Functional analysis — Exercises*

Part 4: Spectrum, eigenvalues, adjoint operators; Riesz–Schauder theorem; Fredholm operators and the Fredholm alternative

● **Problem 4.1.** Let $1 \le p < \infty$ and $(a_n)_{n=1}^\infty$ be a bounded sequence of scalars. Show that the norm of the diagonal operator $T \in \mathscr{L}(\ell_p)$ given by $Tx = (a_n x_n)_{n=1}^\infty$ ($x = (x_n)_{n=1}^\infty \in \ell_p$) equals $\|(a_n)_{n=1}^\infty\|_\infty$, and that this operator is compact if and only if $\lim_{n\to\infty} a_n = 0$.

● **Problem 4.2.** Decide whether there exists $T \notin \mathscr{K}(\ell_2)$ with $T^2 = 0$.

● **Problem 4.3.** Let $W \in \mathscr{L}(\ell_2)$ be the weighted shift operator on ℓ_2 defined by

$$W(x_1, x_2, \ldots) = \Big(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\Big).$$

Show that W is compact, but $\sigma_{\mathrm{p}}(W) = \varnothing$, and hence $\sigma(W) = \{0\}$. Find a nontrivial closed invariant subspace for W, i.e. a closed subspace $0 \ne M \subsetneq \ell_2$ satisfying $W(M) \subseteq M$.

● **Problem 4.4.** Let X be a Banach space and $P \in \mathscr{L}(X)$ be a projection onto a proper subspace $Y \subseteq X$, that is, $P\colon X \to Y$ is surjective and $P^2 = P$. Show that $\sigma(P) = \sigma_{\mathrm{p}}(P) = \{0, 1\}$ and for $\lambda = 0, 1$ we have

$$(P - \lambda I)^{-1} = \frac{1}{\lambda(1-\lambda)} P - \frac{1}{\lambda} I.$$

● **Problem 4.5.** Let $(a_n)_{n=1}^\infty$ be a bounded scalar sequence and $T \in \mathscr{L}(\ell_2)$ be the associated diagonal operator as described in Problem 4.1. Find $\sigma(T)$ and $\sigma_{\mathrm{p}}(T)$.

● **Problem 4.6.** Let K be any nonempty compact subset of the scalar field. Show that there is an operator $T \in \mathscr{L}(\ell_2)$ with $\sigma(T) = K$.

● **Problem 4.7.** On the complex Banach space ℓ_2 consider the right shift operator $R \in \mathscr{L}(\ell_2)$ given by $R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ and the diagonal operator $D \in \mathscr{L}(\ell_2)$ associated with the sequence $d_n = 2^{-n}$ ($n \in \mathbb{N}$), that is, $D(x_1, x_2, \ldots) = (\frac{1}{2}x_1, \frac{1}{4}x_2, \ldots)$. Define a weighted shift operator $T = R \circ D$. Prove that T is compact, one-to-one and satisfies

$$\lim_{n\to\infty} \|T^n\|^{1/n} = 0.$$

The *spectral radius* of any operator $T \in \mathscr{L}(X)$ is defined as

$$\rho(T) = \sup\{|\lambda| \colon \lambda \in \sigma(T)\}.$$

According to Proposition 4.6, we have $\rho(T) \le \|T\|$. Although the invertibility, spectrum and hence the spectral radius are all defined in purely algebraic terms, the spectral radius can be defined by topological means, more precisely, by the norms of iterates of the given operator. This is the *Gelfand theorem* which gives a formula for $\rho(T)$ if T is defined on a complex Banach space:

$$\rho(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|T^n\|^{1/n}.$$

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* Evaluation: ●=2pt, ●=3pt, ●=4pt

● **Problem 4.8.** Show that the *Volterra operator* $V \in \mathscr{L}(L_2[0,1])$ defined by

$$Vf(x) = \int_0^x f(t)\,\mathrm{d}t$$

is compact.

It is easily seen that for each $\alpha \in [0,1]$, the subspace M_α consisting of all L_2-functions on $[0,1]$ that vanish a.e. on $[0,\alpha]$ is an invariant subspace for V. That these are in fact all invariant subspaces for V is a deep theorem which involves some analytic functions and the Paley–Wiener theorem; see [H. Radjavi, P. Rosenthal, *Invariant subspaces*, Dover Publications 2003; Thm. 4.14].

● **Problem 4.9.** For each pair of natural numbers m, n calculate the Fredholm index of an operator $A\colon \mathbb{C}^n \to \mathbb{C}^m$ and observe that it depends on m and n, but not on A.

● **Problem 4.10.** For any given function $g \in C[0,1]$, let $T\colon C^{(1)}[0,1] \to C[0,1]$ be an operator defined by

$$Tf(x) = f'(x) + g(x)f(x).$$

Show that T is a Fredholm operator and find its index.

Let X and Y be normed spaces. For every $T \in \mathscr{L}(X,Y)$, we define its *adjoint operator* $T^* \in \mathscr{L}(Y^*, X^*)$ by

$$\langle x, T^*y^* \rangle = \langle Tx, y^* \rangle. \tag{1}$$

In Problem 4.11 we want to show that the above equation determines T^* uniquely. Observe that in the case where X and Y are finite-dimensional, and T is given by a matrix from $\mathbb{M}_{m\times n}(\mathbb{K})$, the adjoint T^* corresponds to the transposed matrix. (For complex Hilbert spaces $\mathcal{H}$, the adjoint is defined in a slightly different way, by identifying $\mathcal{H}^* \cong \mathcal{H}$, and then the adjoint corresponds to the conjugate transpose.)

Quite obviously, we have:

- $(T+S)^* = T^* + S^*$,
- $(\lambda T)^* = \lambda T^*$,
- $(UT)^* = T^*U^*$

for all $T, S \in \mathscr{L}(X,Y)$, $U \in \mathscr{L}(Y,Z)$ and $\lambda \in \mathbb{K}$. Hence, taking the adjoint is a linear operation in $\mathscr{L}(X,Y)$. Some properties of T naturally translate into properties of T^*. For example, using the Open Mapping Theorem, one can show that if X and Y are Banach spaces (completeness is here essential), then $T \in \mathscr{L}(X,Y)$ is surjective if and only if T^* is one-to-one and has a norm closed range (see [W. Rudin, *Functional analysis*; Thm. 4.15]). In Problem 4.13 we want to show some easier, basic properties. For any subspaces $M \subseteq X$ and $N \subseteq X^*$, we define their *annihilator* and *preannihilator* by

$$M^\perp = \{x^* \in X^*\colon \langle x, x^* \rangle = 0 \text{ for all } x \in M\}, \quad {}^\perp N = \{x \in X\colon \langle x, x^* \rangle = 0 \text{ for all } x^* \in N\}.$$

● **Problem 4.11.** Let X, Y be normed spaces and $T \in \mathscr{L}(X,Y)$. Show that T^* defined above is the unique bounded linear operator satisfying equation (1) and that $\|T\| = \|T^*\|$.

● **Problem 4.12.** Given an operator $T \in \mathscr{L}(X)$, determine T^* in the following cases:

(a) $X = C[0,1]$, $\quad Tf = f \circ \varphi$, where $\varphi\colon [0,1] \to [0,1]$ is a fixed continuous map,

(b) $X = L_2[0,1]$, $\quad Tf = fg$, where $g \in L_\infty[0,1]$ is fixed,

(c) $X = \ell_2$, $\quad T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$.

● **Problem 4.13.** Let X and Y be normed spaces and $T \in \mathscr{L}(X, Y)$. Verify that

$$\ker T^* = T(X)^\perp \quad \text{and} \quad \ker T = {}^\perp T^*(Y^*).$$

Conclude that if $X \sim Y$, then $X^* \sim Y^*$. Also, notice that an operator $S \in \mathscr{L}(X)$ is invertible if and only if $S^* \in \mathscr{L}(X^*)$ is invertible and that we have

$$\sigma(S) = \sigma(S^*).$$

● **Problem 4.14.** Show that for each $n \in \mathbb{Z}$ there exists a Fredholm operator $T \in \mathscr{L}(\ell_2)$ with Fredholm index $i(T) = n$.

● **Problem 4.15.** Let $1 \le p \le \infty$ and let L and R stand for the left and right shift operator on the complex Banach space ℓ_p, respectively, i.e. $L(x_1, x_2, \dots) = (x_2, x_3, \dots)$ and $R(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Show that:

(i) $\sigma_{\mathrm{p}}(L) = D$;
(ii) $\sigma_{\mathrm{p}}(R) = \varnothing$;
(iii) $\sigma(L) = \sigma(R) = \overline{D}$.

(D is the open unit disc.)

● **Problem 4.16.** Let X be a Banach space and $T, S \in \mathscr{L}(X)$.

(a) Give an example which shows that $ST = I$ does not imply that $TS = I$.

(b) Assume that $T \in \mathscr{K}(X)$. Show that $S(I - T) = I$ if and only if $(I - T)S = I$, and that each of these equalities implies that the operator $I - (I - T)^{-1}$ is compact.

Recall that if X and Y are Banach spaces, then an operator from $\mathscr{L}(X, Y)$ is invertible if and only if it is bounded below and its range is dense in Y. The part of the spectrum of $T \in \mathscr{L}(X, Y)$ consisting of those numbers $\lambda \in \mathscr{K}$ for which $T - \lambda I$ is not bounded below is called the *approximate spectrum* of T, and denoted by $\sigma_{\mathrm{ap}}(T)$. Notice that $\lambda \in \sigma_{\mathrm{ap}}(T)$ if and only if there exists a sequence $(x_n)_{n=1}^\infty \subset S_X$ such that $Tx_n - \lambda x_n \to 0$, which justifies calling every element of $\sigma_{\mathrm{ap}}(T)$ an *approximate eigenvalue* of T.

● **Problem 4.17.** Let X be a Banach space and $T \in \mathscr{L}(X)$. Show that

$$\partial\sigma(T) \subseteq \sigma_{\mathrm{ap}}(T).$$

● **Problem 4.18.** Let $1 \le p \le \infty$ and let $(a_n)_{n=1}^\infty$ be an increasing sequence of positive numbers with $r = \lim_{n\to\infty} a_n < \infty$. Define an operator on the complex Banach space ℓ_p, $A \in \mathscr{L}(\ell_p)$ by

$$A(x_1, x_2, \dots) = (0, a_1x_1, a_2x_2, \dots).$$

Show that:

(i) $\sigma(A) = \{\lambda \in \mathbb{C} \colon |\lambda| \le r\}$;
(ii) $\sigma_{\mathrm{p}}(A) = \varnothing$;
(iii) $\sigma_{\mathrm{ap}}(A) = \partial\sigma(A)$;
(iv) if $|\lambda| < r$, then the range of $A - \lambda I$ is closed and has codimension 1.

● **Problem 4.19.** Consider the multiplication operator $T \in \mathscr{L}(L_2[0,1])$ given by the formula $Tf(t) = tf(t)$. Show that:

(i) $\sigma_{\mathrm{p}}(T) = \varnothing$;

(ii) $\sigma(T) = \sigma_{\mathrm{ap}}(T) = [0,1]$.

● **Problem 4.20.** Find an operator $T \in \mathscr{L}(\ell_2)$ such that $\sigma(T) = \{0,1\}$ and $\sigma_{\mathrm{p}}(T) = \varnothing$.

● **Problem 4.21.** Prove that

$$c_0 \overset{1}{\hookrightarrow} \mathscr{K}(\ell_2), \quad \ell_2 \overset{1}{\hookrightarrow} \mathscr{K}(\ell_2), \quad \ell_\infty \not\hookrightarrow \mathscr{K}(\ell_2).$$

● **Problem 4.22.** Let X and Y be Banach spaces. Show that an operator $T \in \mathscr{L}(X,Y)$ is Fredholm if and only if there exists $S \in \mathscr{L}(Y,X)$ such that both $(TS - I)$ and $(ST - I)$ are compact.

The solution of Problem 4.22 basically shows that every Fredholm operator $T \in \mathscr{L}(X,Y)$ admits the following 'block matrix' decomposition: there are closed subspaces $X_0 \subseteq X$ and $Y_1 \subseteq Y$ such that $X = \ker T \oplus X_0$, $Y = T(X) \oplus Y_1$, $T_0 \coloneqq T|_{X_0}$ is an isomorphisms from X_0 onto $T(X)$ and hence T can be written in a matrix form as

$$T = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

You can use this decomposition, as well as the assertion of Problem 4.9, to obtain the stability effect for Fredholm operators formulated in Problem 4.23.

For any $T \in \mathscr{L}(X)$, we define its *essential spectrum* by

$$\sigma_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{K} \colon T - \lambda I \text{ is not a Fredholm operator}\}.$$

Since $\mathscr{K}(X)$ is a closed subspace of $\mathscr{L}(X)$, we can consider the quotient space $\mathscr{C}(X) = \mathscr{L}(X)/\mathscr{K}(X)$ equipped with the distance norm, that is,

$$\|\pi(A)\| = \inf\{\|A + K\| \colon K \in \mathscr{K}(X)\} \quad (A \in \mathscr{L}(X)),$$

where $\pi\colon \mathscr{L}(X) \to \mathscr{C}(X)$ is the canonical quotient map. In this way, $\mathscr{C}(X)$ becomes a Banach space and since $\mathscr{K}(X)$ is a two-sided ideal in $\mathscr{L}(X)$, it also has a natural algebra structure (in fact, a Banach algebra structure, i.e. the norm is submultiplicative) and we call $\mathscr{C}(X)$ the *Calkin algebra*. Now, notice that the assertion of Problem 4.22 can be restated as follows: an operator $T \in \mathscr{L}(X)$ is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra $\mathscr{C}(X)$ and, moreover, $\sigma_{\mathrm{ess}}(T) = \sigma(\pi(T))$, where at the right-hand side we have the usual spectrum with respect to the algebra $\mathscr{C}(X)$, i.e. the set of those $\lambda \in \mathbb{K}$ for which $\pi(T) - \lambda\pi(I)$ is not invertible in this algebra.

● **Problem 4.23.** Let X, Y be Banach spaces and $T \in \mathscr{L}(X,Y)$. Show that there is $\varepsilon > 0$ such that every operator $S \in \mathscr{L}(X,Y)$ satisfying $\|S - T\| < \varepsilon$ is Fredholm and $i(S) = i(T)$.

● **Problem 4.24.** Show that the formula

$$Tf(x) = f(x) + \int_0^x f(t)\,\mathrm{d}t$$

defines an isomorphism of $C[0,1]$ onto itself.

● **Problem 4.25.** Let μ be a σ-finite positive measure on a measure space Ω and let $\mu \otimes \mu$ be the product measure on $\Omega \times \Omega$. Assume that $K \in L_2(\mu \otimes \mu)$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and consider the equation

$$(*) \qquad \lambda f(t) - \int\limits_{\Omega} K(t,s) f(s) \, \mathrm{d}\mu(s) = g(t).$$

Prove that either for every $g \in L_2(\mu)$ equation $(*)$ has a unique solution $f \in L_2(\mu)$, or for some $g \in L_2(\mu)$ it has infinitely many solutions in $L_2(\mu)$, while for some other g it has no solution in $L_2(\mu)$.

● **Problem 4.26.** Let $a \in \mathbb{R}$, $a \neq 0$ and $\tau > 0$. Prove that for any fixed real-valued function $u \in C[0,\tau]$ the integral equation

$$x(t) = u(t) + \int_0^t x(s) \, \mathrm{d}s \quad (0 \le t \le \tau)$$

has a unique continuous solution $x\colon [0,\tau] \to \mathbb{R}$. Determine that solution.

● **Problem 4.27.** Let X be a complex Banach space and $\{T_j \colon 1 \le j \le m\} \subset \mathscr{L}(X)$ be a finite family of commuting operators, that is, $T_i T_j = T_j T_i$ for all $1 \le i, j \le m$. Show that there exist complex numbers $\lambda_1, \dots, \lambda_m$ and a sequence $(x_n)_{n=1}^{\infty} \subset S_X$ such that

$$\lim_{n \to \infty} \|T_j x_n - \lambda_j x_n\| = 0 \quad \text{for every } 1 \le j \le m.$$