Functional analysis — Exercises[*]

Part 5: Inner product spaces and Hilbert spaces; the projection theorem; Gram–Schmidt orthogonalization; adjoint operators

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When dealing with Hilbert spaces, by an *adjoint* operator we mean the adjoint operator in the Hilbert space theoretical setting. This is slightly different from the notion of adjoint operator between Banach spaces which we have already introduced in Part 4 (see the remarks before Problem 4.11). It is not difficult to show (see Problem 5.16) that given any operator $T \in \mathscr{L}(\mathcal{H}, \mathcal{K})$ between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, there exists a unique operator $T^* \in \mathscr{L}(\mathcal{K}, \mathcal{H})$ such that

$$(Tx, y)_{\mathcal{K}} = (x, T^*y)_{\mathcal{H}}$$

(at the left-hand side we have the inner product in $\mathcal{K}$, at the right-hand side the inner product in $\mathcal{H}$), and this operator we call the *adjoint* of T. This definition looks like exactly the same as the one given for operators on Banach spaces. Note, however, that in this case T^* acts formally between $\mathcal{K}$ and $\mathcal{H}$, not $\mathcal{K}^*$ and $\mathcal{H}^*$. Recall that by the Riesz representation theorem, the dual of any Hilbert space can be isometrically identified with the space itself. Hence, we applied the earlier definition of adjoint with suitably identifying $\mathcal{H} \cong \mathcal{H}^*$ and $\mathcal{K} \cong \mathcal{K}^*$.

In particular, for any $T \in \mathscr{L}(\mathcal{H})$, we have $T^* \in \mathscr{L}(\mathcal{H})$, so it makes sense to speak about *self-adjoint* operators between Hilbert spaces, that is, operators for which $T = T^*$. In the finite-dimensional case $\mathcal{H} = \mathbb{K}^n$, $\mathcal{K} = \mathbb{K}^m$, every operator $T \in \mathscr{L}(\mathbb{K}^n, \mathbb{K}^m)$ can be represented by an $m \times n$ matrix from $\mathbb{M}_{m,n}(\mathbb{K})$ with respect to arbitrarily chosen orthonormal bases. Then, taking the adjoint T^* corresponds to taking the transposed matrix if $\mathbb{K} = \mathbb{R}$, whereas in the case $\mathbb{K} = \mathbb{C}$ it corresponds to the conjugate transpose (or Hermitian transpose): $(a_{i,j})^* = (\overline{a}_{j,i})$

For any operators $T, S \in \mathscr{L}(\mathcal{H})$ we have:

- $T^{**} = T$, where the second adjoint is defined as $T^{**} = (T^*)^*$,
- $(T + S)^* = T^* + S^*$,
- $(\lambda T)^* = \overline{\lambda} T^*$,
- $(UT)^* = T^* U^*$.

Notice that the operation $T \mapsto T^*$, in the Hilbert space theoretic setting, is conjugate linear, while in the Banach space theoretic setting it was simply linear. It should be clear from the context that whenever speaking about operators between Hilbert spaces, any adjoint in understood in the way described above.

● **Problem 5.1.** Let $\mathcal{H}$ be a Hilbert space and $T \in \mathscr{L}(\mathcal{H})$. Show that

$$\ker T = T^*(\mathcal{H})^{\perp} \quad \text{and} \quad \ker T^* = T(\mathcal{H})^{\perp}.$$

(The objects at the righ-hand sides can be equivalently understood as annihilators, as well as orthogonal complements.)

● **Problem 5.2.** Let X be an inner product space over $\mathbb{K}$. Show that the following polarization formulas hold true:

- $(x, y) = \frac{1}{4}\big(\|x + y\|^2 - \|x - y\|^2\big)$ if $\mathbb{K} = \mathbb{R}$,
- $(x, y) = \frac{1}{4}\big(\|x + y\|^2 - \|x - y\|^2 + \mathrm{i}\|x + \mathrm{i}y\|^2 - \mathrm{i}\|x - \mathrm{i}y\|^2\big)$ if $\mathbb{K} = \mathbb{C}$.

● **Problem 5.3.** Using the polarization identities prove the following Jordan–von Neumann theorem: If $(X, \|\cdot\|)$ is a real or complex normed space which satisfies the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad (x, y \in X),$$

then X is an inner product space, that is, there exists an inner product $(\cdot\,,\cdot)$ on X such that $(x, x) = \|x\|^2$ for every $x \in X$.

[*] Evaluation: ●=2pt, ●=3pt, ●=4pt

● **Problem 5.4.** Prove that in any inner product space, the following generalized parallelogram law holds true:

$$\sum_{\varepsilon_i=\pm 1}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^2 = 2^n\sum_{i=1}^n \|x_i\|^2.$$

● **Problem 5.5.** Apply the Gram–Schmidt orthogonalization process to the three vectors $\{1, x, x^2\}$ in the Hilbert space $L_2([-1,1])$. Use it to find the distance from x^3 to $\operatorname{lin}\{1, x, x^2\}$, that is, compute

$$\min_{a,b,c\in\mathbb{C}}\int_{-1}^1 \left|x^3 - a - bx - cx^2\right|^2 \,\mathrm{d}x.$$

The Gram–Schmidt process applied to the sequence of monomials $\{1, x, x^2, x^3, \ldots\}$ yields the so-called *Legendre polynomials* $(P_n)_{n=0}^\infty$. The usual convention is that they are normalized by the condition $P_n(1) = 1$ and then we have $P_0(x) = 1$, $P_1(x) = x$ and the recursion formula

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad (n \geq 1).$$

● **Problem 5.6.** In the Hilbert space $L_2[0,1]$ we consider the subspace

$$V = \Big\{f \in L_2[0,1]\colon \int_0^1 tf(t)\,\mathrm{d}t = 0 \text{ and } \int_0^1 t^3 f(t)\,\mathrm{d}t = 0\Big\}.$$

Determine $P_V(g)$ and $\operatorname{dist}(g, V)$, where $g(t) = t^2$ and $P_V\colon L_2[0,1] \to V$ is the orthogonal projection onto V.

● **Problem 5.7.** In the two-dimensional Hilbert space $\mathbb{R}^2$ we define subspaces

$$M = \{(x,0) \in \mathbb{R}^2\colon x \in \mathbb{R}\}, \quad N = \{(x, x\tan\theta)\colon x \in \mathbb{R}\},$$

where $\theta \in (0, \frac{\pi}{2})$ is a fixed angle. Find a formula for a projection $E_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $E_\theta(\mathbb{R}^2) = M$ and $\ker(E_\theta) = N$. Show that $\|E_\theta\| = 1/\sin\theta$.

● **Problem 5.8.** Let $\mathcal{H}$ be a Hilbert, $T \in \mathscr{L}(\mathcal{H})$ and suppose that the range of T is one-dimensional. Show that there are vectors $x, y \in \mathcal{H}$ such that

$$Tz = (z, x)y \qquad (z \in \mathcal{H}).$$

We denote such an operator by $y \otimes x$. Determine T^*.

● **Problem 5.9.** Let $\mathcal{H}$ be a Hilbert space and $P \in \mathscr{L}(\mathcal{H})$. Show that P is an orthogonal projection onto a closed subspace of $\mathcal{H}$ if and only if P is a self-adjoint idempotent, i.e. $P^2 = P = P^*$.

● **Problem 5.10.** Show that the sequence of functions $(f_n)_{n=1}^\infty$ given by $f_n(t) = n^2te^{-nt}$ is pointwise convergent on $[0,1]$ but it is not convergent in the space $L_2[0,1]$.

Let $\mathcal{H}$ be a Hilbert space. An operator $U \in \mathscr{L}(\mathcal{H})$ is called a *unitary* operator provided that U is invertible and $U^{-1} = U^*$, in other words, $UU^* = I = U^*U$. Two operators $S, T \in \mathscr{L}(\mathcal{H})$ are called *unitarily equivalent* if there exists a unitary operator $U \in \mathscr{L}(\mathcal{H})$ such that $USU^* = T$. Unitary operators are exactly Hilbert space automorphisms, or equivalently: surjective isometries, as it is stated in the next problem. Of course, one can also consider unitary operators between two different Hilbert spaces as bounded linear isomorphisms which preserve inner products.

One of the most important example of a unitary operator is the one given by Fourier coefficients with respect to the trigonometric system. Namely, on the Hilbert space $L_2([0,2\pi],(2\pi)^{-1}\mathrm{d}t)$ (the normalized Lebesgue measure) we define an operator Φ into the Hilbert space

$$\ell_2(\mathbb{Z}) = \Big\{(a_n)_{n=-\infty}^{+\infty} \subset \mathbb{C} \colon \sum_{n=-\infty}^{+\infty} |a_n|^2 < \infty\Big\}$$

by the formula

$$\Phi(f) = \big(\widehat{f}(n)\big)_{n=-\infty}^{\infty}, \quad \text{where} \quad \widehat{f}(n) = (f, e^{\mathrm{i}nt}) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-\mathrm{i}nt}\,\mathrm{d}t.$$

However, in order to show that Φ is unitary, we need to know that the trigonometric system $(e^{\mathrm{i}nt})_{n=-\infty}^{\infty}$ is complete, i.e. it forms an orthonormal basis of $L_2([0,2\pi],\mathrm{d}t/2\pi)$.

● **Problem 5.11.** Let $\mathcal{H}$ be a Hilbert space and $U \in \mathscr{L}(\mathcal{H})$. Show that the following assertions are equivalent:

(i) U is unitary,
(ii) U is surjective and $(Ux, Uy) = (x, y)$ for all $x, y \in \mathcal{H}$,
(iii) U is surjective and $\|Ux\| = \|x\|$ for every $x \in \mathcal{H}$.

● **Problem 5.12.** Let $R \in \mathscr{L}(\ell_2)$ be the forward shift on the Hilbert space ℓ_2. Verify that $R^*R = I$ although R is not unitary as it is not surjective.

● **Problem 5.13.** Let $(X, \mathfrak{M}, \mu)$ be a measure space and $M_\varphi \in \mathscr{L}(L_2(\mu))$ be a multiplication operator defined by $M_\varphi(f) = \varphi \cdot f$, where $\varphi \in L_\infty(\mu)$. Determine the spectrum $\sigma(M_\varphi)$ and all functions φ for which M_φ is unitary.

● **Problem 5.14.** Let $(X, \mathfrak{M}, \nu)$ be a σ-finite measure space and $(E_n)_{n=1}^{\infty}$ be a collection of pairwise disjoint measurable sets with $\nu(E_n) < \infty$ for each $n \in \mathbb{N}$. Define μ on $\mathfrak{M}$ by

$$\mu(A) = \sum_{n=1}^{\infty} \frac{\nu(A \cap E_n)}{2^n(\nu(E_n)+1)}.$$

Show that for any $A \in \mathfrak{M}$, $\mu(A) = 0$ if and only if $\nu(A) = 0$, which means that $\mu \ll \nu$ and $\nu \ll \mu$. Find the Radon–Nikodym derivatives $\mathrm{d}\mu/\mathrm{d}\nu$ and $\mathrm{d}\nu/\mathrm{d}\mu$. Notice that the latter one is not μ-integrable but only 'locally' integrable, i.e. integrable on every set of finite μ-measure.

● **Problem 5.15.** The *Rademacher sequence* $(r_k)_{k=0}^{\infty}$ of functions on $[0,1]$ is defined by the formula

$$r_k(t) = \operatorname{sgn}(\sin 2^k \pi t),$$

that is,

$$r_k(t) = \begin{cases} 1 & \text{for} \quad t \in \bigcup_{s=1}^{2^{k-1}} [\frac{2s-2}{2^k}, \frac{2s-1}{2^k}) \\ -1 & \text{for} \quad t \in \bigcup_{s=1}^{2^{k-1}} [\frac{2s-1}{2^k}, \frac{2s}{2^k}). \end{cases}$$

Verify that $(r_k)_{k=0}^{\infty}$ is an orthonormal set in $L_2[0,1]$ but it does not form an orthonormal basis.

● **Problem 5.16.** Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces over $\mathbb{K}$ and suppose that $\psi\colon \mathcal{H}\times\mathcal{K}\to\mathbb{K}$ is a bounded sesquilinear form, i.e. ψ is linear in the first variable, conjugate linear in the second variable and there is a constant $C<\infty$ such that $|\psi(x,y)|\le C\|x\|\|y\|$ for all $(x,y)\in\mathcal{H}\times\mathcal{K}$. Prove that there exists a unique operator $T\in\mathscr{L}(\mathcal{H},\mathcal{K})$ such that

$$\psi(x,y)=(Tx,y)_{\mathcal{K}} \qquad (x\in\mathcal{H},\ y\in\mathcal{K}).$$

● **Problem 5.17.** Suppose $\mathcal{H}$ is a Hilbert space and $T\in\mathscr{L}(\mathcal{H})$ satisfies $(Tx,x)\ge\|x\|^2$ for each $x\in\mathcal{H}$. Prove that T is an isomorphism of $\mathcal{H}$ onto itself.

● **Problem 5.18.** Define

$$\mathcal{W}_n=\left\{z_0+z_1t+\ldots+z_nt^n\colon z_i\in\mathbb{C}\text{ for }0\le i\le n\right\}$$

as the vector space of complex polynomials of real variable t, with degree not greater than $n\in\mathbb{N}$. We equip $\mathcal{W}_n$ with an inner product defined by

$$(P,Q)=\int_0^\infty P(t)\overline{Q(t)}\mathrm{e}^{-t}\,\mathrm{d}t.$$

(a) Find an orthonormal basis of $\mathcal{W}_2$.
(b) Determine $P(g)$, where $g(t)=t^2$ and P is the orthogonal projection from $\mathcal{W}_2$ onto $\mathcal{W}_1$.

● **Problem 5.19.** Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Show that there exists a continuous one-to-one mapping $\gamma\colon[0,1]\to\mathcal{H}$ such that

$$(\gamma(b)-\gamma(a))\perp(\gamma(d)-\gamma(c))\quad\text{for all }0\le a\le b\le c\le d\le 1.$$

(We call γ a curve with *orthogonal increments*.)

● **Problem 5.20.** Let M be a dense subspace of a Hilbert space $\mathcal{H}$ and $T\in\mathscr{L}(M,\mathcal{H})$ be a self-adjoint, *positive semidefinite* operator, i.e. $(Tx,x)\ge 0$ for every $x\in M$. Prove that the following assertions are equivalent:

(a) $\overline{T(M)}=\mathcal{H}$;
(b) $\ker T=\{0\}$;
(c) $(Tx,x)>0$ for every $x\in M$, $x\ne 0$.

● **Problem 5.21.** Let X be an inner product space, $\varnothing\ne M\subseteq X$ and let $x_0\in X$.
(a) Prove that if $y\in M$ satisfies $(x_0-y,z-y)\le 0$ for every $z\in M$, then y is smallest distance projection of x_0, that is, $\operatorname{dist}(x_0,M)=\|x_0-y\|$.
(b) Assuming that M is a linear subspace of X, prove that

$$\|x_0-y\|=\operatorname{dist}(x_0,M)\quad\Longleftrightarrow\quad(x_0-y,z-y)=0\quad\text{for every }z\in M.$$

(c) For any $w\in X$, $w\ne 0$ and $\alpha\in\mathbb{K}$ consider the hyperplane $H_{w,\alpha}=\{x\in X\colon (w,x)=\alpha\}$. Show that the orthogonal projection $\Pi_{w,\alpha}$ of X onto $H_{w,\alpha}$ is given by

$$\Pi_{w,\alpha}(x)=x+\frac{\alpha-(w,x)}{\|w\|^2}w\quad(x\in X).$$

Problem 5.22. Let $(x_n)_{n=1}^\infty$ be an orthonormal basis of a Hilbert space $\mathcal{H}$ and assume that $(y_n)_{n=1}^\infty \subset \mathcal{H}$ is an orthonormal sequence satisfying

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < \infty.$$

Show that $(y_n)_{n=1}^\infty$ is a basis of $\mathcal{H}$.

Problem 5.23. Prove that the duality $L_p(\mu)^* \cong L_q(\mu)$ described in Theorem 6.6 holds true for $1 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and for an arbitrary positive measure μ. To this end, show first that every $\Lambda \in L_p(\mu)^*$ 'lives on' a σ-finite set in the following sense: there exists a sequence $(E_n)_{n=1}^\infty$ of sets of finite measure such that for every $f \in L_p(\mu)$ vanishing on $\bigcup_{n=1}^\infty E_n$ we have $\Lambda f = 0$.

Problem 5.24. Let $\mathfrak{M}$ be a σ-algebra of subsets of X and μ, ν be probability measures on $\mathfrak{M}$. Using the Radon–Nikodym theorem prove that if for some $\alpha \in (0, 1)$ we have $|\alpha\mu - (1-\alpha)\nu|(X) = 1$, then $\mu \perp \nu$.

The next three problems concern the so-called *Bergman space* defined as follows. Let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ be the open unit disc and let $L_a^2(\mathbb{D})$ be the set of all complex analytic functions f on $\mathbb{D}$ satisfying

$$\int_{\mathbb{D}} |f(z)|^2 \frac{\mathrm{d}A}{\pi} < \infty,$$

where $\mathrm{d}A$ is the two-dimensional Lebesgue measure, so $\mathrm{d}A/\pi$ is the normalized area measure on the disc $\mathbb{D}$. We equip $L_a^2(\mathbb{D})$ with usual pointwise operations and the standard L_2-norm, as every element of $L_a^2(\mathbb{D})$ is a representatie of an element of the Hilbert space $L_2(\mathbb{D}, \mathrm{d}A/\pi)$. Hence, $L_a^2(\mathbb{D})$ also inherits the inner product

$$(f, g) = \int_{\mathbb{D}} f(z)\overline{g(z)}\, \frac{\mathrm{d}A}{\pi}.$$

Problem 5.25. Using the Cauchy integral formula show that every analytic function on a closed disc $\overline{D}(a, R)$ satisfies the formula

$$f(a) = \frac{1}{\pi R^2} \int_{B(a,R)} f \, \mathrm{d}A.$$

Conclude that for every $f \in L_a^2(\mathbb{D})$ and any $z \in \mathbb{D}$ we have

$$|f(z)| \leq \frac{1}{1 - |z|} \|f\|_{L_a^2(\mathbb{D})}.$$

Problem 5.26. Using the estimate in Problem 5.25 prove that $L_a^2(\mathbb{D})$ is a Hilbert space.

Problem 5.27. In the Bergman space consider the functions defined by $e_n = \sqrt{n+1} z^n$ for $n = 0, 1, 2, \ldots$ Show that $(e_n)_{n=0}^\infty$ is an orthonormal basis of $L_a^2(\mathbb{D})$.

● **Problem 5.28.** Let M be a finite-dimensional subspace of $L_2[0,1]$ and assume that there exists a constant $c>0$ such that $\|f\|_2 \geq c\|f\|_\infty$ for every $f \in M$, where at the right-hand side we have the norm in $L_\infty[0,1]$. Prove that

$$\dim M \leq \frac{1}{c^2}.$$

● **Problem 5.29.** Determine whether it is true that in any inner product space (not necessarily complete) every maximal orthonormal set must be linearly dense.

We know from Theorem 5.13 that if $\{e_\alpha\}$ is an orthonormal set in a Hilbert space $\mathcal{H}$, then $\{e_\alpha\}$ is total (i.e. linearly dense, $\overline{\operatorname{lin}\{e_\alpha\}} = \mathcal{H}$) if and only if it is maximal. It is easily seen that one implication holds true in every inner product space. Namely, if $\{e_\alpha\}$ is total, then it must be maximal, in other words, it forms an orthonormal basis. However, when proving the converse we build a certain series and use the completeness of $\mathcal{H}$ to conclude that its sum actually lives in $\mathcal{H}$. So, the question asked above is whether one can avoid assuming completeness.

● **Problem 5.30.** Prove that the *Hilbert matrix* $(a_{ij})_{0\leq i,j<\infty}$ given by

$$a_{ij} = \frac{1}{i+j+1} \qquad (0 \leq i,j < \infty)$$

defines a bounded linear operator on ℓ_2 of norm not exceeding π. In other words, there exists $T \in \mathscr{L}(\ell_2)$ such that

$$T(x_0, x_1, x_2, \ldots) = \Big(\sum_{j=0}^{\infty} a_{0j}x_j, \sum_{j=0}^{\infty} a_{1j}x_j, \ldots\Big) \quad \text{for every } (x_n)_{n=0}^{\infty} \in \ell_2$$

and $\|T\| \leq \pi$.

In fact, one can show that $\|T\| = \pi$. When solving this problem one should first derive the following *Schur criterion*: Let $(\alpha_{ij})_{0\leq i,j<\infty}$ be an infinite matrix of nonnegative numbers and assume that there exist positive sequences $(p_i)_{i=0}^{\infty}$, $(q_j)_{j=0}^{\infty}$ and $\beta, \gamma > 0$ such that $\sum_i \alpha_{ij}p_i \leq \beta q_j$ for each $j = 0,1,2,\ldots$ and $\sum_j \alpha_{ij}q_j \leq \gamma p_i$ for each $i = 0,1,2,\ldots$ Then, the matrix $(\alpha_{ij})_{0\leq i,j<\infty}$ corresponds to an operator $A \in \mathscr{L}(\ell_2)$ satisfying $\|A\| \leq \sqrt{\beta\gamma}$.