkowski functionals of balanced, convex, absorbing subsets of X.

● **Problem 2.7.** Let C be a convex neighborhood of 0 in a normed space X. Show that its Minkowski functional $\mu_C\colon X\to[0,\infty)$ has the following properties:

(a) $\mu_C(tx)=t\mu_C(x)$ for all $t\geqslant 0$, $x\in X$ (positive homogeneity),

(b) $\mu_C(x+y)\leqslant\mu_C(x)+\mu_C(y)$ for all $x,y\in X$ (subadditivity),

(c) μ_C is (norm) continuous,

(d) $\{x\in X\colon \mu_C(x)<1\}=\operatorname{int} C$, $\quad\{x\in X\colon\mu_C(x)\leqslant 1\}=\overline{C}$.

● **Problem 2.8.** Let X be the normed space $(\ell_1, \|\cdot\|_\infty)$, that is, $X = \ell_1$ as a vector space and is equipped with the supremum norm (note that we have the set inclusion $\ell_1 \subset \ell_\infty$). Show that the set $C = \{x \in X\colon \|x\|_1 \leqslant 1\}$ is a closed, convex, absorbing subset of X with empty interior. What is the Minkowski functional μ_C?

In view of Problem 2.26, the space X cannot be a Banach space. In other words, ℓ_1 is not complete with respect to the ℓ_∞-norm (which, anyway, can be easily justified by observing that ℓ_1 is not a closed subspace of ℓ_∞).

● **Problem 2.9.** Let X be a normed space and $x^* \in S_{X^*}$. Show that for every $x \in X$ we have

$$\operatorname{dist}(x, \ker x^*) = |\langle x, x^* \rangle|.$$

Compare it with Corollary 2.6 and try to prove a more quantitative version of that statement (what is the optimal norm of the functional x^* there?).

● **Problem 2.10.** Prove that if H is a closed hyperplane of a normed space X, then there exists a bounded *projection* P from X onto H, i.e. a bounded linear operator $P\colon X \to H$ such that $P|_H = \mathrm{id}_H$ (equivalently: P is bounded, surjective and idempotent, i.e. $P \circ P = P$).

● **Problem 2.11.** Let $X = \mathbb{R}^2$ equipped with the norm $\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p)^{1/p}$, where $p \in (1,\infty)$. Using the Lagrange multipliers calculate directly the dual norm on X^*. Conclude that $(\ell_p^2)^*$ can be identified with ℓ_q^2, where $\frac{1}{p} + \frac{1}{q} = 1$.

● **Problem 2.12.** Consider the 2-dimensional real space $\mathbb{R}^2$ equipped with a norm $\|\cdot\|$ given below, its linear subspace Y and a linear functional $f\colon Y \to \mathbb{R}$. Find at least one norm preserving extension of f, that is, a linear functional $\widetilde{f}\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying $\widetilde{f}|_Y = f$ and $\|f\| = \|\widetilde{f}\|$.

(a) $\|(x,y)\| = \|(x,y)\|_1 = |x| + |y|, \quad Y = \{(x,y) \in \mathbb{R}^2\colon y = 3x\}, \quad f(x,y) = -x$

(b) $\|(x,y)\| = \|(x,y)\|_2 = \sqrt{x^2+y^2}, \ Y = \{(x,y) \in \mathbb{R}^2\colon y = -2x\}, \ f(x,y) = y$

In both cases decide whether the extension you found is unique.

● **Problem 2.13.** Show that there exists $\varphi \in (\ell_\infty)^*$ which is not of the form

$$\varphi(\mathbf{x}) = \sum_{n=1}^\infty x_n y_n \quad (\mathbf{x} = (x_n)_{n=1}^\infty \in \ell_\infty)$$

for any sequence $(y_n)_{n=1}^\infty \in \ell_1$.

This means that ℓ_1 is not *reflexive*—a notion we will introduce later (see also the remark after Problem 2.14).

● **Problem 2.14.** (a) For a given normed space X and a linear functional φ, calculate $\|\varphi\|$ and show that in each case its norm is not attained which means that there is no $x \in B_X$ such that $\varphi(x) = \|\varphi\|$:

- $X = c_0, \quad \varphi(x_1, x_2, \ldots) = \sum_{n=1}^\infty 2^{-n} x_n,$
- $X = c, \quad \varphi(x_1, x_2, \ldots) = \sum_{n=1}^\infty 2^{-n} x_n - \lim_{n\to\infty} x_n,$

- $X = \ell_1, \quad \varphi(x_1, x_2, \dots) = \sum_{n=1}^{\infty} \Big(1 - \frac{1}{n}\Big) x_n.$

(b) Give similar examples of not norm-attaining functionals on the spaces $C[0,1]$ and $L_1([0,1])$.

From this assertion it follows that none of the spaces: c_0, c, ℓ_1, $C[0,1]$, $L_1([0,1])$ is *reflexive*, because one of many existing characterizations of reflexivity says that a Banach space X is reflexive if and only if every $x^* \in X^*$ attains its norm at a certain point $x \in S_X$. This is the classical James' theorem; we will discuss the notion of reflexivity later.

The following series of problems (2.15–2.20) provide a more general version of the Hahn–Banach theorem, a couple of separation theorems (the most classical ones) and some interesting applications. When solving Problems 2.16–2.20 you can use the Hahn–Banach theorem formulated in Problem 2.15 if necessary. Also, the notion of Minkowski functional is here a basic tool, so you can freely use the assertions of Problem 2.7.

Problem 2.15. Imitating the proof of the classical version of the Hahn–Banach theorem, derive the following, purely algebraic version:

(a) Let X be a vector space over $\mathbb{R}$ and $M \subseteq X$ be a subspace of X. Let also $p\colon X \to [0,\infty)$ be a positively homogeneous, subadditive functional on X. If $f\colon M \to \mathbb{R}$ is a linear functional satisfying $f(x) \leqslant p(x)$ for every $x \in M$, then there exists a linear functional $F\colon X \to \mathbb{R}$ such that $F|_M = f$ and $-p(-x) \leqslant F(x) \leqslant p(x)$ for every $x \in X$.

(b) Let X be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $M \subseteq X$ be a subspace of X. Let also p be a seminorm on X and $f\colon M \to \mathbb{K}$ a linear functional satifying $|f(x)| \leqslant p(x)$ for every $x \in M$. Then, there exists a linear functional $F\colon X \to \mathbb{K}$ such that $F|_M = f$ and $|F(x)| \leqslant p(x)$ for every $x \in X$.

For any unexplained terminology, see Problem 2.7 and the preceding remarks.

Problem 2.16. Let C be a closed convex set in a normed space X. Prove that for every $x_0 \in X \setminus C$ there exists $x^* \in X^*$ such that

$$\sup\{\mathrm{Re}\,\langle x, x^*\rangle \colon x \in C\} < \langle x_0, x^*\rangle.$$

Deduce from this assertion that every closed convex set $C \subseteq X$ is the intersection of some collection of closed halfspaces, where by a *closed halfspace* we mean any subset of X which has the form

$$H(x^*, t) = \{x \in X \colon \mathrm{Re}\,\langle x, x^*\rangle \leqslant t\} \quad (x^* \in X^*,\, t \in \mathbb{R}).$$

Problem 2.17. Let B be a closed, convex and balanced subset of a normed space X. Show that for every $x_0 \in X \setminus B$ there is a functional $x^* \in X^*$ such that $|\langle x, x^*\rangle| \leqslant 1$ for each $x \in B$, but $\langle x_0, x^*\rangle > 1$.

You can (and probably should) use the separation theorem formulated in Problem 2.16.

Problem 2.18. Let A and B be nonempty, disjoint, convex subsets of a normed space X and assume that A is open. Prove that there exist $x^* \in X^*$ and $\gamma \in \mathbb{R}$ such that

$$\mathrm{Re}\,\langle x, x^*\rangle < \gamma \leqslant \mathrm{Re}\,\langle y, x^*\rangle \quad \text{for all } x \in A,\, y \in B.$$

Problem 2.19. Let $(X, \|\cdot\|)$ be a normed space and Y a subspace of X. Suppose $|||\cdot|||$ is a norm on Y which is equivalent to the norm inherited from X (i.e. $\|\cdot\|$ restricted to Y). Prove that there exists an equivalent norm $|\cdot|$ on X which induces on Y the norm $|||\cdot|||$.

Problem 2.20. Prove that on the real Banach space ℓ_∞ there exists a functional, called a *Banach limit*, $\mathrm{LIM}\colon \ell_\infty \to \mathbb{R}$ such that $\|\mathrm{LIM}\| = 1$ and for every $x \in \ell_\infty$ we have:

(i) $\liminf_{n\to\infty} x(n) \leqslant \mathrm{LIM}(x) \leqslant \limsup_{n\to\infty} x(n)$,

(ii) $\mathrm{LIM}(\tau x) = \mathrm{LIM}(x)$,

where $\tau\colon \ell_\infty \to \ell_\infty$ denotes the *shift* operator defined by $(\tau x)(n) = x(n+1)$.

Problem 2.21. We consider the Banach space $C^{(1)}([0,1])$ of continuously differentiable real-valued functions on $[0,1]$, where the norm is given by $\|f\|_{(1)} = \|f\|_\infty + \|f'\|_\infty$ (see Problem 1.16).

(a) Show that the subspace of all polynomials on $[0,1]$ is dense in $C^{(1)}([0,1])$.

(b) Define

$$f_n(t) = \frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2} \quad \text{for } n \in \mathbb{N},\ t \in [0,1].$$

Examine the convergence of $(f_n)_{n=1}^\infty$ in $C[0,1]$ and $C^{(1)}([0,1])$.

(c) Determine the norm of the linear functional

$$\phi(f) = \int_0^1 t f(t)\,\mathrm{d}t,$$

considered on the spaces $(C[0,1], \|\cdot\|_\infty)$ and $(C^{(1)}([0,1]), \|\cdot\|_{(1)})$.

Problem 2.22. Let X be a normed space such that X^* is separable. Prove that X is separable.

Of course, in X^* we consider the topology given by the norm defined on the space of continuous functionals. The converse statement is not true as, for example, we have $(\ell_1)^* \cong \ell_\infty$ (which will be shown in the lecture). Also, $(C[0,1])^*$ is not separable (consider the uncountable collection of functionals $\{\delta_t : t \in [0,1]\}$, $\delta_t f = f(t)$) but $C[0,1]$ is separable, because the set of all polynomials with rational coefficients is dense.

Problem 2.23. For $\alpha \geqslant 0$, consider a normed space $(\mathcal{C}_\alpha, \|\cdot\|_\alpha)$, where $\mathcal{C}_\alpha$ stands for the vector space of continuous functions $f\colon [0,\infty) \to \mathbb{R}$ satisfying

$$\|f\|_\alpha := \sup_{t\geqslant 0} \mathrm{e}^{\alpha t}\,|f(t)| < +\infty.$$

(a) Show that $(\mathcal{C}_\alpha, \|\cdot\|_\alpha)$ is a Banach space.

(b) Let $\alpha, \beta \geqslant 0$ and $h \in \mathcal{C}_\beta$. Show that the formula $Tf = f \cdot h$ defines an operator T from $\mathscr{L}(\mathcal{C}_\alpha, \mathcal{C}_{\alpha+\beta})$ and calculate $\|T\|$.

(c) Let $0 \leqslant \alpha < \beta$ and define $S\colon \mathcal{C}_\alpha \to \mathcal{C}_\alpha$ by

$$Sf(t) = \int_0^t \mathrm{e}^{-\beta(t-s)} f(s)\,\mathrm{d}s.$$

Prove that $\|S\| = (\beta - \alpha)^{-1}$.

Problem 2.24. Give an example of an operator $T \in \mathscr{L}(\ell_2, \ell_2)$ for which there exists a dense linear subspace $Y \subset \ell_2$ such that $T|_Y$ is injective, yet T fails to be injective on ℓ_2.

Problem 2.25. Consider the real normed space ℓ_2^2, that is, $\mathbb{R}^2$ with the Euclidean norm. As we know, each linear endomorphism of this space is of the form

$$T(x, y) = (ax + by, cx + dy) \quad \text{for some } a, b, c, d \in \mathbb{R}.$$

Show that

$$\|T\| = \frac{1}{2}\Big(\sqrt{(a+d)^2 + (b-c)^2} + \sqrt{(a-d)^2 + (b+c)^2}\Big).$$

Hint: It is quite convenient to identify $\mathbb{R}^2$ with the complex plane and write T in a complex form, that is, $Tz = uz + w\overline{z}$ for certain $u, w \in \mathbb{C}$ given in terms of a, b, c, d. Then, we want to determine $\|T\| = \max_{|z|=1} |Tz|$. This is computing the norm of T more or less directly.

Another, more theoretical approach is to use the so-called *Singular Value Decomposition* (SVD). It is a result from linear algebra which says that every matrix $A \in \mathbb{M}_n(\mathbb{K})$ (in our case $n = 2$) can be written as

$$A = U\Sigma V^*,$$

where Σ is a diagonal, positive semidefinite matrix and U, V are unitary matrices, i.e. $UU^* = U^*U = I$ and $VV^* = V^*V = I$, where * denotes the Hermitian transpose (just transpose in the real case). Moreover, if we write $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_n \geqslant 0$, then

$$\|A \colon \ell_2^2 \to \ell_2^2\| = \sigma_1.$$

The numbers σ_i are eigenvalues of the matrix $(A^*A)^{1/2}$ and we call them *singular values* of A.

Problem 2.26. Prove that every closed, convex and absorbing subset of a Banach space contains a neighborhood of zero.

Problem 2.27. Let X be an infinite-dimensional normed space over $\mathbb{C}$. Prove that there exists a real linear subspace M of X and a $\mathbb{C}$-linear continuous functional $f \colon M \to \mathbb{R}$ which cannot be continuously extended to any $\mathbb{C}$-linear functional on the whole of X.

Important note on terminology: We say that a functional f defined on a *real* vector space M is $\mathbb{C}$-*linear* provided that $f(wx + zy) = wf(x) + zf(y)$ for all vectors $x, y \in M$ and all complex numbers w, z such that $wx + zy \in M$.

Problem 2.28. Let $U \subseteq X$ be an open convex subset of a Banach space X and let $f \colon U \to \mathbb{R}$ be a continuous convex function, that is, $f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in U$ and $\lambda \in [0, 1]$. For any $x_0 \in U$, we call the set

$$\partial f(x_0) = \Big\{x^* \in X^* \colon f(x) \geqslant f(x_0) + \langle x - x_0, x^* \rangle \text{ for every } x \in U\Big\}$$

a *subdifferential* of f at x_0. Prove that $\partial f(x_0) \neq \varnothing$ for each $x_0 \in U$.