Functional analysis — Exercises*

Part 3: (Isometric) isomorphisms; duality; weak convergence; Riesz–Markov–Kakutani representation theorem; compact operators

In the problems below, all $C(K)$-spaces consist of real-valued continuous functions and are considered as Banach spaces over $\mathbb{R}$. Recall that, at this point, we have proved the Riesz Representation Theorem for $(C[a,b])^*$ and the Riesz–Markov–Kakutani for $(C_0(X))^*$ only in the real case. We will use the following common notation: if X is any topological space and $x \in X$, then δ_x stands for Dirac's measure concentrated at x, i.e. for any Borel set $E \subseteq X$, $\delta(E) = 1$ if $x \in E$ and $\delta(E) = 0$ otherwise. It is identified with the functional $\langle f, \delta_x \rangle = f(x)$ acting on continuous functions on X.

● **Problem 3.1.** (a) Show that the Dirac functional $\delta_0 \in \mathcal{M}[0,1]$ is not of the form

$$\langle f, \delta_0 \rangle = \int_0^1 f(t)g(t)\,\mathrm{d}t \quad (f \in C[0,1])$$

for any $g \in C[0,1]$.
(b) Define $\psi\colon C[0,1] \to \mathbb{R}$ by

$$\psi(f) = \frac{f(0)+f(1)}{2} + \int_0^1 tf(t)\,\mathrm{d}t.$$

Determine the measure from the Riesz–Markov–Kakutani theorem corresponding to ψ, i.e. a regular Borel measure μ on $[0,1]$ such that $\psi(f) = \int_{[0,1]} f\,\mathrm{d}\mu$ for $f \in C[0,1]$. Calculate $\|\psi\|$.

● **Problem 3.2.** By the Riesz Representation Theorem, for every $\mathbb{R}$-linear continuous functional $\varphi\colon C[0,1] \to \mathbb{R}$ there exists a unique function $g_\varphi \in \mathrm{NBV}([0,1])$ representing φ in terms of the Riemann–Stieltjes integral with respect to g_φ. Determine g_φ in each of the following cases:

(a) $\varphi = \delta_{t_0}$ for a fixed $t_0 \in [0,1]$,

(b) $\varphi(f) = \displaystyle\int_0^1 f(t)\,\mathrm{d}t$,

(c) $\varphi(f) = \displaystyle\int_0^1 f(t)\cos \pi t\,\mathrm{d}t$,

(d) $\varphi(f) = f(0) - \displaystyle\int_0^{1/2} f(2t)\,\mathrm{d}t$.

It is useful to recall the following fact from the theory of Riemann–Stieltjes integral: If $g\colon [0,1] \to \mathbb{R}$ is absolutely continuous and $f \in C[0,1]$, then we have the following equality between the Riemann–Stieltjes and Lebesgue integrals:

$$\int_0^1 f\,\mathrm{d}g = \int_{[0,1]} f(t)g'(t)\,\mathrm{d}t.$$

(By the absolute continuity, $g'(t)$ exists a.e. on $[0,1]$.)

● **Problem 3.3.** Assume μ is a regular Borel measure on $[0,1]$ such that

$$\int_{[0,1]} x^n\,\mathrm{d}\mu(x) = 0 \quad \text{for } n = 0,1,2,\ldots$$

Show that $\mu = 0$.

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*Evaluation: ●=2pt, ●=3pt, ●=4pt

● **Problem 3.4.** (a) Fix $N \in \mathbb{N}$. Show that there exists a regular signed Borel measure μ on $[0,1]$ such that for every real polynomial $P(x)$ of degree at most N we have

$$\int_{[0,1]} P \, \mathrm{d}\mu = \sum_{k=1}^{N} P^{(k)}\Big(\frac{k}{N}\Big).$$

(b) Decide whether there is a regular signed Borel measure μ on $[0,1]$ such that the above formula is valid for any real polynomial $P(x)$ of arbitrary degree $N = \deg P$.

In solutions of the next two problems you can use *Landau's theorem* (which can be read out from the duality $\ell_p^* \cong \ell_q$): If $p \in (1,\infty)$ and (x_n) is a scalar sequence such that for every $(y_n) \in \ell_p$ the series $\sum_n x_n y_n$ converges, then $(x_n) \in \ell_q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

● **Problem 3.5.** For a parameter $\alpha \in \mathbb{R}$ consider a linear operator T defined on ℓ_3 by the formula

$$Tx = (n^\alpha x_n)_{n=1}^\infty \quad \text{for } x = (x_n)_{n=1}^\infty \in \ell_3.$$

Decide for which values of α we have $T \in \mathscr{L}(\ell_3, \ell_1)$ and in such cases calculate $\|T\|$.

● **Problem 3.6.** Consider a map T which to every sequence $x = (x_n)_{n=1}^\infty$ of real numbers assign the sequence

$$Tx = (n^{-1/4} x_n)_{n=1}^\infty.$$

Determine all $p > 1$ for which $T \in \mathscr{L}(\ell_p, \ell_1)$. For all such p calculate $\|T\|$.

Let $(X, \|\cdot\|)$ be a normed space. We call X *strictly convex* if the unit sphere S_X does not contain any nontrivial line segments, i.e. for any distinct points $x, y \in S_X$ we have $\|(x+y)/2\| < 1$. This can be rephrased as follows: For any $A \subset X$ and $x \in A$ we call x an *extreme point* of A if it is not in the interior of any segment with endpoints in A, i.e. $\lambda y + (1-\lambda) z = x$ for some $y, z \in A$ and $\lambda \in (0,1)$ implies $y = z = x$. We denote by $\operatorname{ext}(A)$ the set of all extreme points of A. The space X is strictly convex if and only if each point of the unit sphere of X is extreme, i.e. $\operatorname{ext}(S_X) = S_X$.

● **Problem 3.7.** Let X and Y be normed spaces. For any $1 \le p \le \infty$ we form the 'ℓ_p-direct sum' of X and Y, that is, the algebraic direct sum $X \oplus Y$ equipped with the norm $\|(x,y)\| = \|(\|x\|, \|y\|)\|_p = (\|x\|^p + \|y\|^p)^{1/p}$. We denote this normed space by $X \oplus_p Y$. Show that for $1 \le p, q \le \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ (for $p = 1$ we take $q = \infty$ and vice versa), we have

$$\big(X \oplus_p Y\big)^* \cong X^* \oplus_q Y^*.$$

● **Problem 3.8.** Prove the following Taylor–Foguel theorem which characterizes normed spaces X for which the Hahn–Banach extensions of all functionals defined on subspaces of X are uniquely determined. Namely, X^* is strictly convex if and only if for every linear subspace $Y \subset X$ and any functional $f \in Y^*$ there exists the *unique* functional $F \in X^*$ satisfying $F|_Y = f$ and $\|F\| = \|f\|$.

● **Problem 3.9.** Determine all $n \in \mathbb{N}$ for which $\ell_1^n \cong \ell_\infty^n$. For all such values of n, describe an operator which gives the isometric isomorphism.

We say that a sequence $(x_n)_{n=1}^\infty$ of elements of a Banach space X is *weakly convergent* to $x \in X$ if $\lim_{n\to\infty}\langle x_n, x^*\rangle = \langle x, x^*\rangle$ for every $x^* \in X^*$. This defines the convergence with respect to a *weak topology* which is defined as the smallest topology τ on X such that all functionals from X^* are τ-continuous.

We say that X has the *Schur property* if every weakly convergent sequence in X is norm convergent. *Warning.* If X has this property, it does not mean that the norm topology coincides with the weak topology; it just means that the classes of convergent sequences are the same. In fact, the weak topology never coincides with the norm topology (i.e. it is strictly smaller) unless $\dim X < \infty$.

● **Problem 3.10.** Let $1 < p < \infty$. Give examples of sequences $(x_n)_{n=1}^\infty \subset \ell_p$ and $(f_n)_{n=1}^\infty \subset L_p[0,1]$ which are weakly convergent to 0 but not convergent in norm.

● **Problem 3.11.** Let X, Y be Banach spaces. By $\mathscr{K}(X,Y)$ we denote the space of *compact operators* between X and Y, that is, those operators $T \in \mathscr{L}(X,Y)$ for which $\overline{T(B_X)}$ is compact. Show that $\mathscr{K}(X,Y)$ is closed in $\mathscr{L}(X,Y)$.

In general, when solving problems on compact sets and compact operators, it is good to keep in mind the following topological fact: A metric space is compact if and only if it is complete and totally bounded (i.e. it admits a finite ε-net for each $\varepsilon > 0$). Therefore, a subset of a Banach space is relatively compact (i.e. has a compact closure) if and only if it is totally bounded. This is a fundamental fact whenever we want to decide whether a given operator is compact.

● **Problem 3.12.** In the space $C^{(1)}[0,1]$ of continuously differentiable functions we consider the norm $\|f\|_{(1)} = \|f\|_\infty + \|f'\|_\infty$ (or any equivalent norm, e.g. $\|f\| = \max\{\|f\|_\infty, \|f'\|_\infty\}$). Show that the identity operator

$$\iota\colon (C^{(1)}[0,1], \|\cdot\|_{(1)}) \longrightarrow (C[0,1], \|\cdot\|_\infty), \quad \iota f = f,$$

is compact.

● **Problem 3.13.** Define

$$Tx = (2^{-n+1}x_n)_{n=1}^\infty \quad \text{for } x = (x_n)_{n=1}^\infty \in \ell_\infty.$$

Show that $T \in \mathscr{K}(\ell_\infty, \ell_1)$.

● **Problem 3.14.** (a) Show that the completion of any normed space X (which is unique up to an isometric isomorphism) can be constructed as follows: Take $\iota\colon X \to X^{**}$ to be the canonical embedding in the bidual, i.e. $\langle x^*, \iota(x)\rangle = \langle x, x^*\rangle$ ($x \in X$, $x^* \in X^*$) and let $\mathfrak{X} = \overline{\iota(X)}$ be the closure of its range inside X^{**}. Justify that $\mathfrak{X}$ is the completion of X, i.e. that it is complete and X is (isometrically isomorphic to) a dense subspace of $\mathfrak{X}$.

(b) Of course, we can identify any vector $x \in X$ with its image $\iota(x)$ in X^{**}. Prove that every element $\widetilde{x} \in \mathfrak{X}$ can be written in the form $\widetilde{x} = \sum_{n=1}^\infty x_n$, where $x_n \in X$ ($n \in \mathbb{N}$) and the series is absolutely convergent, i.e. $\sum_{n=1}^\infty \|x_n\| < \infty$. Moreover, $\|\widetilde{x}\| = \inf \sum_{n=1}^\infty \|x_n\|$, where the infimum is taken over all series in X summing up to $\widetilde{x}$.

Let $\mathcal{F}$ be a field of subsets of some set X. We denote by $\mathsf{ba}(\mathcal{F})$ the collection of all bounded, finitely additive set functions $\mu\colon \mathcal{F} \to \mathbb{R}$. Exactly as in the standard case of σ-additive measures, we define a *variation* of μ by the formula

$$|\mu|(E) = \sup\left\{ \sum_{i=1}^n |\mu(A_i)| \colon n \in \mathbb{N},\ A_i \in \mathcal{F},\ A_i \cap A_j = \varnothing \text{ for } 1 \leq i \neq j \leq n \text{ and } \bigcup_{i=1}^n A_i = E \right\}.$$

(*Warning.* The Hahn decomposition theorem may fail if $\mathcal{F}$ is not a σ-algebra.) It is not difficult to verify that $\mathsf{ba}(\mathcal{F})$ becomes a normed space when equipped with the total variation norm, i.e. $\|\mu\| = |\mu|(X)$. Imitating the classical theory of Lebesgue integral over σ-additive measures, one can easily define an integral with respect to any finitely additive measure $\mu \in \mathsf{ba}(\mathcal{F})$. First, we do it in an obvious way for simple functions: If $f = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{E_i}$, where E_i's are pairwise disjoint in $\mathcal{F}$, then we set $T_\mu(f) = \sum_{i=1}^{n} \alpha_i \mu(E_i)$. Then, T_μ is a linear functional on the space of $\mathcal{F}$-measurable simple functions, and its norm $\|T_\mu\| = \|\mu\|$ provided that on the domain of T_μ we consider the supremum norm. So, T_μ extends uniquely to the completion of the space of simple functions which we denote by $B(\mathcal{F})$ and observe that it is exactly the space of all real-valued functions on X which are uniform limits of $\mathcal{F}$-measurable simple functions. A standard exercise from measure theory shows that in the case where $\mathcal{F}$ is a σ-algebra, $B(\mathcal{F})$ is the space of all bounded $\mathcal{F}$-measurable functions.

● **Problem 3.15.** Show that for every functional $\Lambda \in \ell_\infty^*$ on the real Banach space ℓ_∞ there exists a unique $\mu \in \mathsf{ba}(2^{\mathbb{N}})$ such that

$$\Lambda x = \int_{\mathbb{N}} x \, \mathrm{d}\mu \quad \text{for every } x \in \ell_\infty$$

and, moreover, $\|\Lambda\| = |\mu|(\mathbb{N})$. Conclude that we have an isometric isomorphism $\ell_\infty^* \cong \mathsf{ba}(2^{\mathbb{N}})$.

● **Problem 3.16.** Imitating the proof given in the lecture, derive the complex version of Proposition 3.6, that is, if $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $L_p[0,1]$, $L_q[0,1]$ are over $\mathbb{C}$, then

$$(L_p[0,1])^* \cong L_q[0,1].$$

● **Problem 3.17.** (a) Prove that the ℓ_p-norms are decreasing with respect to p. More precisely, for any $s \in (0, \infty)$ and any scalar sequence $x = (x_n)_{n=1}^{\infty}$ set

$$\|x\|_s = \left(\sum_{n=1}^{\infty} |x_n|^s \right)^{1/s}.$$

(We do not assume anything on x, so $\|x\|_s$ may happen to be $+\infty$. Recall that such a function satisfies the triangle inequality if (and only if) $s \geq 1$.) Then, show that

$$\|x\|_q \leq \|x\|_p \quad \text{for all } 0 < p \leq q < \infty.$$

(b) Prove that for any $1 \leq p \leq q \leq +\infty$ the Banach space $\mathscr{L}(\ell_p, \ell_q)$ is not separable.

● **Problem 3.18.** Let $a, b \in \mathbb{R}$, $a < b$, $k \in \mathbb{N}$. We consider the Banach space $C^{(k)}[a,b]$ of k-times continuously differentiable real-valued functions on $[a,b]$ with the norm

$$\|f\|_{(k)} = \sum_{i=0}^{k} \|f^{(i)}\|_\infty.$$

(a) Prove that a functional $\varphi \colon C^{(k)}[a,b] \to \mathbb{R}$ belongs to $(C^{(k)}[a,b])^*$ if and only if there exist $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in \mathbb{R}$ and a regular signed Borel measure μ on $[a,b]$ such that

$$\varphi(f) = \sum_{i=0}^{k-1} \alpha_i f^{(i)}(a) + \int_{[a,b]} f^{(k)}(t) \, \mathrm{d}\mu(t).$$

(b) Derive a formula for $\|\varphi\|$ in terms of $|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{n-1}|$ and $\|\mu\| = |\mu|([a,b])$ (the total variation of μ).

● **Problem 3.19.** Prove that
$$C[0,1] \sim C^{(1)}([0,1]).$$
(On the latter space we consider e.g. the norm $\|\cdot\|_{(1)}$ defined in Problem 3.18.)

● **Problem 3.20.** Let K be a compact Hausdorff space. Prove that $(f_n)_{n=1}^{\infty} \subset C(K)$ is weakly convergent to a function $f \in C(K)$ if and only if it satisfies the following two conditions:

- $\sup_n \|f_n\|_\infty < +\infty$,
- $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in K$.

That is, weakly convergent sequences in $C(K)$-spaces are exactly those which are uniformly bounded[1] and pointwise convergent.

● **Problem 3.21.** Prove that ℓ_1 has the Schur property.

● **Problem 3.22.** Prove that the sequence $(\delta_{1/n})_{n=1}^{\infty}$ in $\mathcal{M}[0,1]$ is not weakly convergent.

Hint: If that sequence was weakly convergent, then its potential weak limit could be only δ_0 (consider functionals on $\mathcal{M}[0,1]$ given by continuous functions). To show that $\delta_{1/n} \not\to \delta_0$ weakly, you do not need to know the whole dual space $(\mathcal{M}[0,1])^*$. Try to exhibit a subspace of this dual which is of the form $\ell_1(\Gamma)$, for a certain index set Γ (see Example 1.2(3)). Then, use the Schur property; you are allowed to apply the assertion of Problem 3.21.

● **Problem 3.23.** Prove that $c_0 \sim c$, but $c_0 \not\cong c$.

● **Problem 3.24.** Prove that $c_0 \not\hookrightarrow \ell_1$ and $\ell_p \not\hookrightarrow \ell_1$ for $p \in (1,\infty)$.

● **Problem 3.25.** Let X be a Banach spaces. Prove that all closed hyperplanes of X are mutually isomorphic.

● **Problem 3.26.** Decide whether an operator $T \in \mathscr{L}(C[0,1])$ is compact, where:

(a) $Tf(x) = xf(x)$,

(b) $Tf(x) = \displaystyle\int_0^1 \mathrm{e}^{tx} f(t)\,\mathrm{d}t$.

● **Problem 3.27.** Let $X = c_0$ or $X = \ell_p$ with $1 \le p < \infty$. For $n \in \mathbb{N}$ we define $R_n \in \mathscr{L}(X)$ to be the 'n^{th} tail' operator, i.e.
$$R_n(x) = (\underbrace{0,\ldots,0}_{n-1}, x_n, x_{n+1}, \ldots) \quad \text{for } x = (x_n)_{n=1}^{\infty} \in X.$$
Show that a nonempty set $A \subset X$ is relatively compact if and only if $R_n(x) \longrightarrow 0$ uniformly for $x \in A$ which means that for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\|R_n(x)\| < \varepsilon$ for all $x \in A$, $n \ge n_\varepsilon$.

[1] It follows immediately from the Banach–Steinhaus theorem, that every weakly convergent sequence in any Banach space must be bounded. However, you should prove this in the particular case of $C(K)$ without appealing to that theorem.

Using the above characterization it is easy to construct, for every $1 \le p < \infty$, a compact set $K \subset \ell_p$ for which there does not exist any $x = (x_n)_{n=1}^{\infty} \in \ell_p$ such that $|y_n| \le |x_n|$ for all $(y_n)_{n=1}^{\infty} \in K$. Thus, global majorization by a certain element of ℓ_p is not a necessary condition for (relative) compactness, although it is plainly a sufficient condition. However, it is also easy to observe that this condition is both necessary and sufficient for relative compactness in the space c_0.

● **Problem 3.28.** Define $X = \mathbb{R}^2$ to be the plane with a topology given by the condition: a set $U \subset X$ is open if and only if its intersection with every horizontal line is open in the natural topology. Show that X is a locally compact Hausdorff space. For any $f \in C_c(X)$ there are only finitely many x's such that $f(x, y) \neq 0$ for at least one y; denote them by $x_1, \ldots, x_n$ and define

$$\Lambda f = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x_i, y) \, \mathrm{d}y.$$

This defines a positive linear functional Λ on $C_c(X)$. Let μ be the measure corresponding to Λ via the Riesz–Markov–Kakutani theorem. Show that for the real line $E \subset X$ we have $\mu(E) = \infty$, however, $\mu(K) = 0$ for each compact set $K \subset E$.

This exercise shows that, in general, the conditions in the Riesz–Markov–Kakutani Theorem 3.16 do not guarantee that the underlying measure is regular. On the other hand, for measures representing elements of $(C_0(X))^*$ regularity was obtained because such measures are finite (more precisely, have finite variation), i.e. signed or complex-valued, without the value ∞.

● **Problem 3.29.** Prove Pitt's theorem: If $1 \le p < q < \infty$, then $\mathscr{L}(\ell_q, \ell_p) = \mathscr{K}(\ell_q, \ell_p)$.

● **Problem 3.30.** Define an operator T on the space $L_2(0, \infty)$ by

$$(Tf)(x) = \frac{1}{x} \int_0^x f(t) \, \mathrm{d}t.$$

Prove that $T \in \mathscr{L}(L_2(0, \infty))$, $\|T\| \le 2$ and that T is not compact.